$$= \left| -\frac{(n+1)x}{1+(n+1)^2x^2} \right| = \frac{(n+1)x}{1+(n+1)^2x^2} < \varepsilon$$

if $\quad (n+1)^2x^2\varepsilon - (n+1)x + \varepsilon > 0$

or if $\quad (n+1) > \dfrac{x+\sqrt{x^2-4x^2\varepsilon^2}}{2x^2\varepsilon}$ or if $n > -1 + \dfrac{1+\sqrt{1-4\varepsilon^2}}{2x\varepsilon}$

Now if $x \to 0$, then $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad x \quad \forall \in [0, 1]$$

So the convergence is non-uniform in [0, 1]. Here $x = 0$ is a point of non-uniform convergence.

Example 7. *Test the uniform convergence of the series* $\sum_{n=1}^{\infty} \left[\dfrac{2n^2x^2}{e^{n^2x^2}} - \dfrac{2(n-1)^2x^2}{e^{(n-1)^2x^2}} \right]$ *in [0, 1].*

Sol. Here $\quad f_n(x) = \dfrac{2n^2x^2}{e^{n^2x^2}} - \dfrac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$

$\therefore \quad f_1(x) = \dfrac{2x^2}{e^{n^2}} - 0$

$$f_2(x) = \frac{2\,.\,2^2x^2}{e^{2^2x^2}} - \frac{2x^2}{e^{x^2}}$$

$$f_3(x) = \frac{2\,.\,3^2x^2}{e^{3^2x^2}} - \frac{2\,.\,2^2x^2}{e^{2^2x^2}}$$

....

$$f_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

$\Rightarrow \quad S_n(x) = \dfrac{2n^2x^2}{e^{n^2x^2}}$

$\therefore \quad S(x) = \lim\limits_{n\to\infty} S_n(x) = \lim\limits_{n\to\infty} \dfrac{2n^2x^2}{e^{n^2x^2}} \qquad \left| \text{Form } \dfrac{\infty}{\infty} \right.$

$$= \lim_{n\to\infty} \frac{4nx^2}{2nx^2e^{n^2x^2}} = \lim_{n\to\infty} \frac{2}{e^{n^2x^2}} = 0 \ \forall\, x$$

Now the series $\sum_{n=1}^{\infty} f_n(x)$ will be uniformly convergent in [0, 1] if for a given $\varepsilon > 0$, there is always a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, 1]$$

i.e., $\quad \dfrac{2n^2x^2}{e^{n^2x^2}} < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, 1] \qquad ...(1)$

But, in particular, if we take $x = \dfrac{1}{n}$ which is a point of [0, 1] for all $n \in$ N, then the inequality (1) gives $\dfrac{2}{e} < \varepsilon$ which shows that we take $\varepsilon < \dfrac{2}{e}$, the above inequality will not hold.

Hence the given series is non-uniformly convergent in [0, 1].

Example 8. *Show that the series*

$$\frac{x}{1+x^2}+\left(\frac{2^2x}{1+2^3x^2}-\frac{x}{1+x^2}\right)+\left(\frac{3^2x}{1+3^3x^2}-\frac{2^2x}{1+2^3x^2}\right)+\dots$$

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does not converge uniformly on [0, 1].

Sol. Here
$$f_1(x)=\frac{x}{1+x^2}$$
$$f_2(x)=\frac{2^2x}{1+2^3x^2}-\frac{x}{1+x^2}$$
$$f_3(x)=\frac{3^2x}{1+3^3x^2}-\frac{2^2x}{1+2^3x^2}$$
....
$$f_n(x)=\frac{n^2x}{1+n^3x^2}-\frac{(n-1)^2x}{1+(n-1)^3x^2}$$

$\therefore$
$$S_n(x)=\frac{n^2x}{1+n^3x^2}$$

$$S(x)=\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\frac{n^2x}{1+n^3x^2}=\lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^3}+x^2}=0\ \forall\, x\in[0,1]$$

$$|S_n(x)-S(x)|=\left|\frac{n^2x}{1+n^3x^2}-0\right|=\frac{n^2x}{1+n^3x^2}$$

Let
$$y=\frac{n^2x}{1+n^3x^2}$$

then
$$\frac{dy}{dx}=\frac{(1+n^3x^2).n^2-n^2x.2n^3x}{(1+n^3x^2)^2}=\frac{n^2(1-n^3x^2)}{(1+n^3x^2)^2}$$

For max. or min. $\frac{dy}{dx}=0 \Rightarrow 1-n^3x^2=0\Rightarrow x=\frac{1}{n^{3/2}}$

Also
$$\frac{d^2y}{dx^2}=\frac{n^2[(1+n^3x^2)^2(-2n^3x)-(1-n^3x^2).2(1+n^3x^2).2n^3x]}{(1+n^3x^2)^4}$$
$$=\frac{n^2[-2n^3x(1+n^3x^2)-4n^3x(1-n^3x^2)]}{(1+n^3x^2)^3}$$
$$=\frac{-2n^5x(1+n^3x^2)+2(1-n^3x^2)]}{(1+n^3x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{3/2}}}=\frac{-2n^5.\frac{1}{n^{3/2}}(1+1)}{(1+1)^3}=-\frac{1}{2}n^{7/2}<0$$

$\Rightarrow$ y is maximum when $x=\frac{1}{n^{3/2}}$ and maximum value of $y=\frac{n^2.\frac{1}{n^{3/2}}}{1+1}=\frac{1}{2}\sqrt{n}$

$\therefore$
$$M_n=\max_{x\in[0,1]}|S_n(x)-S(x)|=\frac{1}{2}\sqrt{n}\to\infty \text{ as } n\to\infty.$$

Since M_n does not tend to zero as $n \to \infty$, the sequence $< S_n >$ and hence the given series is non-uniformly convergent on [0, 1]. Here 0 is a point of non-uniform convergence.

Example 9. *Show that the series* $\frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \left(\frac{3x^2}{1+3x} - \frac{2x^2}{1+2x}\right) + \ldots\ldots$ *converges uniformly on [0, 1].*

Sol. Here,

$$f_1(x) = \frac{x^2}{1+x}$$

$$f_2(x) = \frac{2x^2}{1+2x} - \frac{x^2}{1+x}$$

$$f_3(x) = \frac{3x^2}{1+3x} - \frac{2x^2}{1+2x}$$

....

$$f_n(x) = \frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}$$

$$\therefore \quad S_n(x) = \frac{nx^2}{1+nx}$$

$$S(x) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{nx^2}{1+nx} = \begin{cases} x, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given, then for $0 < x \le 1$, we have

$$| S_n(x) - S(x) | = \left| \frac{nx^2}{1+nx} - x \right| = \left| \frac{-x}{1+nx} \right| = \frac{x}{1+nx} < \varepsilon$$

if $\quad 1 + nx > \frac{x}{\varepsilon} \quad$ or if $\quad nx > \frac{x}{\varepsilon} - 1 \quad$ or if $\quad n > \frac{1}{\varepsilon} - \frac{1}{x}$

If we choose a positive integer m just $\ge \frac{1}{\varepsilon} - \frac{1}{x}$, then

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad 0 < x \le 1$$

For $x = 0$, $| S_n(x) - S(x) | = 0 < \varepsilon \ \forall\, n \ge 1$.

Hence the series converges uniformly on [0, 1].

Example 10. *Test for uniform convergence the series* $\sum_{n=0}^{\infty} xe^{-nx}$ *in the closed interval [0, 1].*

Sol. Here $\quad S_n(x) = \sum_{n=0}^{n-1} xe^{-nx} = x + xe^{-x} + xe^{-2x} + \ldots\ldots + xe^{-(n-1)x}$

which is a geometric series

$$= \frac{x(1-e^{-nx})}{1-e^{-x}} = \frac{x\left(1-\frac{1}{e^{nx}}\right)}{1-\frac{1}{e^x}} = \frac{xe^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right)$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} \frac{xe^x}{x-1}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Now for $0 < x \le 1$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{xe^x}{e^x - 1}\left(1 - \frac{1}{e^{nx}}\right) - \frac{xe^x}{e^x - 1}\right| = \left|\frac{-xe^x}{(e^x - 1)e^{nx}}\right| = \frac{xe^x}{(e^x - 1)e^{nx}} < \varepsilon$$

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if $$\frac{(e^x - 1)e^{nx}}{xe^x} > \frac{1}{\varepsilon}$$

or if $$\log (e^x - 1) + nx - \log x - x > \log \frac{1}{\varepsilon}$$

or if $$\log \left\{x + \frac{x^2}{2!} + \ldots\ldots\right\} - \log x + nx - x > \log \frac{1}{\varepsilon}$$

or if $$\log \left\{1 + \frac{x}{2!} + \ldots\ldots\right\} + nx - x > \log \frac{1}{\varepsilon}$$

or if $$n > \frac{\log \frac{1}{\varepsilon} + x - \log\left\{1 + \frac{x}{2!} + \ldots\ldots\right\}}{x}$$

This shows that when $x \to 0$, $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad \forall\, x \in [0, 1].$$

Hence the series is non-uniformly convergent in any interval containing 0.

Example 11. *Show that $x = 0$ is a point of non-uniformly convergence of the series*

$$\sum_{n=1}^{\infty} \frac{-2x(1 + x)^{n-1}}{[1 + (1 + x)^{n-1}][1 + (1 + x)^n]}.$$

Sol. Here $$f_n(x) = \frac{-2x(1 + x)^{n-1}}{[1 + (1 + x)^{n-1}][1 + (1 + x)^n]} = \frac{2}{1 + (1 + x)^n} - \frac{2}{1 + (1 + x)^{n-1}}$$

$\Rightarrow$ $$f_1(x) = \frac{2}{1 + (1 + x)} - \frac{2}{1 + 1}$$

$$f_2(x) = \frac{2}{1 + (1 + x)^2} - \frac{2}{1 + (1 + x)}$$

$$f_3(x) = \frac{2}{1 + (1 + x)^3} - \frac{2}{1 + (1 + x)^2}$$

....

$$f_n(x) = \frac{2}{1 + (1 + x)^n} - \frac{2}{1 + (1 + x)^{n-1}}$$

$\therefore$ $$S_n(x) = \frac{2}{1 + (1 + x)^n} - 1$$

$$S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \left[\frac{2}{1 + (1 + x)^n} - 1\right] = \begin{cases} -1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ 1 & \text{when } x < 0 \end{cases}$$

Thus for $x > 0$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{2}{1 + (1 + x)^n} - 1 + 1\right| = \frac{2}{1 + (1 + x)^n} < \varepsilon$$

if $$1 + (1 + x)^n > \frac{2}{\varepsilon} \quad \text{or if} \quad n \log (1 + x) > \log \left(\frac{2}{\varepsilon} - 1\right)$$

of if $$n > \frac{\log\left(\frac{2}{\varepsilon}-1\right)}{\log(1+x)}$$

This shows that if $x \to 0$, $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series since no value of m can be chosen such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \geq m \text{ and for every } x \text{ near } x = 0.$$

Example 12. *Prove that the series* $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots$ *converges in the interval [0, k], k > 0 but the series is not uniformly convergent in [0, k].*

Sol. Here $S_n(x)$ = Sum to n terms of the series

$$= x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots + \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4\left[1-\frac{1}{(1+x^4)^n}\right]}{1-\frac{1}{1+x^4}} = (1+x^4)\left[1-\frac{1}{(1+x^4)^n}\right]$$

$$\therefore \qquad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4, & \text{when } 0 < x \leq k \\ 0, & \text{when } x = 0 \end{cases}$$

As $S(x)$ exists for all values of x in $[0, k]$, $k > 0$, the series is convergent in this interval.

To text for uniform convergence, we have for $0 < x \leq k$ and for a given $\varepsilon > 0$,

$$|S_n(x) - S(x)| = \left|(1+x^4)\left\{1-\frac{x^4}{(1+x^4)^n}\right\}-(1+x^4)\right| = \frac{1+x^4}{(1+x^4)^n} = \frac{1}{(1+x^4)^{n-1}} < \varepsilon$$

if $(1+x^4)^{n-1} > \frac{1}{\varepsilon}$ or if $(n-1)\log(1+x^4) > \log\frac{1}{\varepsilon}$

or if $n - 1 > \frac{\log\frac{1}{\varepsilon}}{\log(1+x^4)}$ or if $n > 1 + \frac{\log\frac{1}{\varepsilon}}{\log(1+x^4)}$

This shows that if $x \to 0$, $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[h, k]$ where $0 < h < k$ since in this case.

$|S_n(x) - S(x)| < \varepsilon$ for all $n \geq m$ and for every x in $[h, k]$ where m is a positive integer just

$$\geq 1 + \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(1+h^4)}.$$

Example 13. *Discuss the uniform convergence of the series* $\sum_{n=1}^{\infty} x^n(1-x)$ *on [0, 1].*

Sol. Here $S_n(x) = x(1-x) + x^2(1-x) + x^3(1-x) \ldots\ldots + x^n(1-x)$

$$= \frac{x(1-x)(1-x^n)}{1-x} = x(1-x^n)$$

$$\therefore \qquad S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} x(1 - x^n) = \begin{cases} 0 & \text{when } x = 0 \text{ or } 1 \\ x & \text{when } 0 < x < 1 \end{cases}$$

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Now if $0 < x < 1$, then for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = | x(1 - x^n) - x | = x^{n+1} < \varepsilon$$

if $$\left(\frac{1}{x}\right)^{n+1} > \frac{1}{\varepsilon} \quad \text{or} \quad \text{if } n + 1 > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} \quad \text{or} \quad \text{if} \quad n > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} - 1$$

This shows that if $x \to 1$, $n \to \infty$. Thus it is not possible to find a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \text{for every } x \text{ in } [0, 1].$$

Here $x = 1$ is the point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[0, b[$ where $0 < b < 1$ since in this case, we can choose a positive integer m just $\geq \dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{b}} - 1$ such that $| S_n(x) - S(x) | < \varepsilon$ $\forall\, n \geq m$ and $\forall\, x$ in $[0, b]$.

Example 14. *Show that if $0 < r < 1$, then each of the following series is uniformly convergent on R :*

$$\sum_{n=1}^{\infty} r^n \cos nx$$

Sol. Here $\qquad f_n(x) = r^n \cos nx$

$$| f_n(x) | = | r^n \cos nx | = r^n \, | \cos nx | \qquad (\because \quad r > 0)$$

$$\leq r^n = M_n \;\forall\, x \in R.$$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric series with $0 < r < 1$, it is convergent.

Hence by Weierstrass's M-test, the given series converges uniformly on R.

Example 15. *Show that the following series are uniformly convergent for all real x.*

$$\sum_{n=1}^{\infty} \frac{\sin (x^2 + n^2 x)}{n(n+2)}.$$

Sol. Here $\qquad f_n(x) = \dfrac{\sin (x^2 + n^2 x)}{n(n+2)}$

$$| f_n(x) | = \left| \frac{\sin (x^2 + n^2 x)}{n(n+2)} \right|$$

$$= \frac{| \sin (x^2 + n^2 x) |}{n(n+2)} \leq \frac{1}{n(n+2)} \leq \frac{1}{n^2} = M_n \;\forall\, x \in R$$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 16. *Show that the following series are uniformly and absolutely convergent for all real values of x and p > 1,* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$.

Sol. Here $$f_n(x) = \frac{\sin nx}{n^p}$$

$$| f_n(x) | = \left| \frac{\sin nx}{n^p} \right| \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n \ \forall\, x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, by Weierstrass's M-test, the given series converges uniformly and absolutely for all real values of x.

Example 17. *Test for uniform convergence the series*

(*i*) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ (*ii*) $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$.

Sol. (*i*) Here $f_n(x) = \frac{x}{(n+x^2)^2}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n+x^2)^2 \,.\, 1 - x \,.\, 2(n+x^2) \,.\, 2x}{(n+x^2)^4} = \frac{(n+x^2) - 4x^2}{(n+x^2)^3} = \frac{n-3x^2}{(n+x^2)^3}$$

For max. or min. $\frac{df_n(x)}{dx} = 0 \Rightarrow n - 3x^2 = 0 \Rightarrow x = \sqrt{\frac{n}{3}}$

Also $$\frac{d^2 f_n(x)}{dx^2} = \frac{(n+x^2)^3 \,.\, (-6x) - (n-3x^2) \,.\, 3(n+x^2)^2 \,.\, 2x}{(n+x^2)^6}$$

$$= \frac{-6x[(n+x^2)+(n-3x^2)]}{(n+x^2)^4}$$

$$\left.\frac{d^2 f_n(x)}{dx^2}\right]_{x=\sqrt{\frac{n}{3}}} = \frac{-6\sqrt{\frac{n}{3}}\left(n+\frac{n}{3}\right)}{\left(n+\frac{n}{3}\right)^4} = -\frac{27\sqrt{3}}{32 n^{5/2}} < 0$$

$\Rightarrow$ $f_n(x)$ is maximum at $x = \sqrt{\frac{n}{3}}$ and the maximum value of $f_n(x)$ is

$$\frac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16 n^{3/2}}$$

$$\Rightarrow \quad | f_n(x) | \le \frac{3\sqrt{3}}{16 n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

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(*ii*) Here $$f_n(x) = \frac{x}{n(1+nx^2)}$$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{1}{n} \cdot \frac{(1+nx^2).1 - x.2nx}{(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}$$

For max. or min. $\frac{df_n(x)}{dx} = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$

Also $$\frac{d^2f_n(x)}{dx^2} = \frac{1}{n} \cdot \frac{(1+nx^2)^2 .(-2nx) - (1-nx^2).2(1+nx^2).2nx}{(1+nx^2)^4}$$

$$= -\frac{2x[(1+nx^2)+2(1-nx^2)]}{(1+nx^2)^3}$$

$$\left.\frac{d^2f_n(x)}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = -\frac{\frac{2}{\sqrt{n}}[1+1]}{(1+1)^3} = -\frac{1}{2\sqrt{n}} < 0$$

$\Rightarrow$ $f_n(x)$ is maximum at $x = \frac{1}{\sqrt{n}}$ and the maximum value of $f_n(x)$ is $\frac{\frac{1}{\sqrt{n}}}{n(1+1)} = \frac{1}{2n^{3/2}}$.

$$\Rightarrow \quad | f_n(x) | \le \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

Example 18. *Show that the series* $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ *converges in* $[1, \infty)$.

Sol. Here $$f_n(x) = \frac{1}{1+n^2x}$$

$$\therefore \quad | f_n(x) | = \left| \frac{1}{1+n^2x} \right| \le \frac{1}{1+n^2} < \frac{1}{n^2} = M_n \ \forall \ x \in [1, \infty).$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of $x \in [1, \infty)$.

Example 19. *Show that the series* $\sum_{n=1}^{\infty} \frac{a_n x^{2n}}{1+x^{2n}}$ *is uniformly convergent for all real* x *if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Sol. Here $$f_n(x) = \frac{a_n x^{2n}}{1+x^{2n}}$$

Since $$\frac{x^{2n}}{1+x^{2n}} < 1 \quad \forall \ x \in R$$

$$\therefore \quad | f_n(x) | = \left| \frac{a_n x^{2n}}{1+x^{2n}} \right| = | a_n | . \frac{x^{2n}}{1+x^{2n}} < | a_n | = M_n \text{ for all } x \in R$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 20. *Show that the series* $\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ *is uniformly convergent for all real x if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Sol. Here $$f_n(x) = \frac{a_n x^n}{1+x^{2n}}$$

Let $$y = \frac{x^n}{1+x^{2n}}$$

Then $$\frac{dy}{dx} = \frac{(1+x^{2n}) \cdot nx^{n-1} - x^n \cdot 2n\, x^{2n-1}}{(1+x^{2n})^2}$$

$$= \frac{nx^{n-1}(1+x^{2n} - 2x^{2n})}{(1+x^{2n})^2} = \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow x = 1$

Also $$\frac{d^2y}{dx^2} = \frac{(1+x^{2n})^2 \cdot [n(n-1)x^{n-2}(1-x^{2n}) + nx^{n-1}(-2nx^{2n-1})] - nx^{n-1}(1-x^{2n}) \cdot 2(1+x^{2n}) \cdot 2nx^{2n-1}}{(1+x^{2n})^4}$$

$$= \frac{(1+x^{2n})[n(n-1)x^{n-2}(1-x^{2n}) - 2n^2x^{3n-2}] - 4n^2x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1} = \frac{2[0-2n^2]-0}{(2)^3} = -\frac{n^2}{2} < 0$$

$\Rightarrow$ $y = \frac{x^n}{1+x^{2n}}$ is maximum at $x = 1$ and the maximum value of y is $\frac{1}{2}$.

$\therefore$ $$|f_n(x)| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{x^n}{1+x^{2n}}\right| |a_n|$$

$$\leq \frac{1}{2} |a_n| < |a_n| = M_n \text{ for all real values of } x.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 21. *If the series* Σa_n *converges absolutely then prove that* $\Sigma a_n \cos nx$ *and* $\Sigma a_n \sin nx$ *are uniformly convergent on R.*

Sol. Let us test $\Sigma a_n \cos nx$.

Here $$f_n(x) = a_n \cos nx$$

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$\therefore$ $|f_n(x)| = |a_n \cos nx| = |a_n|\,|\cos nx| \le |a_n| = M_n$ for all real values of x

Since Σa_n is absolutely convergent, therefore, $\Sigma M_n = \Sigma |a_n|$ is convergent. Hence by Weierstrass's M-test, the series $\Sigma a_n \cos nx$ is uniformly convergent on R.

Similarly, $\Sigma a_n \sin nx$ is uniformly convergent on R.

Example 22. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$ *is absolutely and uniformly convergent for all real x if p > 1.*

Sol. Here $$f_n(x) = \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$$

Since $$\frac{x^{2n}}{1+x^{2n}} < 1 \quad \forall\, x \in R$$

$$\therefore \quad |f_n(x)| = \left|\frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{n^p} = M_n \text{ for all } x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, therefore, by Weierstrass's M-test, the given series is absolutely and uniformly convergent for all real x if $p > 1$.

Example 23. *Show that* $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *is uniformly convergent for all real x and p > 1.*

Sol. Here $$f_n(x) = \frac{1}{n^p + n^q x^2}$$

Since $x^2 \ge 0$ for all real x.

$$\therefore \quad n^q x^2 \ge 0 \Rightarrow n^p + n^q x^2 \ge n^p \Rightarrow \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p}$$

$$\therefore \quad |f_n(x)| = \left|\frac{1}{n^p + n^q x^2}\right| \le \frac{1}{n^p} = M_n \text{ for all real } x.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real x and $p > 1$.

Example 24. *Show that the series* $\sum_{n=1}^{\infty} \frac{x}{n^p + n^q x^2}$ *is uniformly convergent for all real x if p + q > 2.*

Sol. Here $$f_n(x) = \frac{x}{n^p + n^q x^2}$$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n^p + n^q x^2) \cdot 1 - x \cdot 2n^q x}{(n^p + n^q x^2)^2} = \frac{n^p - n^q x^2}{(n^p + n^q x^2)^2}$$

For max. or min. $\frac{df_n(x)}{dx} = 0$

$$\Rightarrow \quad n^p - n^q x^2 = 0 \Rightarrow x^2 = n^{p-q} \Rightarrow x = n^{\frac{p-q}{2}}$$

Also $$\frac{d^2 f_n(x)}{dx^2} = \frac{(n^p + n^q x^2)^2 . (-2n^q x) - (n^p - n^q x^2) . 2(n^p + n^q x^2) . 2n^q x}{(n^p + n^q x^2)^4}$$

$$= -\frac{2n^q x[(n^p + n^q x^2) + 2(n^p - n^q x^2)]}{(n^p + n^q x^2)^3}$$

$$\left.\frac{d^2 f_n(x)}{dx^2}\right]_{x = n^{\frac{p-q}{2}}} = -\frac{2n^q . n^{\frac{p-q}{2}} (n^p + n^p)}{(n^p + n^p)^3} = -\frac{1}{2} n^{\frac{q-3p}{2}} < 0$$

$\Rightarrow$ $f_n(x)$ is maximum at $x = n^{\frac{p-q}{2}}$ and the maximum value of $f_n(x)$ is $\dfrac{n^{\frac{p-q}{2}}}{n^p + n^p} = \dfrac{1}{2n^{\frac{p+q}{2}}}$

$$\Rightarrow \qquad |f_n(x)| = \left|\frac{x}{n^p + n^q x^2}\right| \le \frac{1}{2n^{\frac{p+q}{2}}} < \frac{1}{n^{\frac{p+q}{2}}} = M_n.$$

Since $\sum M_n = \sum \dfrac{1}{n^{\frac{p+q}{2}}}$ is convergent if $\dfrac{p+q}{2} > 1$

i.e., if $p + q > 2$, therefore, by Weierstrass's M-test, the given series is convergent for all real x if $p + q > 2$.

Example 25. *Show that the series* $1 + \dfrac{e^{-2x}}{2^2 - 1} + \dfrac{e^{-4x}}{4^2 - 1} + \dfrac{e^{-6x}}{6^2 - 1} + \ldots\ldots$ *is uniformly convergent for* $x \ge 0$.

Sol. Neglecting the first term, we have $f_n(x) = \dfrac{e^{-2nx}}{(2n)^2 - 1} = \dfrac{e^{-2nx}}{4n^2 - 1}$

For all $x \ge 0$, we have $\quad e^{2nx} \ge 1 \quad \Rightarrow \quad e^{-2nx} \le 1$

Also $\quad 3n^2 > 1 \ \forall\ n \quad \Rightarrow \quad 4n^2 > n^2 + 1 \quad \Rightarrow \quad 4n^2 - 1 > n^2$

$$\therefore \qquad |f_n(x)| = \left|\frac{e^{-2nx}}{4n^2 - 1}\right| < \frac{1}{n^2} = M_n.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for $x \ge 0$.

Example 26. *Test for uniform convergence the series*

$$\frac{1}{(1+x)^3} + \frac{2}{(2+x)^3} + \frac{3}{(3+x)^3} + \ldots\ldots, x \ge 0.$$

Sol. The given series is $\sum_{n=1}^{\infty} \dfrac{n}{(n+x)^3}$

Here $$f_n(x) = \frac{n}{(n+x)^3}$$

$$\forall\ x \ge 0, \qquad |f_n(x)| = \left|\frac{n}{(n+x)^3}\right| = \frac{n}{(n+x)^3} < \frac{n}{n^3} = \frac{1}{n^2} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all $x \ge 0$.

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Example 27. *Show that the series* $\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots\ldots$ *is uniformly convergent in (– 1, 1).*

Sol. The given series is $\sum_{n=1}^{\infty} \frac{2^n x^{2^n-1}}{1+x^{2^n}} = \sum_{n=1}^{\infty} f_n(x)$

$$|f_n(x)| = \left|\frac{2^n \cdot x^{2^n-1}}{1+x^{2^n}}\right| \le 2^n \cdot k^{2^n-1} = M_n \text{ for } |x| \le k < 1 \quad \ldots(1)$$

Now $\qquad M_n = 2^n k^{2^n-1} \Rightarrow M_{n+1} = 2^{n+1} \cdot k^{2^{n+1}-1}$

$$\therefore \lim_{n\to\infty} \frac{M_n}{M_{n+1}} = \lim_{n\to\infty} \frac{k^{2^n-1}}{2 \cdot k^{2^{n+1}-1}} = \lim_{n\to\infty} \frac{1}{2k^{2^{n+1}-2^n}} = \lim_{n\to\infty} \frac{1}{2k^{2n}} = \infty, \text{ since } k < 1$$

By ratio test, the series $\sum_{n=1}^{\infty} M_n$ is convergent. Hence, by Weierstrass's M-test, the given series is uniformly convergent in (– 1, 1).

Example 28. *Test the series $\Sigma f_n(x)$ for uniform convergence where*

$$f_n(x) = \frac{1}{(x^2+n)(x^2+n+1)}$$

Sol. $$|f_n(x)| = \left|\frac{1}{(x^2+n)(x^2+n+1)}\right| < \frac{1}{n^2} = M_n \ \forall\, x \in R$$

Since $\Sigma M_n = \sum \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real values of x.

Example 29. *Test for uniform convergence the series*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots \text{ in } [-1, 1].$$

Sol. Neglecting 1, we have $f_n(x) = \frac{x^n}{n!}$

$$|f_n(x)| = \left|\frac{x^n}{n!}\right| \le \frac{1}{n!} \text{ for } -1 \le x \le 1$$

Since $n! \ge 2^n$ for $n > 3$, therefore, we have $|f_n(x)| \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = M_n$

But we know that $\Sigma M_n = \sum \left(\frac{1}{2}\right)^n$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent in [– 1, 1].

Example 30. *Prove that if δ is any fixed positive number less than unity, the series $\sum_{n=1}^{\infty} (n+1)x^n$ converges uniformly in (– δ, δ).*

Sol. Here $\qquad -\delta < x < \delta$ and $0 < \delta < 1 \Rightarrow |x| < \delta < 1$

Also $\qquad f_n(x) = (n+1)\, x^n$

$\therefore \qquad |f_n(x)| = |(n+1)\, x^n| < (n+1)\, \delta^n = M_n$

$$\lim_{n \to \infty} \frac{M_n}{M_{n+1}} = \lim_{n \to \infty} \frac{(n+1)\,\delta^n}{(n+2)\,\delta^{n+1}} = \frac{1}{\delta} > 1$$

$\therefore$ By ratio test, ΣM_n is convergent. Hence the given series is uniformly convergent in $(-\delta, \delta)$.

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3.14. ABEL'S TEST

If (i) $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$,

(ii) the sequence $< g_n(x) >$ is monotonic decreasing for all $x \in [a, b]$, and

(iii) there exists a positive real number k such that $| g_n(x) | < k \ \forall x \in [a, b]$ and $n \in N$ then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

Proof. $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$

$\Rightarrow$ By Cauchy's criterion, for each $\varepsilon > 0$ and $\forall x \in [a, b]$, there exists a positive integer m (depending only on ε) such that

$$| f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) | < \frac{\varepsilon}{k} \quad \forall n \geq m, p \in N$$

$$\Rightarrow \quad \left| \sum_{r=n+1}^{n+p} f_r(x) \right| < \frac{\varepsilon}{k} \quad \forall n \geq m, p \in N$$

Also, the sequence $< g_n(x) >$ is monotonic decreasing on $[a, b]$

and $\quad | g_n(x) | < k \quad \forall x \in x[a, b]$ and $n \in N$

$\therefore$ By Abel's lemma, we get

$$\Rightarrow \left| \sum_{r=n+1}^{n+p} f_r(x)\, g_r(x) \right| < \frac{\varepsilon}{k} \cdot k = \varepsilon \quad \forall n \geq m, p \in N \text{ and } x \in [a, b]$$

$$\Rightarrow | f_{n+1}(x)\, g_{n+1}(x) + f_{n+2}(x)\, g_{n+2}(x) + \ldots\ldots + f_{n+p}(x)\, g_{n+p}(x) | < \varepsilon$$
$$\forall n \geq m, p \in N \text{ and } x \in [a, b]$$

Hence by Cauchy's criterion, $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

3.15. DIRICHLET'S TEST

If (i) there exists a positive real number k such that

$$| S_n(x) | = \left| \sum_{r=1}^{\infty} f_r(x) \right| < k \ \forall x \in [a, b], n \in N \text{ and}$$

(ii) $< g_n(x) >$ is a positive monotonic decreasing sequence converging uniformly to zero on $[a, b]$ then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

Proof. Since $| S_n(x) | < k \quad \forall x \in [a, b], n \in N$

$\therefore \quad \forall x \in [a, b], n \geq m_1, p \in N$, we have

$$| S_{n+p}(x) - S_n(x) \leq | S_{n+p}(x) | + | S_n(x) | < k + k = 2k$$

$$\Rightarrow \qquad \left| \sum_{r=n+1}^{n+p} f_r(x) \right| < 2k \quad \forall\, n \geq m_1,\, p \in \mathrm{N},\, x \in [a, b]$$

Also, the sequence $< g_n(x) >$ is positive monotonic decreasing on $[a, b]$.

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$\therefore$ By Abel's lemma, we get

$$\left| \sum_{r=n+1}^{n+p} f_r(x)\, g_r(x) \right| < 2k g_{n+1}(x) \ \forall\, n \geq m_1,\, p \in \mathrm{N} \text{ and } x \in [a, b] \qquad \ldots(1)$$

Since $< g_n(x) >$ converges uniformly to zero on $[a, b]$.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$$| g_n(x) | < \frac{\varepsilon}{2k} \quad \forall\, n \geq m_2 \qquad \ldots(2)$$

Let $m = \max \{m_1, m_2\}$, then both (1) and (2) hold for $n \geq m$.

From (1) and (2), we have

$$\left| \sum_{r=n+1}^{n+p} f_r(x)\, g_r(x) \right| < 2k \cdot \frac{\varepsilon}{2k} = \varepsilon \quad \forall\, n \geq m,\, p \in \mathrm{N},\, x \in [a, b]$$

Hence $\Sigma f_n(x) g_n(x)$ is uniformly convergent on $[a, b]$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that the series* $\sum \dfrac{(-1)^{n-1}}{n} x^n$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $\qquad f_n(x) = \dfrac{(-1)^{n-1}}{n}$ and $g_n(x) = x^n$

The series $\Sigma f_n(x)$ is convergent by Leibnitz's test. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Also, for $0 \leq x \leq 1$, $x^n > x^{n+1} \quad \forall\, n \in \mathrm{N}$ and $| g_n(x) | = | x^n | = | x |^n \leq 1$

$\therefore \quad < g_n(x) >$ is monotonic decreasing and bounded on $[0, 1]$ for all $n \in \mathrm{N}$.

Hence by Abel's test, the series $\sum f_n(x) g_n(x) = \sum \dfrac{(-1)^{n-1}}{n} x^n$ is uniformly convergent on $[0, 1]$.

Example 2. *If* Σa_n *is convergent, then show that* $\sum \dfrac{a_n}{n^x}$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $\qquad f_n(x) = a_n$ and $g_n(x) = \dfrac{1}{n^x}$

The series $\Sigma f_n(x) = \Sigma a_n$ is given to be convergent. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Since $< n^x >$ increases on $[0, 1]$, $< \dfrac{1}{n^x} >$ decreases on $[0, 1]$.

Also, $\quad | g_n(x) | = \dfrac{1}{n^x} \le \dfrac{1}{n^0} = 1.$

$\therefore$ The sequence $< g_n(x) >$ is monotonic decreasing and bounded on [0, 1] for all $n \in \mathrm{N}$.

Hence by Abel's test, the series $\sum f_n(x)g_n(x) = \sum \dfrac{a_n}{n^x}$ is uniformly convergent on [0, 1].

Example 3. *Show that the series* $\sum \dfrac{(-1)^{n-1}}{n+x^2}$ *is uniformly convergent for all values of* x.

Sol. Let $\quad f_n(x) = (-1)^{n-1}$ and $g_n(x) = \dfrac{1}{n+x^2}$

Now $\quad \mathrm{S}_n(x) = \sum_{r=1}^{\infty} f_r(x) = 1 - 1 + 1 - 1 + \ldots\ldots + (-1)^{n-1} = \begin{cases} 1, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$

$\Rightarrow \quad \mathrm{S}_n(x)$ is bounded for all n and for all x.

Also $< g_n(x) > = < \dfrac{1}{n+x^2} >$ is a positive monotonic decreasing sequence converging to 0 for all values of x.

Hence by Dirichlet's test, the series $\sum f_n(x)g_n(x) = \sum \dfrac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all x.

Example 4. *Show that the series* $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \ldots\ldots$ *converges uniformly in* $(0, 2\pi)$.

Sol. The given series is $\sum \dfrac{\cos nx}{n}$.

Let $\quad f_n(x) = \cos nx$ and $g_n(x) = \dfrac{1}{n}$

Now $\quad \mathrm{S}_n(x) = \sum_{r=1}^{\infty} f_r(x) = \cos x + \cos 2x + \cos 3x + \ldots\ldots + \cos nx$

$$= \frac{\cos\left[x + \frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{\cos\frac{n+1}{2}x\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore \quad | \mathrm{S}_n(x) | = \frac{\left|\cos\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|} \quad \text{or} \quad |\mathrm{S}_n(x)| \le \left|\operatorname{cosec}\frac{x}{2}\right|$$

But cosec $\frac{x}{2}$ is bounded for all values of x in $(0, 2\pi)$. Let k be the least upper bound of cosec $\frac{x}{2}$ in $(0, 2\pi)$, then $|S_n(x)| < k$ for all $x \in (0, 2\pi)$.

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Also $< g_n(x) > = < \frac{1}{n} >$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series $\sum f_n(x) g_n(x) = \sum \frac{\cos nx}{n}$ converges uniformly in $(0, 2\pi)$.

Example 5. *Test the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ *for uniform convergence on [0, 1].*

Sol. Let $\qquad f_n(x) = \sin nx \quad \text{and} \quad g_n(x) = \frac{1}{n}$

Now $\qquad S_n(x) = \sum_{r=1}^{n} f_r(x) = \sin x + \sin 2x + \sin 3x + \ldots + \sin nx$

$$= \frac{\sin\left[x + \frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{n+1}{2}x\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore \qquad |S_n(x)| = \frac{\left|\sin\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

But cosec $\frac{x}{2}$ is bounded on $(0, 1]$ which is a subset of $\left(0, \frac{\pi}{2}\right)$.

Let k be the least upper bound of cosec $\frac{x}{2}$ on $(0, 1]$

When $x = 0$, $S_n(x) = 0 + 0 + 0 + \ldots.. + 0 = 0$

$$\therefore \qquad |S_n(x)| < k \quad \forall\, x \in [0, 1] \quad \text{and} \quad n \in \mathbb{N}.$$

Also $< g_n(x) > = < \frac{1}{n} >$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series $\Sigma f_n(x) g_n(x) = \sum \frac{\sin nx}{n}$ converges uniformly on $[0, 1]$.

Example 6. *Prove that the series* $\sum (-1)^n \frac{x^2 + n}{n^2}$ *converges uniformly in every bounded interval, but does not converge absolutely for any value of x.*

Sol. Let the bounded interval be $[a, b]$ so that there exists a positive number k such that for all

$$x \in [a, b], \ | x | < k.$$

Let $$f_n(x) = (-1)^n \quad \text{and} \quad g_n(x) = \frac{x^2 + n}{n^2}$$

$$S_n(x) = \sum_{r=1}^{\infty} f_r(x) = -1 + 1 - 1 + 1 \ldots\ldots + (-1)^n = \begin{cases} -1, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\Rightarrow$ $S_n(x)$ is bounded for all $x \in [a, b]$ and for all $n \in N$.

Also $$g_n(x) = \frac{x^2 + n}{n^2} < \frac{k^2 + n}{n^2}$$

$\therefore$ $< g_n(x) >$ is a positive monotonic decreasing sequence converging to 0 uniformly for $x \in [a, b]$.

Hence by Dirichlet's test, the series $\Sigma f_n(x)\, g_n(x)$

$$= \sum (-1)^n \cdot \frac{x^2 + n}{n^2} \text{ converges uniformly on } [a, b].$$

Now $$\sum \left| (-1)^2 \frac{x^2 + n}{n^2} \right| = \sum \frac{x^2 + n}{n^2} = \sum \frac{1}{n^2}$$

which diverges. Hence the given series is not absolutely convergent for any value of x.

3.16. UNIFORM CONVERGENCE AND CONTINUITY

Theorem 1. *If a sequence of continuous functions $< f_n >$ is uniformly convergent to a function f on $[a, b]$, then f is continuous on $[a. b]$.*

Proof. Let $\varepsilon > 0$ be given.

Since $< f_n >$ is uniformly convergent of f on $[a, b]$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{3} \quad \forall\, n \geq m \quad \text{and} \quad x \in [a, b] \qquad \ldots(1)$$

Let c be any point of $[a, b]$, then from (1), in particular, we have

$$| f_n(c) - f(c) | < \frac{\varepsilon}{3} \ \forall\, n \geq m \qquad \ldots(2)$$

Since f_n is continuous on $[a, b]$ for each $n \in N$.

$\therefore$ f_n is continuous at $c \in [a, b]$.

$\Rightarrow$ there exists $\delta > 0$ such that $| f_n(x) - f_n(c) | < \frac{\varepsilon}{3}$ whenever $| x - c | < \delta$ ...(3)

Now $$\begin{aligned} | f(x) - f(c) | &= | f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c) |. \\ &\leq | f(x) - f_n(x) | + | f_n(x) - f_n(c) | + | f_n(c) - f(c) | \\ &= | f_n(x) - f(x) | + | f_n(x) - f_n(c) | + | f_n(c) - f(c) | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } | x - c | < \delta \end{aligned}$$

[by (1), (2) and (3)]

$\Rightarrow$ f is continuous at c.

Since $c \in [a, b]$ is arbitrary, f is continuous on $[a, b]$.

NOTES

Theorem 2. *If a series* $\sum_{n=1}^{\infty} f_n$ *of continuous functions is uniformly convergent to a function f on [a, b] then the sum function f is also continuous on [a, b].*

Proof. Let $S_n(x) = \sum_{r=1}^{\infty} f_r(x)$ for all $n \in N$. Let $\varepsilon > 0$ be given.

Since Σf_n converges uniformly to f on $[a, b]$, there exists a positive integer m such that

$$| S_n(x) - f(x) | < \frac{\varepsilon}{3} \quad \forall\, n \geq m \quad \text{and} \quad x \in [a, b] \qquad ...(1)$$

Let c be any point of $[a, b]$ then from (1), in particular, we have

$$| S_n(c) - f(c) | < \frac{\varepsilon}{3} \quad \forall\, n \geq m \qquad ...(2)$$

Since f_n is continuous on $[a, b]$ $\forall\, n \in N$

$\therefore$ $S_n = f_1 + f_2 + ... + f_n$ is continuous on $[a, b]$ for all $n \in N$

$\Rightarrow$ S_n is continuous at $c \in [a, b]$

$\Rightarrow$ there exists $\delta > 0$ such that $| S_n(x) - S_n(c) | < \frac{\varepsilon}{3}$ whenever $| x - c | < \delta$ ...(3)

$$\begin{aligned} \text{Now } | f(x) - f(c) | &= | f(x) - S_n(x) + S_n(x) - S_n(c) + S_n(c) - f(c) | \\ &\leq | f(x) - S_n(x) | + | S_n(x) - S_n(c) | + | S_n(c) - f(c) | \\ &= | S_n(x) - f(x) | + | S_n(x) - S_n(c) | + | S_n(c) - f(c) | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } | x - c | < \delta \quad \text{[by (1), (2) and (3)]} \end{aligned}$$

$\Rightarrow$ f is continuous at c.

Since $c \in [a, b]$ is arbitrary, f is continuous on $[a, b]$.

Remark. Uniform convergence of the sequence $< f_n >$ is only a sufficient but not a necessary condition for the continuity of the limit function f, *i.e.*, if the limit function f is continuous on $[a, b]$, then it is not necessary that the sequence $< f_n >$ is uniformly convergent on $[a, b]$. Theorem 1 shows that if the limit function f is discontinuous then the sequence $< f_n >$ of continuous functions cannot be uniformly convergent on $[a, b]$. Thus the theorem provides a very good negative test for uniform convergence of a sequence. Similarly, if the sum function f is discontinuous then the series Σf_n of continuous functions cannot be uniformly convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test for uniform convergence and continuity the sequence* $< f_n >$ *where* $f_n(x) = x^n,\ 0 \leq x \leq 1$.

Sol. The limit function f is given by

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = 0 \text{ for } 0 \leq x < 1$$

When $x = 1$, the sequence $< f_n >$ converges to 1.

$$\therefore \qquad f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Clearly is discontinuous at $x = 1$ and hence f is discontinuous on [0, 1].

Also $\qquad f_n(x) = x^n,\ 0 \le x \le 1$ is continuous on [0, 1] $\forall\ n \in$ N.

Since $< f_n >$ is a sequence of continuous functions and its limit function f is discontinuous on [0, 1].

$\therefore$ The sequence $< f_n >$ cannot converge uniformly on [0, 1].

Example 2. *Test the uniform convergence and continuity of* $< f_n >$ *where*

$$f_n(x) = \frac{1}{1 + nx},\ 0 \le x \le 1.$$

Sol. The limit function f is given by $f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{1}{1+nx} = 0$ for $0 < x \le 1$

When $x = 0$, the sequence $< f_n >$ converges to 1.

$$\therefore \qquad f(x) = \begin{cases} 0, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x = 0 \end{cases}$$

Clearly, f is discontinuous at $x = 0$ and hence f is discontinuous on [0, 1].

Also $\qquad f_n(x) = \dfrac{1}{1+nx},\ 0 \le x \le 1$ is continuous on [0, 1] $\forall\ n \in$ N.

Since $< f_n >$ is a sequence of continuous functions and its limit function f is discontinuous on [0, 1].

$\therefore$ The sequence $< f_n >$ cannot converge uniformly on [0, 1].

Example 3. *If* $f_n(x) = \dfrac{1}{x+n}$, $x \ge 0$ *then show that* $< f_n >$ *converges uniformly to the continuous function 0.*

Sol. Here $f_n(x)$ is continuous $\forall\ n \in$ N and $x \ge 0$

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x+n} = 0 \quad \forall\ x \ge 0.$$

f being a constant function is continuous for all $x \ge 0$.

But continuous of f is no guarantee for uniform convergence of $< f_n >$.

Now proceeding as in Example 3, Illustrative Examples—A, $< f_n >$ is uniformly convergent for $x \ge 0$.

Hence $< f_n >$ converges uniformly to the continuous function 0.

Example 4. *Examine for uniform convergence and continuity of the limit function of the sequence* $< f_n >$, *where* $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $0 \le x \le 1$.

Sol. Here $f_n(x)$ is continuous $\forall\ n \in$ N and $x \in$ [0, 1]

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1+n^2x^2} = \lim_{n \to \infty} \frac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0 \quad \forall\ x \in [0, 1]$$

Clearly, f is continuous for all $x \in$ [0, 1].

Now proceeding as in Example 8, Illustrative Examples – A, 0 is a point of non-uniform convergence of $< f_n >$ on [0, 1].

Hence $< f_n >$ is not uniformly convergent on [0, 1].

NOTES

Example 5. *Show that the series* $\sum_{n=1}^{\infty} (1-x)\, x^n$ *is not uniformly convergent on* [0, 1].

Sol. The terms of the series are continuous functions and converge pointwise to S(x), where

$$S(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{if } x = 1 \end{cases}$$

Since the sum function S(x) is discontinuous at $x = 0 \in [0, 1]$, the given series is not uniformly convergent on [0, 1].

Example 6. *Show that the series* $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots$ *is not uniformly convergent on* [0, 1].

Sol. The terms of the series are continuous functions.

Here. $$S_n(x) = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots + \frac{x^4}{(1+x^4)^{n-1}} \quad \text{(which is a G.P.)}$$

$$= \frac{x^4\left[1 - \frac{1}{(1+x^4)^n}\right]}{1 - \frac{1}{1+x^4}} = (1+x^4)\left[1 - \frac{1}{(1+x^4)^n}\right]$$

$$= (1+x^4) - \frac{1}{(1+x^4)^{n-1}}$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4, & \text{if } 0 < x \le 1 \\ 0 \quad , & \text{if } x = 0 \end{cases}$$

which is discontinuous at $x = 0 \in [0, 1]$.

Hence the given series is not uniformly convergent on [0, 1].

Example 7. *Show that the series* $\sum_{n=1}^{\infty} f_n$, *where* $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$ *is not uniformly convergent on* [0, 1] *though the sum function is continuous on* [0, 1].

Sol. Here $S_n(x) = \frac{nx}{1+n^2x^2}$ [See Example 5, Illustrative Examples—C]

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0 \ \forall \ x \in [0, 1]$$

Clearly, the sum function S(x) is continuous on [0, 1].

Since $x = 0 \in [0, 1]$ is a point of non-uniform convergence of the sequence of partial sums $< S_n >$, the series is not uniformly convergent on [0, 1].

(See Example 8, Illustrative Examples—A or example 3, Illustrative Examples—B).

3.17. UNIFORM CONVERGENCE AND INTEGRATION

NOTES

Theorem 1. *If a sequence $<f_n>$ converges uniformly to f on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$ and the sequence $< \int_a^b f_n \, dx >$ converges uniformly to $\int_a^b f \, dx$.*

Proof. Let $\varepsilon > 0$ be given.

Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{2(b-a)} \quad \forall \, n \geq m \quad \text{and} \quad x \in [a, b], a \neq b$$

$$\Rightarrow \quad f(x) - \frac{\varepsilon}{2(b-a)} < f_n(x) < f(x) + \frac{\varepsilon}{2(b-a)} \; \forall \, n \geq m, x \in [a, b]$$

$$\left.\begin{aligned} \Rightarrow \quad & f(x) < f_n(x) + \frac{\varepsilon}{2(b-a)} \\ \text{and} \quad & f(x) > f_n(x) - \frac{\varepsilon}{2(b-a)} \end{aligned}\right\} \; \forall \, n \geq m, x \in [a, b] \qquad \ldots(1)$$

Also f_n is integrable on $[a, b]$ for each $n \in \mathbb{N}$

$$\Rightarrow \quad \int_{\underline{a}}^{b} f_n(x) \, dx = \int_a^{\bar{b}} f_n(x) \, dx = \int_a^b f_n(x) \, dx \qquad \ldots(2)$$

Now, from (1), we have

$$\int_a^{\bar{b}} f(x) \, dx < \int_a^{\bar{b}} \left(f_n(x) + \frac{\varepsilon}{2(b-a)} \right) dx = \int_a^{\bar{b}} f_n(x) \, dx + \frac{\varepsilon}{2(b-a)} \cdot (b-a)$$

$$= \int_a^b f(x) \, dx + \frac{\varepsilon}{2} \qquad \text{[using (2)]} \ldots(3)$$

Again, from (1), we have

$$\int_{\underline{a}}^{b} f(x) \, dx > \int_{\underline{a}}^{b} \left(f_n(x) - \frac{\varepsilon}{2(b-a)} \right) dx$$

$$= \int_{\underline{a}}^{b} f_n(x) \, dx - \frac{\varepsilon}{2(b-a)} (b-a) = \int_a^b f_n(x) \, dx - \frac{\varepsilon}{2}$$

$$\Rightarrow \quad -\int_{\underline{a}}^{b} f(x) \, dx < -\int_a^b f_n(x) \, dx + \frac{\varepsilon}{2} \qquad \ldots(4)$$

Adding (3) and (4), we get $\quad 0 \leq \int_a^{\bar{b}} f(x) \, dx - \int_{\underline{a}}^{b} f(x) \, dx < \varepsilon$

Since ε is arbitrary, $\quad \int_a^{\bar{b}} f(x) \, dx - \int_{\underline{a}}^{b} f(x) \, dx = 0 \; \Rightarrow \; \int_a^{\bar{b}} f(x) \, dx = \int_{\underline{a}}^{b} f(x) \, dx$

$\Rightarrow \quad f$ is integrable on $[a, b]$.

NOTES

Again, from (2), (3) and (4), we have

$$\int_a^b f_n(x)\,dx - \frac{\varepsilon}{2} < \int_a^b f(x)\,dx < \int_a^b f_n(x)\,dx + \frac{\varepsilon}{2}\ \forall\ n \geq m$$

$$\Rightarrow \quad \int_a^b f(x)\,dx - \frac{\varepsilon}{2} < \int_a^b f_n(x)\,dx < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}\ \forall\ n \geq m$$

$$\Rightarrow \quad \left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}\ \forall\ n \geq m \Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Theorem 2. *If a series of functions* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *on* $[a, b]$ *and each function* f_n *is integrable on* $[a, b]$, *then* f *is integrable on* $[a, b]$ *and* $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$ *converges uniformly to* $\int_a^b f(x)\,dx$

i.e., $\quad \sum_{n=1}^{\infty} \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$ *i.e., the series is term by term integrable.*

Proof. Let $< S_n >$ denote the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$.

Since $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$.

$\therefore$ The sequence $< S_n >$ converges uniformly to f on $[a, b]$.

Also, S_n being the sum of n integrable functions is integrable for each n.

$\therefore$ By Theorem 1, f is integrable on $[a, b]$

and $\quad \lim_{n\to\infty} \int_a^b S_n(x)\,dx = \int_a^b f(x)\,dx \quad$ *i.e.,* $\quad \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$

Remark. It should be noted that uniform convergence of the sequence $< f_n >$ (or series $\sum_{n=1}^{\infty} f_n$) is only sufficient but not a necessary condition for the validity of term by term integration.

Note. If $< f_n >$ is a sequence of integrable functions converging to f on $[a, b]$ and if $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b f(x)\,dx$, then $< f_n >$ cannot converge uniformly to f.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the sequence* $< f_n >$, *where* $f_n(x) = nx\,e^{-nx^2}$, $n \in N$ *is not uniformly convergent on* $[0, 1]$.

Sol. Here $\quad f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$

$$= \lim_{n \to \infty} \frac{nx}{1 + \frac{nx^2}{2!} + \frac{n^2x^4}{2!} + \ldots\ldots} = 0 \text{ for } x \in [0, 1].$$

NOTES

Also $\int_0^1 f(x)\, dx = 0$

and $\int_0^1 f_n(x)\, dx = \int_0^1 nx\, e^{-nx^2}\, dx = \int_0^n \frac{1}{2} e^{-t}\, dt$ where $t = nx^2$

$$= \frac{1}{2}[1 - e^{-n}]$$

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = \lim_{n \to \infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2} \neq \int_0^1 f(x)\, dx$$

$\Rightarrow$ the sequence $< f_n >$ is not uniformly convergent on [0, 1].

In fact, $x = 0$ is a point of non-uniform convergence.

Example 2. *Examine for term by term integration the series* $\sum_{n=1}^{\infty} f_n$, *where*

$S_n(x) = \sum_{i=1}^{n} f_i = nx\, e^{-nx^2}$, *over the intervals*

(i) $[0, 1]$ *(ii)* $[a, 1]$, $0 < a < 1$.

Sol. Here $f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{nx}{e^{nx^2}} = \lim_{n \to \infty} \frac{nx}{1 + \frac{nx^2}{1!} + \frac{n^2x^4}{2!} + \ldots\ldots} = 0$ for all x

Consider the interval [0, 1]

$$\int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0$$

and $\int_0^1 S_n(x)\, dx = \int_0^1 nx\, e^{-nx^2}\, dx = \int_0^n \frac{1}{2} e^{-t}\, dt$ where $t = nx^2 = \frac{1}{2}(1 - e^{-n})$

$$\lim_{n \to \infty} \int_0^1 S_n(x)\, dx = \lim_{n \to \infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2}$$

Since $\lim_{n \to \infty} \int_0^1 S_n(x)\, dx \neq \int_0^1 f(x)\, dx$ *i.e.,* $\lim_{n \to \infty} \int_0^1 S_n(x)\, dx \neq \int_0^1 \left(\lim_{n \to \infty} S_n(x)\right) dx$

the series $\sum_{n=1}^{\infty} f_n$ does not admit of term by term integration over the interval [0, 1].

Now consider the interval $[a, 1]$, $0 < a < 1$

$$\int_a^1 f(x)\, dx = \int_a^1 0\, dx = 0$$

and $\int_a^1 S_n(x)\, dx = \int_a^1 nx\, e^{-nx^2}\, dx = \int_{na^2}^n \frac{1}{2} e^{-t}\, dt$ where $t = nx^2$

$$= \frac{1}{2}\left(e^{-na^2} - e^{-n}\right)$$

$$\lim_{n \to \infty} \int_a^1 S_n(x)\, dx = \lim_{n \to \infty} \frac{1}{2}\left(e^{-na^2} - e^{-n}\right) = 0 = \int_a^1 f(x)\, dx$$

Hence term by term integration is justified over the interval $[a, 1]$ where $0 < a < 1$.

Example 3. *Prove that* $\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)}$.

NOTES

Sol. Let $$f_n(x) = \frac{x^n}{n^2}$$

$$| f_n(x) | = \left| \frac{x^n}{n^2} \right| \le \frac{1}{n^2} = M_n \text{ for } 0 \le x \le 1$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent for $0 \le x \le 1$.

∴ The series can be integrated term by term.

$$\Rightarrow \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2} \right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2} dx = \sum_{n=1}^{\infty} \left[\frac{x^{n+1}}{n^2(n+1)} \right]_0^1 = \sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)}.$$

Example 4. *Show that the series* $1 - x + x^2 - x^3 + \ldots, 0 \le x \le 1$ *admits of term by term integration on* $[0, 1]$, *though it is not uniformly convergent on* $[0, 1]$.

Sol. The given series is $1 - x + x^2 - x^3 + \ldots$

When $x = 1$, the series $1 - 1 + 1 - 1 + \ldots$ oscillates.

For $0 \le x < 1$, $\quad 1 - x + x^2 - x^3 + \ldots = \frac{1}{1-(-x)} = \frac{1}{1+x}$

Thus the series is not uniformly convergent on $[0, 1]$, $x = 1$ being the point of non-uniform convergence.

However, integrating term by term over the interval $[0, 1]$ *i.e.*, including 1, we have

$$\int_0^1 1\, dx - \int_0^1 x\, dx + \int_0^1 x^2\, dx - \int_0^1 x^3\, dx + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$$

Also, $$\int_0^1 \frac{1}{1+x} dx = \Big[\log (1+x) \Big]_0^1 = \log 2.$$

Thus the two sides are equal. Hence term by term integration is possible over $[0, 1]$, even though the given series is not uniformly convergent on $[0, 1]$.

Example 5. *Examine for term by term integration the series the sum of whose first n terms is*

$$n^2 x (1-x)^n, \; 0 \le x \le 1.$$

Sol. Here $\quad S_n(x) = n^2 x (1-x)^n$.

When $x = 0$ or 1, $S_n(x) = 0$

When $0 < x < 1$, $\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \dfrac{n^2 x}{(1-x)^{-n}}$ | Form $\frac{\infty}{\infty}$

$$= \lim_{n \to \infty} \frac{2nx}{-(1-x)^{-n} \log(1-x)} \quad \Big| \text{ Form } \frac{\infty}{\infty}$$

$$= \lim_{n \to \infty} \frac{2x}{(1-x)^{-n} [\log(1-x)]^2} = 0$$

$\therefore$ $$f(x) = \lim_{n \to \infty} S_n(x) = 0 \text{ for all } x \in [0, 1]$$

Also, $$\int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0 \text{ and } \int_0^1 S_n(x)\, dx = \int_0^1 n^2 x(1-x)^n\, dx$$

Changing x to $1 - x$ $$= \int_0^1 n^2(1-x)\, x^n\, dx$$

$$= \int_0^1 n^2(x^n - x^{n+1})\, dx$$

$$= n^2 \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n \to \infty} \int_0^1 S_n(x)\, dx = \lim_{n \to \infty} \frac{n^2}{(n+1)(n+2)} = 1$$

Since $\lim_{n \to \infty} \int_0^1 S_n(x)\, dx \neq \int_0^1 \left(\lim_{n \to \infty} S_n(x) \right) dx$, term by term integration is not justified on $[0, 1]$.

Example 6. *Show that the series for which*

(*i*) $S_n(x) = \dfrac{1}{1 + nx}$ (*ii*) $S_n(x) = nx(1-x)^n$

can be integrated term by term on $[0, 1]$, though they are not uniformly convergent on $[0, 1]$.

Sol. (*i*) Here $$S_n(x) = \frac{1}{1 + nx}$$

so that $$f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{1}{1 + nx} = \begin{cases} 0, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x = 0 \end{cases}$$

For $0 < x \le 1$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - f(x) | = \left| \frac{1}{1 + nx} - 0 \right| = \frac{1}{1 + nx} < \varepsilon \text{ if } 1 + nx > \frac{1}{\varepsilon} \quad \text{or} \quad \text{if} \quad n > \frac{\frac{1}{\varepsilon} - 1}{x}$$

If $x \to 0$, $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series. Thus the series does not converge uniformly on $[0, 1]$.

Now $$\int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0$$

and $$\int_0^1 S_n(x)\, dx = \int_0^1 \frac{dx}{1 + nx} = \left[\frac{\log(1 + nx)}{n} \right]_0^1 = \frac{\log(1 + n)}{n}$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{\log(1+n)}{n} \qquad \Big|\text{Form } \frac{\infty}{\infty}$$

NOTES

$$= \lim_{n\to\infty}\frac{\frac{1}{1+n}}{1} = 0$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty}[S_n(x)]\,dx$, the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

(*ii*) Here $S_n(x) = nx(1-x)^n$

When $0 < x < 1$, we have $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\frac{nx}{(1-x)^{-n}} \qquad \Big|\text{Form } \frac{\infty}{\infty}$

$$= \lim_{n\to\infty}\frac{x}{-(1-x)^{-n}\log(1-x)} = 0$$

Also $S_n(x) = 0$ for $x = 0$ or 1

$\therefore$ $f(x) = 0$ for every x in [0, 1].

Now $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ and $\int_0^1 S_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$

Changing x to $1-x$ $= \int_0^1 n(1-x)\,x^n\,dx = \int_0^1 n(x^n - x^{n+1})\,dx$

$$= n\left[\frac{1}{n+1}-\frac{1}{n+2}\right] = \frac{n}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{n}{(n+1)(n+2)} = 0$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty}[S_n(x)]\,dx$

the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

(See Example 4, Illustrative Examples—B)

Example 7. *Test for uniform convergence and term by term integration the series*

$\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$. *Also show that* $\int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx = \frac{1}{2}$.

Sol. The series $\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$ is uniformly convergent.

(See Example 17 (*i*), Illustrative Examples—C)

Hence it is integrable term by term between any finite limits.

$$\Rightarrow \int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx = \lim_{n\to\infty}\int_0^1\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\,dx$$

$$= \lim_{n\to\infty}\sum_{n=1}^{\infty}\int_0^1 x(n+x^2)^{-2}\,dx$$

$$= \lim_{n \to \infty} \sum_{n=1}^{\infty} \left[\frac{(n+x^2)^{-1}}{-2}\right]_0^1 = \lim_{n \to \infty} \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{n \to \infty} \frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1-\frac{1}{n+1}\right) = \frac{1}{2}.$$

Example 8. *Show that the series* $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ *can be integrated term by term on [0, 1] although it is not uniformly convergent on [0, 1].*

Sol. Proceeding as in Example 5, Illustrative Examples—C, we have

$$S_n(x) = \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n \to \infty} S_n(x) = 0 \ \forall \ x \in [0, 1]$$

$x = 0$ is a point of non-uniform convergence of the series.

Now $$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$$

and $$\int_0^1 S_n(x)\,dx = \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \frac{1}{2n}\int_0^1 \frac{2n^2x}{1+n^2x^2}\,dx$$

$$= \frac{1}{2n}\left[\log(1+n^2x^2)\right]_0^1 = \frac{1}{2n}\log(1+n^2)$$

$$\lim_{n \to \infty} \int_0^1 S_n(x)\,dx = \lim_{n \to \infty} \frac{\log(1+n^2)}{2n} \qquad \left|\ \text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n \to \infty} \frac{\frac{2n}{1+n^2}}{2} = \lim_{n \to \infty} \frac{n}{1+n^2} \qquad \left|\ \text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n \to \infty} \frac{1}{2n} = 0$$

Since $\lim_{n \to \infty} \int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n \to \infty} [S_n(x)]\,dx$, the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

3.18. UNIFORM CONVERGENCE AND DIFFERENTIATION

Theorem 1. *If a sequence of functions* $< f_n >$ *is such that*

(i) *each* f_n *is differentiable on* $[a, b]$ (ii) *each* f_n' *is continuous on* $[a, b]$

(iii) $< f_n >$ *converges to* f *on* $[a, b]$

(iv) $< f_n' >$ *converges uniformly to* g *on* $[a, b]$

then f *is differentiable and* $f'(x) = g(x) \ \forall \ x \in [a, b]$.

Proof. Since each f_n' is continuous on $[a, b]$ and $< f_n' >$ converges uniformly to g on $[a, b]$, therefore, g is continuous and hence integrable on $[a, b]$.

NOTES

Also, since $< f_n' >$ converges uniformly to g on $[a, y]$, where $a \le y \le b$.

$$\therefore \quad \lim_{n \to \infty} \int_a^y f_n'(x)\, dx = \int_a^y g(x)\, dx \quad \text{...(1)}$$

By Fundamental theorem of Integral Calculus, we know that

$$\int_a^y f_n'(x)\, dx = f_n(y) - f_n(a)$$

$$\therefore \text{ From (1), } \lim_{n \to \infty} [f_n(y) - f_n(a)] = \int_a^y g(x)\, dx \quad \text{...(2)}$$

Since $< f_n >$ converges to f on $[a, b]$

$$\therefore \quad \lim_{n \to \infty} f_n(y) = f(y) \text{ and } \lim_{n \to \infty} f_n(a) = f(a).$$

$$\therefore \text{ From (2), } \quad f(y) - f(a) = \int_a^y g(x)\, dx \Rightarrow f'(x) = g(y),\ a \le y \le b.$$

Changing y to x, we have $f'(x) = g(x)\ \forall\ x \in [a, b]$.

Theorem 2. *If a series of functions* $\sum_{n=1}^{\infty} f_n$ *is such that*

(*i*) *each* f_n *is differentiable on* $[a, b]$

(*ii*) *each* f_n' *is continuous on* $[a, b]$

(*iii*) $\sum_{n=1}^{\infty} f_n$ *converges to* f *on* $[a, b]$

(*iv*) $\sum_{n=1}^{\infty} f_n'$ *converges uniformly to* g *on* $[a, b]$ *then* f *is differentiable on* $[a, b]$ *for* $f'(x) = g(x)\ \forall\ x \in [a, b]$.

Proof. Let $S_n = f_1 + f_2 + ... + f_n$ so that $< S_n >$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$.

Since $\sum_{n=1}^{\infty} f_n$ converges to f on $[a, b]$, the sequence $< S_n >$ converges to f on $[a, b]$.

Also $S_n' = f_1' + f_2' + ... + f_n'$ the sequence $< S_n' >$ of the partial sums of the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a, b]$, where $g = \sum_{n=1}^{\infty} f_n'$.

$\therefore$ By Theorem 1, $f'(x) = g(x) \quad \forall\ x \in [a, b]$.

Note 1. In the above theorems the derived sequence (series) must be uniformly convergent and the original sequence (series) need only be convergent.

Note 2. The conclusion of Theorem 2 can be written as $f' = \sum_{n=1}^{\infty} f_n' = f_1' + f_2' + ... + f_n'$ so that the series is differentiable term by term.

3.19. WEIERSTRASS APPROXIMATION THEOREM

Theorem 2.8.1. *Let f be any continuous function in C[a, b]. Then for any $\varepsilon > 0$ there exists a polynomial P(x) such that $|p(x) - f(x)| < \varepsilon, \forall x \in [a, b]$.*

Proof. To prove this theorem first we shall prove it for a special case $a = 0$ and $b = 1$. Since we observe that $x = (b - a)x' + a$ provides that a continuous mapping of [0, 1] onto $[a, b]$, so that we can define a function g, by $g(x') = f[(b - a)x' + a]$ is a continuous real function on [0, 1]. So if our theorem is true for the case $a = 0$ and $b = 1$, then there exists a polynomial p' on [0, 1] such that $|g(x') - p'(x)| < \varepsilon, \forall x' \in [0, 1]$. In term of x this inequality reduced to $|f(x) - p'[(x - a)/(b - a)]| < \varepsilon, \forall x \in [a, b]$

or, $|f(x) - p(x)| < \varepsilon, \forall x \in [a, b]$, where $f(x) = p'[(x - a)/(b - a)], \forall x \in [a, b]$ yields our theorem in general case. Accordingly, we may assume that $a = 0$ and $b = 1$.

Let $f \in C[0, 1]$, define a sequence of polynomials $\{B_n\}$ as:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n) \ (0 < x \le 1), n \in N, \text{ where } \binom{n}{k} = n!/k!/(n-k)! \quad \ldots(A)$$

This polynomial $B_n(x)$ is called n^{th} Berstein polynomial for f.

Now by Binomial theorem for any $p, q \in R$, we have

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p + q)^n, n \in N \quad \ldots(i)$$

On differentiating (*i*) with respect to p, we have

$$\sum_{k=0}^{n} \binom{n}{k} k\, p^k q^{n-k} = n(p + q)^{n-1}, n \in N$$

$$\Rightarrow \quad \sum_{k=0}^{n} \left(\frac{k}{n}\right)\binom{n}{k} p^k q^{n-k} = p(p + q)^{n-1}, n \in N \quad \ldots(ii)$$

Again differentiating (*ii*) with respect to p, we have

$$\sum_{k=0}^{n} \binom{n}{k}\left(\frac{k^2}{n}\right) p^{k-1} q^{n-k} = (n - 1)p(p + q)^{n-2} + (p + q)^{n-1}, n \in N$$

$$\Rightarrow \quad \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right)\binom{n}{k} p^k q^{n-k} = \left\{1 - \left(\frac{1}{n}\right)\right\} p^2 (p + q)^{n-2} + \left(\frac{p}{n}\right)(p + q)^{n-1}, n \in N \quad \ldots(iii)$$

Now for $x \in [0, 1]$, $p = x$ and $q = 1 - x$ in (*i*), (*ii*) and (*iii*), we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1, n \in N \quad \ldots(iv)$$

$$\sum_{k=0}^{n} \left(\frac{k}{n}\right)\binom{n}{k} x^k (1 - x)^{n-k} = x, n \in N \quad \ldots(v)$$

$$\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)\binom{n}{k}x^k(1-x)^{n-k} = \left\{1-\left(\frac{1}{n}\right)\right\}x^2+\left(\frac{x}{n}\right), n\in \mathbb{N} \quad \ldots(vi)$$

NOTES

Now by equations (*iv*), (*v*) and (*vi*), [(*iii*) – 2*kx*(*ii*) + x^2(*i*)], we have

$$\sum_{k=0}^{n}\left[\left(\frac{k^2}{n^2}\right)-\left(\frac{2kx}{n}\right)+x^2\right]\binom{n}{k}x^k(1-x)^{n-k}$$

$$= x^2-\left(\frac{x^2}{n}\right)+\left(\frac{x}{n}\right)-2x^2+x^2, \ x\in[0,1] \text{ and } n\in\mathbb{N}$$

$$\Rightarrow \quad \sum_{k=0}^{n}\left[\left(\frac{k}{n}\right)-x\right]^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{x(1-x)}{n}, \ x\in[0,1] \text{ and } n\in\mathbb{N} \quad \ldots(vii)$$

Since we know that "if (M_1, ρ_1) be compact metric space and if f is continuous function from (M_1, ρ_1) to another metric space (M_2, ρ_2) then f is uniformly continuous on M_1."

Hence f is uniformly continuous on C[0, 1], then for given $\varepsilon > 0 \ \exists \ \delta > 0$ such that

$$|f(x)-f(y)| < \frac{\varepsilon}{2} \text{ for } |x-y| < \delta. \quad \forall x, y\in[0,1].$$

Choose positive integer N such that $\quad 1/N^{1/4} < \delta \quad \ldots(viii)$

and such that $\quad 1/\sqrt{N} < \varepsilon/4\|f\| \quad \ldots(ix)$

For $x\in[0,1]$, multiply (*iv*) by $f(x)$ and subtract from (A), we have

$$f(x)-B_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right] = \sum{}' + \sum{}'' \text{ (say)} \quad \ldots(x)$$

where Σ' is the sum over those values of k such that $\left|\left(\frac{k}{n}\right)-x\right| < \frac{1}{n^{1/4}} \quad \ldots(xi)$

while Σ'' is the sum over all other those values of k.

Now if k does not satisfies (*xi*)

i.e., $$\left|\left(\frac{k}{n}\right)-x\right| \geq \frac{1}{n^{1/4}} \Rightarrow (k-nx)^2 \geq \frac{n^2}{\sqrt{n}} = n^{1/3} \quad \ldots(xii)$$

Hence $$|\Sigma''| = \left|\sum{}''\binom{n}{k}x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right]\right|$$

$$\leq \sum{}''\binom{n}{k}x^k(1-x)^{n-k}\left[|f(x)|+\left|f\left(\frac{k}{n}\right)\right|\right]$$

$$\leq 2\|f\|\sum{}''\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leq [2\|f\|1/n^{1/3}]\sum'' (k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leq [2\|f\|/n^{1/3}]\; nx(1-x) \text{ [by } (vii)] \leq 2\|f\|/\sqrt{n}\;.\; x\in [0,1]$$

Now for $n \geq N$ by (ix) $1/\sqrt{n} \leq 1/\sqrt{N} < \varepsilon/4\|f\|$, therefore by $(xiii)$

$$|\Sigma''| \leq 2\|f\|/\sqrt{n} < 2\|f\|\varepsilon/4\|f| = \varepsilon/2, x\in[0,1] \qquad \ldots(xiii)$$

Further for $n \geq N$ if k satisfy (vi), then by $(viii)$, we have

$$|(k/n)-x| < 1/n^{1/4} \leq 1/N^{1/4} < \delta \quad \text{implies } |f(x/n) - f(x)| < \varepsilon/2.$$

Hence,
$$|\Sigma'| = \left|\sum{}'\binom{n}{k}x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right]\right|$$

$$\leq \sum{}'\binom{n}{k}x^k(1-x)^{n-k}\left[f(x)-f\left(\frac{k}{n}\right)\right]$$

$$< \left(\frac{\varepsilon}{2}\right)\left|\sum{}'\binom{n}{k}x^k(1-x)^{n-k} < \frac{\varepsilon}{2} \qquad \text{[by } (iv)]\right.$$

Hence by $f(x)$, we have

$$|f(x) - B_n(x)| \leq |\Sigma'| + |\Sigma''| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, x\in [0,1].\; n \geq N.$$

Since x being arbitrary, consequently,

$$|f(x) - B_n(x)| \leq |\Sigma'| + |\Sigma''| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } x\in [0,1].\; n \geq N.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the sequence $< f_n >$ where $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $0 \leq x \leq 1$ cannot be differentiated term by term at $x = 0$.*

Sol. Here $\qquad f(x) = \lim_{n\to\infty} f_n(x) = 0 \;\; \forall\, x \in [0,1].$

$\therefore \qquad f'(0) = 0$

Also,
$$f_n'(0) = \lim_{h\to\infty}\frac{f_n(0+h)-f_n(0)}{h} = \lim_{h\to 0}\frac{\frac{nh}{1+n^2h^2}-0}{h}$$

$$= \lim_{h\to\infty}\frac{n}{1+n^2h^2} = n$$

$\therefore \qquad f_n'(0) \to \infty$ as $n \to \infty$

$\Rightarrow \qquad f'(0) \neq \lim_{n\to\infty} f_n'(0)$

Hence $< f_n >$ cannot be differentiated term by term at $x = 0$.

NOTES

Example 2. *Show that for the sequence* $<f_n>$ *where* $f_n(x) = \dfrac{x}{1+nx^2}$ *the formula* $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$ *is true if* $x \neq 0$ *and false if* $x = 0$*. Why so ?*

Sol. The sequence $<f_n>$ converges uniformly to zero for all real x. [See Example 1, Illustrative Examples—B]

$$\Rightarrow \qquad f(x) = 0 \ \forall\, x \in \mathbb{R} \qquad \Rightarrow \qquad f'(x) = 0 \ \forall\, x \in \mathbb{R}$$

When $x \neq 0$,
$$f_n'(x) = \frac{(1+nx^2).1 - x.2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} \qquad \Big| \text{Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{-x^2}{2(1+nx^2).x^2} = 0 = f'(x)$$

so that if $x \neq 0$, the formula $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$ is true.

At $x = 0$

$$f_n'(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{h}{1+nh^2} - 0}{h} = \lim_{h\to 0} \frac{1}{1+nh^2} = 1$$

so that $\quad \lim\limits_{n\to\infty} f_n'(0) = 1 \neq f'(0)$.

Hence at $x = 0$, the formula $= \lim\limits_{n\to\infty} f_n'(x) = f'(x)$ is false.

It is so because the sequence $<f_n'>$ is not uniformly convergent in any interval containing zero.

Example 3. *Show that the function represented by* $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$ *is differentiable for every* x *and its derivative is* $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$.

Sol. Here $\qquad f_n(x) = \dfrac{\sin nx}{n^3}$

$$\therefore \qquad f_n'(x) = \frac{\cos nx}{n^2} \Rightarrow \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Since $\left|\dfrac{\cos nx}{n^2}\right| \leq \dfrac{1}{n^2}$ $\forall\, x$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum\limits_{n=1}^{\infty} f_n'$ is uniformly convergent for all x and hence $\sum\limits_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

$$\Rightarrow \quad \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}\right)' = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Example 4. *Show that the differential co-efficient of*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2} \text{ is } -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \text{ for all real } x.$$

Sol. Here $\qquad f_n(x) = \dfrac{1}{n^3 + n^4x^2} = \dfrac{1}{n^3(1+nx^2)}$

$$\Rightarrow \quad f_n'(x) = \frac{1}{n^3}\left[\frac{-2nx}{(1+nx^2)^2}\right] = -\frac{2x}{n^2\,(1+nx^2)^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} f_n'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2\,(1+nx^2)^2}$$

Now $f_n'(x)$ is maximum when $\dfrac{d}{dx} f_n'(x) = 0$

i.e., when $\qquad -\dfrac{2}{n^2} \cdot \dfrac{(1+nx^2)^2 \,.\, 1 - x^2\,(1+nx^2)\,.\,2nx}{(1+nx^2)^4} = 0$

or $\quad$ when $1 - 3nx^2 = 0$ $\quad$ or $\quad$ when $x = \dfrac{1}{\sqrt{3n}}$

$$\therefore \quad \text{Maximum value of } |\, f_n'(x) \,| = \frac{2\,.\,\frac{1}{\sqrt{3n}}}{n^2\,(1+\frac{1}{3})^2} = \frac{3\sqrt{3}}{8n^{5/2}}$$

$$\Rightarrow \quad |\, f_n'(x) \,| \le \frac{3\sqrt{3}}{8n^{5/2}} < \frac{1}{n^{5/2}} \;\forall\; x.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^{5/2}}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all real x and hence $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n' \Rightarrow \left(\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2}\right) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2\,(1+nx^2)^2}.$$

Example 5. *Show that the series for which $S_n(x) = \dfrac{nx}{1+n^2x^2}$, $0 \le x \le 1$ cannot be differentiated term by term at $x = 0$.*

Sol. Here $\qquad f(x) = \lim\limits_{n\to\infty} S_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for $0 \le x \le 1$

$\therefore \qquad f'(0) = 0$

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Also $$S_n'(0) = \lim_{h \to 0} \frac{S_n(0+h) - S_n(0)}{h} = \lim_{h \to 0} \frac{\frac{nh}{1+n^2h^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{n}{1+n^2h^2} = n$$

$\Rightarrow$ $$\lim_{n \to \infty} S_n'(0) = \infty$$

Thus $$f'(0) \neq \lim_{n \to \infty} S_n'(0)$$

Hence the given series cannot be differentiated term by term.

Example 6. *Given the series $\sum_{n=1}^{\infty} f_n$ for which $S_n(x) = \frac{1}{2n^2} \log(1 + n^4x^2)$, $0 \le x \le 1$.*

Show that the series $\sum_{n=1}^{\infty} f_n'$ does not converge uniformly, but the given series can be differentiated term by term.

Sol. Here $$f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{\log(1+n^4x^2)}{2n^2} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n \to \infty} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n} = \lim_{n \to \infty} \frac{n^2x^2}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1$$

$\therefore$ $$f'(x) = 0$$

Also $$\lim_{n \to \infty} S_n'(x) = \lim_{n \to \infty} \left(\frac{1}{2n^2} \cdot \frac{2n^4x}{1+n^4x^2} \right) = \lim_{n \to \infty} \frac{n^2x}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1$$

$\therefore$ $$f'(x) = \lim_{n \to \infty} S_n'(x)$$

Thus term by term differentiation holds.

However, the series $\sum_{n=1}^{\infty} f_n'$ is not uniformly convergent for $0 \le x \le 1$ since the sequence $< S_n' >$ *i.e.*, $< \frac{n^2x}{1+n^4x^2} >$ has $x = 0$ as a point of non-uniform convergence.

TEST YOUR KNOWLEDGE

1. Show that the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ on $[0, k]$ is not uniformly convergent.
2. Show that the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ on $[a, b]$, $a > 0$ is uniformly convergent.

   $\left[\textbf{Hint.} \text{ Maximum value of } \frac{\log \frac{1}{\varepsilon}}{x} \text{ is } \frac{\log \frac{1}{\varepsilon}}{a} \right]$

3. If $f_n(x) = \frac{\sin nx}{n}$, $0 \le x \le 1$, does there exist $m \in \mathbb{N}$ such that

   $| f_n(x) - f(x) | < \frac{1}{10}$ $\forall\, n \ge m$ and $\forall\, x \in [0, 1]$?

   [**Hint.** Here $f(x) = 0\ \forall\, x \in [0, 1]$].

4. Show that the sequence $< f_n >$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$, is not uniformly convergent on any interval containing zero.

5. Show that the sequence $< f_n >$, where $f_n(x) = \dfrac{n^2x}{1+n^3x^2}$, is not uniformly convergent on [0, 1].

$\left[\right.$**Hint.** Here $y = \dfrac{n^2x}{1+n^3x^2}$ is maximum when $x = \dfrac{1}{n^{3/2}}$ and max. value of y is $\dfrac{\sqrt{n}}{2}$.

Also, $x = \dfrac{1}{n^{3/2}} \to 0$ as $n \to \infty$.

M_n does not tend to 0 as $n \to \infty$].

6. Show that the sequence of functions $< f_n >$ where $f_n(x) = \dfrac{nx}{1+n^3x^2}$, $x \in R$ converges uniformly on any closed interval $[a, b]$.

7. Show that the sequence $< f_n >$, where $f_n(x) = \dfrac{x}{(n+x^2)^2}$ is uniformly convergent for all $x \geq 0$.

$\left[\textbf{Hint.}\ \text{Here } f(x) = 0 \text{ and } M_n = \dfrac{3\sqrt{3}}{16n^{3/2}} \to 0 \text{ as } n \to \infty\right]$.

8. Show that the sequence $< f_n >$, where $f_n(x) = \dfrac{x}{n(1+nx^2)}$ is uniformly convergent for all x.

$\left[\textbf{Hint.}\ \text{Here } f(x) = 0 \text{ and } M_n = \dfrac{1}{2n^{3/2}} \to 0 \text{ as } n \to \infty\right]$.

9. The sum to n terms of a series is $S_n(x) = \dfrac{n^2x}{1+n^4x^2}$. Show that the series is non-uniformly convergent on [0, 1].

[**Hint.** See Example 12, Illustrative Examples—A].

10. Show that the series $\sum_{n=1}^{\infty} x^{n-1}(1-x)^2$ converges uniformly to $1 - x$ in [0, 1].

11. Show that the series $\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly to $\dfrac{1}{1-x}$ in $[0, b]$, $0 < b < 1$ but does not converge uniformly on [0, 1].

12. Show that if $0 < r < 1$, then each of the following series is uniformly convergent on R:

(*i*) $\sum_{n=1}^{\infty} r^n \sin nx$ (*ii*) $\sum_{n=1}^{\infty} r^n \cos n^2x$ (*iii*) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

13. Show that the given series are uniformly convergent for all real x:

$$\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n^2+2)}.$$

14. Show that following series are uniformly and absolutely convergent for all real values of x and $p > 1$: $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^p}$.

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15. Show that the series $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots$ converges uniformly on R.

$\left[\textbf{Hint. } f_n(x) = \frac{\cos nx}{n^2}\right]$

16. Show that each of the following series is uniformly convergent for all values of x

(*i*) $\sum \frac{1}{n^4 + n^2x^2}$ (*ii*) $\sum \frac{1}{n^2 + n^4x^2}$ (*iii*) $\sum \frac{1}{n^3 + n^4x^2}$.

$\left[\textbf{Hint. } \forall\, x \in \mathrm{R}, n^4 + n^2x^2 \geq n^4 \text{ so that } | f_n(x) | \leq \frac{1}{n^4}\right]$

17. Prove that if k is any fixed positive number less than unity, then each of the following series is uniformly convergent in $[-k, k]$.

(*i*) $\sum x^n$ (*ii*) $\sum \frac{x^n}{n+1}$.

18. Prove that if k is any fixed number greater than unity, then each of the following series is uniformly convergent for all $x \geq k$.

(*i*) $\sum \frac{1}{x^n}$ (*ii*) $\sum \frac{1}{1+x^n}$.

19. Prove that the series $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots\ldots$ is uniformly convergent in any interval.

20. Show that the sequence $< f_n >$ where $f_n(x) = \tan^{-1} nx$ is not uniformly convergent on $[0, 1]$.

$\left[\textbf{Hint. } f(x) = \begin{cases} \frac{\pi}{2}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases} \text{ is discontinuous at } x = 0.\right]$

21. If $f_n(x) = nx(1-x)^n$, $x \in [0, 1]$ then show that $< f_n >$ is not uniformly convergent on $[0, 1]$ though the limit function is continuous on $[0, 1]$.

[**Hint.** See Example 7. Illustrative Examples—B].

22. Examine whether the series for which $S_n(x) = \frac{1}{n + n^3x^2}$ is differentiable term by term.

Answers

3. $m = 11$

SUMMARY

- Let $< f_n >$ be a sequence of functions on I. If to each $x \in$ I and to each $\varepsilon > 0$, there corresponds to positive integer m such that $| f_n(x) - f(x) | < \varepsilon\ \forall\ n \geq m$ then we say that $< f_n >$ converges pointwise to the function f on I.
- Let $< f_n >$ be a sequence of functions on I. Then $< f_n >$ is said to be uniformly convergent to a function f on I if to each $\varepsilon > 0$, there exists a positive integer m (depending only on ε) such that

$$| f_n(x) - f(x) | < \varepsilon\ \forall\ x \geq m \text{ and } \forall\ x \in \mathrm{I}.$$

The function f is called **uniform limit** of the sequence $< f_n >$ on I.

- Let $< f_n >$ be a sequence of functions defined on I. A point $x \in$ I said to be a point of non-uniform convergence if $< f_n >$ does not converge uniformly in any neighbourhood (however small) of x.
- **Cauchy's criterion for uniform convergence):** A sequence $< f_n >$ of functions defined on I is uniformly convergent on I if and only if for each $\varepsilon > 0$ and for all $x \in$ I, there exists a positive integer m such that

$$| f_{n_1}(x) - f_{n_2}(x) | < \varepsilon \ \forall\ n_1, n_2 \geq m.$$

- **M_n Test:** Let $< f_n >$ be a sequence of functions on I such that

$$\lim_{n \to \infty} f_n(x) = f(x), \ \forall\ x \in \mathrm{I}$$

and let $\mathrm{M}_n = \sup \{| f_n(x) - f(x) | : x \in \mathrm{I}\}$

Then $< f_n >$ converges uniformly on I if and only if $\lim_{n \to \infty} \mathrm{M}_n = 0$.

- If $< f_n >$ is a sequence of real-valued functions on an interval I, then $f_1 + f_2 + \ldots\ldots + f_n + \ldots$ is called a series of real-valued functions defined on I.
- **Cauchy's Criterion for uniform convergence of a series of functions:** A series of functions Σf_n is uniformly convergent on an interval I if and only if for each $\varepsilon > 0$ and for all $x \in$ I, there exists a positive integer m (depending only on ε) such that

$$| f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) | < \varepsilon \ \forall\ n \geq m, p \in N.$$

- **Weierstrass's M-Test:** A series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely) on an interval I if there exists a convergent series $\sum_{n=1}^{\infty} \mathrm{M}_n$ of non-negative terms (*i.e.*, $\mathrm{M}_n \geq 0 \ \ \forall\ n \in \mathrm{N}$) such that

$$| f_n(x) | \leq \mathrm{M}_n \quad \forall\ n \in \mathrm{N} \quad \text{and} \quad \forall\ x \in \mathrm{I}.$$

- **Abel's Test:**

  If (*i*) $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$,

  (*ii*) the sequence $< g_n(x) >$ is monotonic decreasing for all $x \in [a, b]$, and

  (*iii*) there exists a positive real number k such that $| g_n(x) | < k \ \forall\ x \in [a, b]$ and $n \in$ N then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

- **Dirichlet's Test:** If (*i*) there exists a positive real number k such that

$$| \mathrm{S}_n(x) | = \left| \sum_{r=1}^{\infty} f_r(x) \right| < k \ \forall\ x \in [a, b], n \in \mathrm{N} \text{ and}$$

  (*ii*) $< g_n(x) >$ is a positive monotonic decreasing sequence converging uniformly to zero on $[a, b]$ then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

- If a sequence of continuous functions $< f_n >$ is uniformly convergent to a function f on $[a, b]$, then f is continuous on $[a, b]$.

NOTES

- If a series $\sum_{n=1}^{\infty} f_n$ of continuous functions is uniformly convergent to a function f on $[a, b]$ then the sum function f is also continuous on $[a, b]$.
- If a sequence $< f_n >$ converges uniformly to f on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$ and the sequence $< \int_a^b f_n \, dx >$ converges uniformly to $\int_a^b f \, dx$.
- If a series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$ and $\sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx$ converges uniformly to $\int_a^b f(x)\, dx$

  i.e., $\sum_{n=1}^{\infty} \int_a^b f(x)\, dx = \int_a^b f(x)\, dx$ *i.e.*, the series is term by term integrable.
- If a sequence of functions $< f_n >$ is such that
  (*i*) each f_n is differentiable on $[a, b]$
  (*ii*) each f_n' is continuous on $[a, b]$
  (*iii*) $< f_n >$ converges to f on $[a, b]$
  (*iv*) $< f_n' >$ converges uniformly to g on $[a, b]$
  than f is differentiable and $f'(x) = g(x)\ \forall\ x \in [a, b]$
- If a series of functions $\sum_{n=1}^{\infty} f_n$ is such that
  (*i*) each f_n is differentiable on $[a, b]$
  (*ii*) each f_n' is continuous on $[a, b]$
  (*iii*) $\sum_{n=1}^{\infty} f_n$ converges to f on $[a, b]$
  (*iv*) $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a, b]$ then f is differentiable on $[a, b]$ for $f'(x) = g(x)\ \forall\ x \in [a, b]$.
- Let f be any continuous function in $C[a, b]$. Then for any $\varepsilon > 0$ there exists a polynomial $P(x)$ such that $|\, p(x) - f(x) \,| < \varepsilon$. $\forall\ x \in [a, b]$.

❑❑❑

UNIT

4

FUNCTIONS OF SEVERAL VARIABLES

STRUCTURE

4.1. INTRODUCTION

As we studied the partial derivatives of the functions from R^n to R. In this unit our main aim to introduced the theory of derivatives of functions from R^n into R^m. The partial derivative is some what unsatisfactory generalization of the usual derivative because the existence of all the partial derivatives at a particular point does not necessarily imply the continuity of the function at that point, because of in the partial derivatives we treat a function of several variables as a function of one variable at a time. The partial derivatives describes the rate of change of a function in the direction of each co-ordinate axis. There is a slight generalization, called the *directional derivative,* which studies the rate of change of a function in an arbitrary direction. When we have a system of several equations involving several variables and we want to solve these equations for some of these variables in terms of remaining variables, the *implicit function theorem* provided a description of some conditions and conclusions about the solution of these equations.

4.2. FUNCTIONS OF SEVERAL VARIABLES

NOTES

4.2.1 Definition

As you are familiar to the concept of vector spaces and their properties in earlier classes for the convenience here we recall some definitions and preliminaries of the same in the frame work in Euclidean n-space R^n.

4.3. VECTORS

Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots\ldots, x_n$ be any n numbers then the ordered n-tuple $X = (x_1, x_2, \ldots, x_n)$ is called an n-vector. Thus the coordinates of a point in space can be represented by a 3-vector (x, y, z). Similarly $(1, 0, 2, -1)$ and $(2, 7, 5, -3)$ are 4-vectors. The n numbers $x_1, x_2, \ldots\ldots, x_n$ are called the components of the n-vector $X = (x_1, x_2, \ldots\ldots, x_n)$. A vector may be written either as a *row vector* or as a *column vector*. If A be a matrix of order $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector. A vector whose components are all zero is called a zero vector and is denoted by O. Thus $O = (0, 0, 0, \ldots\ldots, 0)$.

Let $X = (x_1, x_2, \ldots\ldots, x_n)$ and $Y = (y_1, y_2, \ldots\ldots, y_n)$ be two vectors.

Then $X = Y$ if and only if their corresponding components are equal.

i.e., if $x_i = y_i$, for $i = 1, 2, \ldots\ldots, n$

$$X + Y = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$$

If k be a scalar, then $kX = (kx_1, kx_2, \ldots\ldots, kx_n)$.

4.4. LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

A set of r n-vectors $X_1, X_2, \ldots\ldots, X_r$ is said to be *linearly dependent* if there exist r scalars (numbers) $k_1, k_2, \ldots\ldots, k_r$ not all zero, such that

$$k_1X_1 + k_2X_2 + \ldots + k_rX_r = O$$

A set of r n-vectors $X_1, X_2, \ldots\ldots, X_r$ is said to be *linearly independent* if every relation of the type

$$k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r = O \text{ implies } k_1 = k_2 = \ldots\ldots = k_r = 0$$

To test the linear dependence of r given vectors, write them as row vectors. Add suitable multiples of one vector to the others so that the resulting $(r - 1)$ vectors have their first component zero. Choose any one of these $(r - 1)$ vectors and add its multiples to the others so that the resulting $(r - 2)$ vectors have their second component zero. In this way continue, reducing the successive components to zero. If the final reduction gives a vector all of whose components are zero, then the original vectors are linearly dependent. However, if the final reduction gives a vector all of whose components are not zero, then the original vectors are linearly independent.

Note. If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the remaining vectors.

Example. *Show that the vectors $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$ and $x_4 = (-3, 7, 2)$ are linearly dependent and find the relation between them.*

Sol. Adding suitable multiples of x_1 to x_2 and x_4 so that the first component reduces to zero, we have

$$x_2 - 2x_1 = (2, -1, 3) - (2, 4, 8) = (0, -5, -5)$$

$$x_4 + 3x_1 = (-3, 7, 2) + (3, 6, 12) = (0, 13, 14)$$

Also, $x_3 = (0, 1, 2)$.

Adding suitable multiples of x_3 to the above vectors so that the second component reduces to zero, we have

$$(x_2 - 2x_1) + 5x_3 = (0, -5, -5) + (0, 5, 10) = (0, 0, 5)$$

$$(x_4 + 3x_1) - 13x_3 = (0, 13, 14) - (0, 13, 26) = (0, 0, -12)$$

To reduce the third component to zero, multiplying the above vectors by 12 and 5 respectively and adding, we have

$$12(x_2 - 2x_1 + 5x_3) + 5(x_4 + 3x_1 - 13x_3) = (0, 0, 60) + (0, 0, -60)$$

$$\Rightarrow \quad -9x_1 + 12x_2 - 5x_3 + 5x_4 = (0, 0, 0)$$

$$\Rightarrow \quad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Thus, there exist numbers $k_1 = 9$, $k_2 = -12$, $k_3 = 5$, $k_4 = -5$ which are not all zero such that

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

Hence the vectors x_1, x_2, x_3 and x_4 are linearly dependent. Also, (1) is the relation between them.

4.5. LINEAR TRANSFORMATIONS

Let a point P(x, y) in a plane transforms to the point P$'(x', y')$ under reflection in the co-ordinate axes, or reflection in the line $y = x \tan \theta$ or rotation of OP through an angle θ about the origin or rotation of axes through an angle θ etc. Then the co-ordinates of P$'$ can be expressed in terms of those of P by the linear relations of the form

$$x' = a_1x + b_1y$$

$$y' = a_2x + b_2y$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ or $X' = AX$

Such transformations are called linear transformations in two dimensions.

Similarly, relations of the form $\left.\begin{aligned} x' &= a_1x + b_1y + c_1z \\ y' &= a_2x + b_2y + c_2z \\ z' &= a_3x + b_3y + c_3z \end{aligned}\right]$

which in matrix notations is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $X' = AX$ gives a linear transformation $(x, y, z) \longrightarrow (x', y', z')$ in three dimensions.

In general, the relation Y = AX, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

NOTES

defines a linear transformation which carries any vector X into another vector Y over the matrix A which is called the linear operator of the transformation.

This transformation is called linear because $Y_1 = AX_1$ and $Y_2 = AX_2$ implies $aY_1 + bY_2 = A(aX_1 + bX_2)$ for all values of a and b.

Thus, if $X = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ then $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

so that $(2, -3) \to (5, -5)$ under the transformation defined by A.

If the transformation matrix A is non-singular, *i.e.*, if $|A| \neq 0$, then the linear transformation is called *non-singular* or *regular*.

If the transformation matrix A is singular, *i.e.*, if $|A| = 0$, then the linear transformation is also called *singular*.

For a non-singular transformation Y = AX, since A is non-singular, A^{-1} exists and we can write the inverse transformation, which carries the vector Y back into the vector X, as $X = A^{-1}Y$.

Note. If a transformation from $(x_1, x_2, \dots\dots, x_n)$ to $(y_1, y_2, \dots\dots, y_n)$ is given by Y = AX and another transformation from $(y_1, y_2, \dots\dots, y_n)$ to $(z_1, z_2, \dots\dots, z_n)$ is given by Z = BY, then the transformation from $(x_1, x_2, \dots\dots, x_n)$ to $(z_1, z_2, \dots\dots, z_n)$ is given by Z = BY = B(AX) = (BA)X.

4.6. ORTHOGONAL TRANSFORMATION

The linear transformation Y = AX, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be *orthogonal* if it transforms $y_1^2 + y_2^2 + \dots\dots + y_n^2$ into $x_1^2 + x_2^2 + \dots\dots + x_n^2$.

The matrix A of this transformation is called an *orthogonal matrix*.

Now, $$X'X = [x_1 x_2 \dots\dots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \dots\dots + x_n^2$$

and similarly $$Y'Y = y_1^2 + y_2^2 + \dots\dots + y_n^2$$

∴ If Y = AX is an orthogonal transformation, then

$$X'X = x_1^2 + x_2^2 + \dots\dots + x_n^2 = y_1^2 + y_2^2 + \dots\dots + y_n^2$$
$$= Y'Y = (AX)'(AX) = (X'A')(AX) \quad [\because (AB)' = B'A']$$
$$= X'(A'A)X$$

which holds only when $A'A = I$ or when $A'A = A^{-1}A$ $\quad [\because A^{-1}A = I]$

or when $A' = A^{-1}$.

Hence *a square matrix A is said to be orthogonal if AA′ = A′A = I*

Also, *for an orthogonal matrix A, A′ = A⁻¹*.

NOTES

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the transformation* $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ *is regular. Write down the inverse transformation.*

Sol. In matrix notation, the given transformation is Y = AX,

where $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1 \neq 0,$$

the matrix A is non-singular and hence, the given transformation is non-singular or regular.

As usual, $$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

The inverse transformation is given by $X = A^{-1}Y$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 2y_1 - 2y_2 - y_3,$$
$$x_2 = -4y_1 + 5y_2 + 3y_3, \quad x_3 = y_1 - y_2 - y_3.$$

Example 2. *Prove that the following matrix is orthogonal* $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Sol. Denoting the given matrix by A, we have

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now, $$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since AA′ = I, A is an orthogonal matrix.

Example 3. *If A is an orthogonal matrix, prove that* $|A| = \pm 1$.

Sol. A is an orthogonal matrix

$\Rightarrow \quad AA' = I \quad \Rightarrow \quad |AA'| = |I|$

$\Rightarrow \quad |A||A'| = 1 \quad \Rightarrow \quad |A||A| = 1 \quad [\because \ |A'| = |A|]$

$\Rightarrow \quad |A|^2 = 1 \quad \Rightarrow \quad \therefore \quad |A| = \pm 1.$

NOTES

Example 4. *Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orgthogonal.*

Sol. (*i*) A is orthogonal $\Rightarrow$ AA′ = I

$\Rightarrow$ $(AA')^{-1} = I^{-1} \Rightarrow (A')^{-1} A^{-1} = I$ $[\because (AB)^{-1} = B^{-1}A^{-1}]$

$\Rightarrow$ $(A^{-1})'A^{-1} = I$

$\Rightarrow$ Product of A^{-1} and its transpose $(A^{-1})'$ is equal to I

$\Rightarrow$ A^{-1} is also orthogonal.

(*ii*) A is orthogonal $\Rightarrow$ AA′ = I $\Rightarrow$ $(AA')' = I'$ $\Rightarrow$ $(A')' A' = I$

$\Rightarrow$ Product of A′ and its transpose $(A')'$ is equal to I.

$\Rightarrow$ A′ is also orthogonal.

TEST YOUR KNOWLEDGE

1. Are the following vectors linearly dependent? If so, find a relation between them.
   (*i*) $x_1 = (1, 3, 2), x_2 = (5, -2, 1), x_3 = (-7, 13, 4)$
   (*ii*) $x_1 = (3, 2, 7), x_2 = (2, 4, 1), x_3 = (1, -2, 6)$
   (*iii*) $x_1 = (2, -1, 3, 2), x_2 = (1, 3, 4, 2), x_3 = (3, -5, 2, 2)$.
   (*iv*) $x_1 = (2, 3, 1, -1), x_2 = (2, 3, 1, -2), x_3 = (4, 6, 2, 1)$.
2. Show that the transformation $y_1 = x_1 - x_2 + x_3$, $y_2 = 3x_1 - x_2 + 2x_3$, $y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular. Find the inverse transformation.
3. Represent each of the transformations $x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$, $y_1 = z_1 + 2z_2$ and $y_2 = 3z_1$ by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.
4. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by Y = AX, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by Z = BY, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$,
   $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$. Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.
5. Which of the following matrices are orthogonal?
   (*i*) $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$.
6. Prove that the matrix $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is orthogonal.

Answers

1. (*i*) Yes ; $3x_1 - 2x_2 - x_3 = 0$ (*ii*) Yes ; $x_1 - x_2 - x_3 = 0$
   (*iii*) Yes ; $2x_1 - x_2 - x_3 = 0$ (*iv*) Yes ; $5x_1 - 3x_2 - x_3 = 0$.
2. $x_1 = \frac{1}{2}(y_1 + y_2 - y_3), x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3), x_3 = -2y_1 + y_3$.
3. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ **4.** Z = (BA)X, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$
5. (*i*) Orthogonal (*ii*) Not orthogonal.

4.7. DIFFERENTIATION

4.7.1 Derivatives in an Open Subset of R^n

Before to arrive at the definition of the derivative of a function in an open subset of R^n. First of all we recall the familiar case for $n = 1$. If f is a real function with domain $(a, b) \in R^1$ and if $x \in (a, b)$, then the derivative of $f(x)$ is denoted by $f'(x)$ and usually defined to be a real number as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists. Thus,

$$f(x + h) - f(x) = f'(x)h + r(h) \quad \ldots(1)$$

where the "remainder" $r(h)$ is small, in the sense that

$$\lim_{h \to 0} [r(h)/h] = 0$$

Here we observe that (1) expresses the difference $f(x + h) - f(x)$ as the sum of a linear function that takes h to $f(x)h$ plus a small remainder. Therefore we can regard the derivative of f at x, not as a real number but as the linear operator on R^1 that takes h to $f'(x)h$.

We observe that every real number α gives rise to a linear operator on R^1; the operator in question is simply multiplication by α. For example define a function $T_\alpha : R^1 \to R^1$ by $T_\alpha(x) = \alpha x \; \forall \; x \in R^1$, clearly T_α is a linear operator. Conversely we see that every linear function that carries R^1 to R^1 is multiplication by some real number. Thus there is a 1-1 correspondence between R^1 and $L(R^1)$ which motivates the preceding statements.

Next we consider a function f from $(a, b) \subset R^1$ into R^m. Here if $f'(x)$ is exists then we define $f'(x)$ to be a vector $y \in R^m$ for which

$$\lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} - y \right\} = 0$$

It can be rewrite as $f(x + h) - f(x) = hy + r(h)/h$, $r(h)/h \to 0$ as $h \to 0$. Here we observe that the main term on the right side is again a linear function of h. Thus we see that every $y \in R^m$ induces a linear transformation of R^1 into R^m by associating to each $h \in R^1$ the vector $hy \in R^m$. This identification of R^m with $L(R^1, R^m)$ allows us to regard $f'(x)$ as a member of $L(R^1, R^m)$. Hence if f is a differentiable mapping of $(a, b) \in R^1$ into R^m and if $x \in (a, b)$ then $f'(x)$ is the linear transformation of R^1 into R^m satisfies

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0$$

or equivalently

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0$$

From the above discussion we are now ready to define the differentiability of a function f for the case $n > 1$.

4.7.2 Definition

Suppose E is an open set in R^n, f maps E into R^m and $x \in E$. If there exists a linear transformation A of R^n into R^m such that

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-Ah|}{|h|}=0 \quad \ldots(ii)$$

NOTES

then f is said to be differentiable at x, and we write $f'(x) = A$. If f is differentiate at every $x \in E$, we say that f is differentiable in E. Here we note that, of course, $h \in R^n$. Also since E is open then if $|h|$ is small enough, then $x + h \in E$. Thus $f(x+h)$ is defined, $f(x+h) \in R^m$, and since $A \in L(R^n, R^m)$, $Ah \in R^m$. Therefore,

$$f(x+h) - f(x) - Ah \in R^m.$$

The norm in the numerator of (*i*) is that of R^m and in the denominator we have the R^n-norm of h.

Now there arise a question that $f'(x) = A$ defined in term of linear transformation is unique or not. The following result deals the answer of this question.

4.7.3 Theorem

Suppose E is an open set in R^n, f maps E into R^m and $x \in E$. If there exists a linear transformation A of R^n into R^m such that

$Lim \ | f(x+h) - f(x) - Ah \ | = 0$

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-Ah|}{|h|}$$

holds with $A = A_1$ and with $A = A_2$. Then $A_1 = A_2$.

Proof. If $B = A_1 - A_2$ then

$$\begin{aligned}|Bh| &= |(A_1 - A_2)h| = |A_1h - A_2h|\\ &= |A_1h - f(x+h) + f(x) + f(x+h) - f(x) - A_2h|\\ &\le |f(x+h) - f(x) - A_1h| + |f(x+h) - f(x) - A_2h|.\end{aligned}$$

implies that $\quad |Bh|/|h| \longrightarrow 0$ as $h \longrightarrow 0$.

Hence for fixed $h \neq 0$ and the linearity of B, it follows that

$$\begin{aligned}\lim_{t\to 0}[|B(th)|/|th|] &= \lim_{t\to 0}[t|B(h)|/t|h|]\\ &= \lim_{t\to 0}[t|B(h)|/|h|] = 0\end{aligned}$$

Shows that the above equality is independent of t. Hence, $Bh = 0$ for every $h \in R^n$. Consequently $B = 0$ or $A_1 = A_2$.

4.7.4 Some Remarks

(*a*) The relation $\displaystyle \lim_{h\to 0}\frac{|f(x+h)-f(x)-Ah|}{|h|}=0 \quad \ldots(ii)$

can be written in the form $f(x+h) - f(x) = f'(x)h + r(h) \quad \ldots(iii)$

where the remainder $r(h)$ satisfies $\text{Lim}_{h\to 0}[r(h)/h] = 0$.

By saying that for fixed x and small h, (*ii*) can be interpret as $f(x+h) - f(x) = f'(x)h$ that is, left hand side of (*ii*) is approximately equal to $f'(x)h$, that is, to the value of a linear transformation applied to h.

(*b*) Suppose f and E are as defined earlier and f is differentiable in E. Then we can, interpret $f'(x)$ for every $x \in E$, a function namely, a linear transformation of R^n into R^m. But f' is also a function: f' maps E into $L(R^n, R^m)$.

(*c*) Since as differentiability implies continuity it follow that f is continuous at any point at which f is differentiable.

(*d*) The derivative defined by (*ii*) or (*iii*) is often called the differential of f at x, or the total derivative of f at x.

(*e*) We have defined derivatives of functions carrying R^n to R^m to be linear transformations of R^n into R^m. What is the derivative of such a linear transformation? For the answer of this question consider $A \in L(R^n, R^m)$ and if $x \in R^n$, then $A'(x) = A$. Since x appears on the left side of $A'(x) = A$, but not on the right. However both $A'(x)$ and A are members of $L(R^n, R^m)$, whereas $Ax \in R^m$. Then by linearity of A, we have

$$A(x + h) - Ax = Ah$$

Therefore for $f(x) = Ax$ from (*ii*), we have

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-Ah|}{|h|} = \lim_{h\to 0}\frac{|Ah-Ah|}{|h|} = 0$$

for all $h \in R^n$, yields that $f'(x) = A'x = A$.

Now we extend the chain rule in present situation.

4.8. CHAIN RULE

4.8.1 Theorem

Suppose E is an open set in R^n. f maps E into R^m, f is differentiate at $x_0 \in E$, g maps an open set containing f(E) into R^k, and g is differentiable at $f(x_0)$. Then the mapping F of E into R^k, defined by

$$F(x) = g(f(x)) \text{ is differentiable at } x_0$$

and
$$F'(x_0) = g'(f(x_0))f'(x_0) \quad \ldots(i)$$

Note. Here we note the right side of (*i*) is the product of two linear transformations.

Proof. Put $y_0 = f(x_0)$, $A = f'(x_0)$, $B = g'(y_0)$, and write

$$u(h) = f(x_0 + h) - f(x_0) - Ah;\ v(k) = g(y_0 + k) - f(y_0) - Bk$$

for all $h \in R^n$ and $k \in R^m$ for which $f(x_0 + h)$ and $g(y_0 + k)$ are defined. Then by differentiability of f and g at x_0 and y_0 respectively, we have

$$\lim_{h\to 0}\frac{|f(x_0+h)-f(x_0)-Ah|}{|h|} = \lim_{h\to 0}\frac{|u(h)|}{|h|} = 0$$

and
$$\lim_{k\to 0}\frac{|g(y_0+k)-g(y_0)-Bk|}{|k|} = \lim_{h\to 0}\frac{|v(k)|}{|k|} = 0$$

which implies that

$|u(h)| = \varepsilon(h)|h|$ and $|v(k)| = \eta(k)|k|$ where $\varepsilon(h) \to 0$ as $h \to 0$ and $\eta(k) \to 0$ as $k \to 0$.

Now for given h put $k = f(x_0 + h) - f(x_0)$. Then $|k| = |Ah + u(h)| \le [\|A\| + \varepsilon(h)]\,|h|$ and

$$F(x_0 + h) - F(x_0) - BAh = g\{f(x_0 + h)\} - g\{f(x_0)\} - BAh = g(y_0 + k) - g(y_0) - BAh$$
$$= v(k) + Bk - Bah = B(k - Ah) + v(k) = Bu(h) + v(k).$$

Hence for $h \neq 0$, we have

$$\frac{|F(x_0+h)-F(x_0)-BAh|}{|h|} = \frac{|Bu(h)+v(k)|}{|h|} \le \frac{|Bu(h)|+|v(k)|}{|h|}$$

$$\leq \frac{\| B \| \varepsilon(h) \| h | + \eta(k) | k |}{| h |} \leq \| B \| \varepsilon(h) + [\| A \| + \varepsilon(h)]\eta(k).$$

NOTES

Letting $h \to 0$, then $\varepsilon(h) \to 0$, $k \to 0$ and $\eta(k) \to 0$.

It follows that

$$\lim_{h \to 0} \frac{| F(x_0 + h) - F(x_0) - BAh |}{| h |} = 0$$

i.e., $F'(x_0) = BA = g'(f(x_0))f'(x_0)$.

4.9. PARTIAL DERIVATIVES

Let f be a function from an open set $E \subset R^n$ into R^m. Let $\{e_1, ..., e_n\}$ and $\{u_1, ..., u_m\}$ be the standard bases of R^n and R^m respectively. Then the components of f are the real functions $f_1, ..., f_m$ defined by

$$f(x) = (f_1(x), f_2(x), ..., f_m(x)) = f_1(x)\ (1, 0, ..., 0) + ... + f_m(x)\ (0, 0, ... , 1)$$

$$= f_1(x)u_1 + ... + f_m(x)y_m = \Sigma f_i(x)u_1. \quad ...(i) \text{ (where } i = 1 \text{ to } m \text{ and } x \in E)$$

or, equivalently, by $f_i(x) = f(x) \cdot u_i$, $1 \leq i \leq m$.

4.9.1 Definition

For $x \in E$, $1 \leq i \leq m$, $1 \leq j \leq n$, we define

$$D_j f_i(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t} \quad ...(ii)$$

provided the limit exists. Write $f_i(x) = f_j(x_i, ..., x_n)$, we see that $D_j f_i$ is the derivative of f_i with respect to x_j, keeping the other variables $x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n$ fixed. The notation $\partial f_i / \partial x_j$ is often used in place of $D_j f_i$ it is called a ***partial derivative*** of f_i with respect to x_j.

In many cases when we deal with functions of one variable, the existence of a derivative is sufficient. On the other hand we need continuity or at least boundness of the partial derivatives while dealing with the function of several variables. However, if f is known to be differentiate at a point x, then its partial derivatives exist at x, and they determine the linear transformation $f'(x)$ completely.

4.9.2 Theorem

Suppose f maps an open set $E \subset R^n$ into R^m and f is differentiable at a point $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)e_j = \Sigma(D_j f_i)(x)u_i \ (1 \leq i \leq m, \ 1 \leq j \leq n) \quad ...(i)$$

where $\{e_1, ..., e_n\}$ and $\{u_1, ..., u_m\}$ are the standard bases of R^n and R^m.

Proof. Since f is differentiable at x, then take $h = te_j$ for fix j, we have

$$f(x + te_j) - f(x) = f'(x)\ (te_j) + r(te_j) \quad ...(ii)$$

where $| r(te_j) | / t \to 0$ as $t \to 0$.

Now by the linearity of $f'(x)$, we have $f'(te_j) = tf'(e_j)$, therefore by (*ii*), we have

$$\lim_{t\to 0} \frac{f(x + tej) - f(x)}{t} = f'(x)e_j \qquad \ldots(iii)$$

On representing f in terms of its components (*iii*) becomes

$$\lim_{t\to 0} \frac{\Sigma[f_i(x + te_j) - f_i(x)]u_i}{t} = f'(x)e_j\ (1 \le i \le m) \qquad \ldots(iv)$$

Since right side of (*iv*) exists therefore each quotient in this sum has a limit as $t \to 0$, so that each $\lim_{t\to 0} \frac{f_i(x + te_j) - f_i(x)}{t}$ exists $t = 0$

i.e., $(D_j f_i)(x)$ exists. Hence from (*ii*), we have $f'(x)e_j = \Sigma(D_j f_i)(x)u_i$ $(1 \le i \le m, 1 \le j \le n)$.

4.9.3 Directional Derivatives

1. Let $[f'(x)]$ be the matrix that represents $f'(x)$ with respect to bases $\{e_1, e_1, \ldots, e_m\}$ and $\{u_1, u_1, \ldots, u_n\}$. Then $f'(x)e_j$ is the j^{th} column vector of $[f'(x)]$ and (*i*) shows that the number $(D_jf_i)(x)$ occupies the spot in the i^{th} row and j^{th} column of $[f'(x)]$. Thus,

$$[f'(x)] = \begin{pmatrix} (D_1f_1)(x) & \ldots & (D_nf_1)(x) \\ \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ (D_1f_m)(x) & \ldots & (D_nf_m)(x) \end{pmatrix} \qquad \ldots(i)$$

If $h = \Sigma h_je_j$ $(1 \le j \le n)$ be any vector in R^n. Then by (*i*), we have

$$\begin{aligned} f'(x)h &= f'(x)\ \Sigma h_je_j \quad (1 \le j \le n) \\ &= \Sigma h_jf'(x)e_j \quad (1 \le j \le n) \\ &= \Sigma h_j\Sigma(D_jf_i)(x)u_i \quad (1 \le i \le m,\ 1 \le j \le n) \\ &= \Sigma\{\Sigma(D_jf_i)(x)\ h_j\}u_i \quad (1 \le i \le m,\ 1 \le j \le n) \qquad \ldots(ii) \end{aligned}$$

2. Let γ be a differentiable mapping of the segment $(a, b) \subset R$ into an open set $E \subset R^n$, that is γ be a differentiable curve in E. Further let f be a real-valued differentiable function with domain in E. Thus we may consider f to be a differentiable mapping of E into R.

Let $$g(t) = f((t)) \quad (a < t < b) \qquad \ldots(iii)$$

Then by the chain rule

$$g'(t) = f'(\gamma(t))\ \gamma'(t) \quad (a < t < b) \qquad \ldots(iv)$$

Here we observe that $\gamma'(t) \in L(R, R^n)$ and $f'(g(t)) \in L(R^n, R)$ therefore $g'(t)$ is a linear operator on R, since if g maps (a, b) into R, furthermore $g'(t)$ can be regarded as a real number which is computed in terms of the partial derivatives of f and the derivatives of the components of γ as follows:

Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, then $\gamma(t) = \gamma_1(t)e_1 + \gamma_2(t)e_2 + \ldots + \gamma_n(t)e_n$, then $\gamma'(t)\ g'(t) = f'(\gamma(t))\ \gamma'(t) = \gamma'_1(t)e_1 + \gamma'_2(t)e_2 + \ldots + \gamma'_n(t)e_n$ and so $[\gamma'(t)]$ is a column matrix of order n by 1 in which $\gamma'_i(t)$ is in the i^{th} row with respect to the standard basis $\{e_1, e_1, \ldots, e_n\}$ of R^n. Again for all $x \in E$, $[f'(x)]$ is a row matrix of order 1 by n which has $(D_jf)(x)$ in the j^{th} column. Hence $[g'(t)]$ is the 1 by 1 matrix of real number.

$$g'(t) = \Sigma(D_jf)\ (\gamma(t))\ \gamma i'(t)\ (1 \le i \le n) \qquad \ldots(v)$$

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Now we define the gradient of f at each $x \in$ E by $(\Delta f)(x) = \Sigma(D_j f)(x)e_i$ $(1 \le i \le n)$ and since $\gamma_i'(t) = \Sigma\gamma_i'(t)\, e_j(1 \le i \le n)$. Hence we have

$$g'(t) = (\Delta f)\,(\gamma(t)) \cdot \gamma'(t)$$

the scalar product of the vectors $(\Delta f)\, \gamma'(t)$ and $\gamma'(t)$. Now we define $\gamma(t) = x + tu$ ...(*vi*)

For $t \in (-\infty, \infty)$ for fix $x \in$ E and unit vector $u \in R^n$. Then $\gamma(t) = u$ for all t. Hence for $t = 0$ (*ix*) becomes

$$g'(0) = (\Delta f)(x) \cdot u \qquad \text{...(vii)}$$

Hence by (*x*), we have

$$g(t) - g(0) = f(\gamma(t)) - f(\gamma(0)) = f(x + tu) - f(x) + tu.$$

Further $\quad g'(0) = \lim_{t\to 0}[g(t) - g(0)]/t = \lim_{t\to 0}[f(x + tu) - f(x)]/t$

Hence by (*vii*) $\lim_{t\to 0}[f(x + tu) - f(x)]/t = (\Delta f)\,(x) \cdot u$ ...(*viii*)

Here in (*viii*) is usually called the directional derivative of f at x in the direction of unit vector u and is denoted by $(D_u f)\,(x)$.

On keeping f and x is fixed and u varies, then (*viii*) shows that the direction derivatives of f at x attains its maximum when u is a positive scalar multiple of $(\Delta f)(x)$. Here we exclude the case $(\Delta f)\,(x) = 0$. If we put $u = \Sigma u_i e_j$ $(1 \le i \le n)$, then $(D_u f)\,(x) = \Sigma(D_j f)(x)u_i$ $(1 \le i \le n)$.

4.9.4 Theorem

Suppose f maps a convex open set $E \subset R^n$ into R^m, f is differential in E, and there is real number M such that

$$\| f'(x) \| \le M$$

for every $x \in E$. Then

$$| f(b) - f(a) | \le M\, | b - a | \text{ for all } a, b \in E.$$

Proof. Fix $a, b \in$ E and define

$$\gamma(t) = (1 - t)a + tb \qquad \text{...(i)}$$

for all $t \in R^1$ such that $\gamma(t) \in$ E. Since if $0 \le t \le 1$ then by convexity of E, $\gamma(t) \in$ E. Now we put $g(t) = f(\gamma(t))$. Then by chain rule and by (*i*), we have

$$g'(t) = f'(\gamma(t)\gamma'(t) = f'(\gamma(t))\;(b - a)$$

So that $\quad | g'(t) | \le \| f'(\gamma(t)) \|\, | b - a | \le M\, | b - a |$ for all $t \in [0, 1]$ ...(*ii*)

Now for all $t \in [0, 1]$ by mean value theorem for vector valued function, we have

$$| g(1) - g(0) | \le (1 - 0)\, | g'(t) | \le M\, | b - a |$$

Now since $g(0) = f(\gamma(0)) = f(a)$ and $g(1) = f(\gamma(1)) = f(b)$. Hence, $f(b) - f(a) | \le M\, | b - a |$. This complete the proof.

4.9.5 Corollary

If $f'(x) = 0$ for all $x \in E$ then f is constant.

Proof. Since if $f'(x) = 0$ for all $x \in$ E then by (*ii*), we have $g'(t) = 0$.

Therefore, $\quad | g(1) - g(0) | \le 0,$

i.e., $\quad g(1) = g(0)$ or $f(b) = f(a)$, proves that f is constant.

4.9.6 Definition (Continuously differentiable function)

A differentiable mapping f of an open set $E \subset R^n$ into R^m is said to be continuously differentiable in E if f' is a continuous mapping of E into $L(R^n, R^m)$.

In other word, a differentiable mapping f of an open set $E \subset R^n$ into R^m is said to be continuously differentiable if for every $x \in E$ and to every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\| f'(y) - f'(x) \| < \varepsilon$ if $y \in E$ and $| x - y | < \delta$.

A continuously differentiable mapping f is also known as a ζ – mapping, or that $f \in \zeta'(E)$.

4.9.7 Theorem

Suppose f maps an open set $E \subset R^n$ into R^m. Then $f \in \zeta'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m$, $1 \le j \le n$.

Proof. Assume first that $f \in \zeta'(E)$ and if $\{e_1, e_2, \ldots, e_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ are the standard basis of R^n and R^m respectively. Since the partial derivative of f_i w.r.t x_j is given by $(D_j f_i)(x) = (f'(x)e_j) \cdot u_i$ for all i, j and for all $x \in E$. Hence,

$$(D_j f_i)(y) - (D_j f_i)(x) = \{[f'(y) - f'(x)]e_j\} - u_i.$$

and since $| u_i | = | e_j | = 1$, it follows that

$$| (D_j f_i)(y) - (D_j f_i)(x) | = | \{[f'(y) - f'(x)]e_j\} \cdot u_i | \le |[f'(y) - f'(x)]e_j \| u_i |$$

(Since $| x \cdot y | \le | x | | y |$)

$$= | [f'(y) - f'(x)]e_j | \le \| f'(y) - f'(x) \| \| e_j \|,$$

(Since $| Ax | \le \| A \| |x|$) ... (*i*)

$$= \| f'(y) - f'(x) \|$$

Since f' is continuous therefore for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$| x - y | < \delta \Rightarrow \| f'(x) - f'(y) \| < \varepsilon \ \forall \ x, y \in E \qquad \ldots (ii)$$

Thus from (*i*) and (*ii*) for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad | D_j f_i (y) - (D_j f_i)(x) | < \varepsilon.$$

Hence $D_j f_i$ is continuous on E, $1 \le i \le m$, $1 \le j \le n$.

For the converse, it suffices to consider the case $m = 1$. Suppose that f maps an open set $E \subset R^n$ into R^m such that its partial derivatives $D_j f$ exists and continuous on E for $1 \le j \le n$. Fix $x \in E$ and $\varepsilon > 0$ be given. Since E is open, there is an open ball $B_r(x) \subset E$, with center at x and radius r, and the continuity of the functions $D_j f$ shows that r can be chosen so that

$$| (D_j f)(y) - (D_j f)(x) | < \varepsilon/n \ (y \in B_r(x),\ 1 \le j \le n) \qquad \ldots (iii)$$

Suppose $h = \Sigma h_j e_j$, $| h | < r$ $(1 \le j \le n)$, put $v_0 = 0$ and $v_k = h_1 e_1 + \ldots + h_k e_k$, for $1 \le k \le n$. Then,

$$\begin{aligned} f(x + h) - f(x) &= f(x + h_1 e_1 + \ldots + h_n e_n) - f(x) \\ &= [f(x + h_1 e_1 + \ldots + h_n e_n) - f(x + h_1 e_1 + \ldots + h_{n-1} e_{n-1})] \\ &\quad + [f(x + h_1 e_1 + \ldots + h_{n-1} e_{n-1}) - f(x + h_1 e_1 + \ldots + h_{n-2} e_{n-2})] \\ &\quad + \ldots + [f(x + h_1 e_1) - f(x + 0)]. \\ &= [f(x + v_n) - f(x + v_{n-1})] + [f(x + v_{n-1}) - f(x + v_{n-2})] \\ &\quad + \ldots + [f(x + v_1) - f(x + v_0)] \\ &= \Sigma [f(x + v_j) - f(x + v_{j-1})] \ (1 \le j \le n) \qquad \ldots (iv) \end{aligned}$$

Since $| v_k | < r$ for $1 \le k \le n$ and since $B_r(x)$ is convex, the segments with end points $x + v_{j-1}$ and $x + v_j$ lie in $B_r(x)$. Since $v_j = v_{j-1} + h_j e_j$, the mean value theorem shows that the j^{th} summand in (*iv*) is given by $f(x + v_j) - f(x + v_{j-1}) = h_j (D_j f)(x - v_{j-1} \theta_j h_j e_j)$ for some $0 < \theta_j < 1$, and then $| h_j (D_j f)(x - v_j - \theta_j h_j e_j) - h_j (D_j f)(x) | = | h_j | \ | (D_j f)(x - v_{j-1} \theta h_j e_j) - (D_j f)(x) | < | h_j | \varepsilon/n$, by (*iii*).

By (*iv*), it follows that for $1 \le j \le n$,

$$| \Sigma h_j (D_j f)(x - v_{j-1} \theta_j h_j e_j) - \Sigma h_j (D_j f)(x) | < \Sigma | h_j | \varepsilon/n \le | h | \varepsilon < r\varepsilon,$$

Or, $|f(x+h)| - f(x) - \Sigma h_j(D_jf)(x)| < r\varepsilon$. This inequality shows that f is differentiable at x and that $f'(x)$ is the linear function which assigns the number $\Sigma h_j(D_jf)(x)$ to the vector $h = \Sigma h_j e_j$.

The matrix $[f'(x)]$ consists of the row $(D_1f)(x), \ldots, (D_nf)(x)$, and since $(D_1f), \ldots, (D_nf)$ are continuous functions on E.

4.10. DERIVATIVES OF HIGHER ORDER

Let f is a real valued function defined in an open set $E \subset R^n$, is called the function of n-variables, with partial derivatives $D_1f, D_2f, \ldots, D_nf$. If the functions D_jf are themselves differentiable, then the second-order partial derivative $D_iD_j f (i, j = 1, \ldots, n)$. For $i = j$, the second order partial derivative of f are written as D_i^2f. If all these functions $D_{ij}f$ are continuous in E, we say that f is of class ζ'in E, or that $f \in \zeta'$ of f are defined by $D_{ij}f = (E)$. A mapping f of E into R^m is said to be of class ζ'' if each component f_k $(k = 1, 2, \ldots, m)$ of $f = (f_1, f_2, \ldots, f_m)$ is of class ζ'' *i.e.*, $f \in \zeta''$ (E). The third ***order partial derivatives*** of f are defined by $D_{ijk}f = D_iD_jD_kf$ $(i, j, k = 1, \ldots, n)$. For $j = k$, $i = j$ and $i = j = k$ third order partial derivative written as $D_iD_j^2f$, $D_i^2D_kf$ and D_i^3f respectively. Similarly fourth order, fifth order, ... partial order can be defined. $D_1^{m_1} D_2^{m_2} \ldots D_n^{m_n}$ is called the m^{th} order partial derivatives, where $m = m_1 + m_2 + \ldots + m_n$. If all m^{th} order partial derivatives are continuous in E, we call $f \in \zeta''(E)$.

Repeated partial derivatives: If the derivatives $(D_{ij}f)(x) = (D_iD_j)(x)$ exists, it is called a *repeated partial derivatives*.

4.10.1 Interchange of Order of Differentiation

In general, the order of repeated partial derivatives cannot be interchanged, *i.e.*, it can happen that $D_{ij}f \neq D_{ji}f$ at some points, although both derivatives exist.

For example consider the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, (x, y) \neq 0 \text{ ,we have} \end{cases}$$

$$0, (x, y) = (0, 0)$$

$$(D_xf)(x, y) = y\frac{(3x^2 - y^2)(x^2 + y^2) - 2x^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$= y\frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}, (x, y) \neq 0,$$

and
$$D_xf(0, 0) = \lim_{h \to 0}\frac{f(h, 0) - f(0, 0)}{h} = 0.$$

Hence $(D_xf)(0, y) = -y$ for all y, therefore

$(D_{xy}f)(0, y) = -1$ and $(D_{xy}f)(0, 0) = -1$. But

$$(D_yf)(x, y) = x\frac{(x^2 - 3y^2)(x^2 + y^2) - 2y^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$= x\frac{x^4 - 4x^2y^2 - y^4}{(x^2 + y^2)^2}, (x, y) \neq (0, 0), \text{and}$$

and
$$D_yf(0, 0) = \lim_{k \to 0}\frac{f(0, k) - f(0, 0)}{k} = 0.$$

Hence $(D_y f)(x, 0) = x$ for all x, therefore

$(D_{yx} f)(x, 0) = 1$ and $(D_{yx} f)(0, 0) = 1$. Thus $(D_{xy} f)(0, 0) \neq (D_{yx} f)(0, 0)$.

However, our next results give the criteria that $D_{ij} f = D_{ij} f$, whenever these derivatives are continuous.

4.10.2 Theorem (Mean Value)

Suppose f is defined in an open set $E \subset R^2$ and $D_1 f$, $D_{21} f$ exist at every point of E. Suppose $Q \subset E$ is a closed rectangle with sides parallel to the coordinate axes, having (a, b) and $(a + h, b + k)$ as opposite vertices $(h \neq 0, k \neq 0)$. Put

$$\Delta(f, Q) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

Then there is a point (x, y) in the interior of Q such that

$$\Delta(f, Q) = hk(D_{21} f)(x, y)$$

Proof. Put $u(t) = f(t, b + k) - f(t, b)$. Now by the mean value theorem for real valued function stated as "If f is a continuous real valued function on $[a, b]$, which is differentiable in (a, b) then there exists a point x in (a, b) such that $f(b) - f(a) = (b - a) f'(x)$". Two applications of this theorem are, there is a x between a and $a + h$, and a y between b and $b + k$, such that

$$\Delta(f, Q) = u(a + h) - u(a) = hu'(x)$$
$$= h[(D_1 f)(x, b + k) - (D_1 f)(x, b)] = hk(D_{21} f)(x, y)$$

4.10.3 Theorem

Suppose f is a real value function defined in an open set $E \subset R^2$ suppose that $D_1 f$, $D_{21} f$ and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$. Then $D_{12} f$ exists at (a, b) and $(D_{12} f)(a, b) = (D_{21} f)(a, b)$.

OR, Suppose f is a real value function defined in open set $E \subset R^2$, suppose that the partial derivatives $D_1 f$, $D_2 f$, $D_{12} f$, and $D_{21} f$ and continuous in E. Then $(D_{12} f)(a, b) = (D_{21} f)(a, b)$; $(a, b) \in E$, or $D_{21} f = D_{12} f$ if $f \in \zeta''(E)$.

Proof. Let $(a, b) \in E$ be fix arbitrary element, since E is open and so there exists an open ball $B(0, r)$ such that $(a, b) + (h, k) \in E$ for all $(h, k) \in B(0, r)$. Now define a function Δ: $B(0, r) \to r$ by $\Delta(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$ for $(h, k) \in B(0, r)$.

Consider the function $u(t) = f(t, b + k) - f(t, b)$. Then by mean value theorem
$u(a + h) - u(a) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$

$$= hu'(a + \theta_1 h), \theta_1 \in (0, 1)$$
$$= h[(D_1 f)(a + \theta_1, h, b + k) - (D_1 f)(a + \theta_1 h, b)]$$

Further $\Delta(h, k) = u(a + h) - u(a) = hu'(x)$

$$= h[(D_1 f)(a + \theta_1 h, b + k) - (D_1 f)(a + \theta_1 h, b)]$$
$$= hk[(D_{21} f)(a + \theta_1 h, b + \theta_2 k), \theta_2 \in (0, 1).$$

Now by the continuity of $D_{21} f$, we have $\lim_{(h,k)\to(0,0)} = \dfrac{\Delta(h,k)}{hk} = (D_{21} f)(a, b)$. Similarly,

$\lim_{(h,k)\to(0,0)} = \dfrac{\Delta(h,k)}{hk} = (D_{12} f)(a, b)$. Hence $(D_{12} f)(a, b) = (D_{21} f)(a, b)$ $\forall$ $(a, b) \in E$.

This theorem can be extended to the function of n-variables as:

"Suppose f is a real value function defined in an open set $E \subset R^n$, has continuous partial derivatives of order two $(D_{ij} f)(x) = (D_{ij} f)(x)$".

NOTES

4.10.4 Taylor's Theorem

In this section we extend the one dimensional Taylor's theorem to real-valued function f defined on subset of R^n. Here we use some notations:

$L(x, y) = \{(1 - t)x + ty\}: t \in [0, 1]\}$, the line segment joining the point x, y in R^n.

An open ball $B(x_0, r)$ contains line segment $L(x, y)$, whenever $x, y \in B(x_0, r)$.

A real-valued function $f: E \subset R^n \to R$ of n-variables defined on an open set E of R^n is said to be of class ζ if all repeated partial derivatives $D_{ijk\ldots n} f$ are continuous in E and we write $f \in \zeta(E)$.

If x is a point in R^n where second order, third order, ... partial derivatives of f exist at x and if $u = (u_1, u_2, \ldots, u_n)$ is an arbitrary point in R^n, we write

$$(D_u^2 f)(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij} f)(x) u_i u_j,\ (D_u^3 f)(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (D_{ij} f)(x) u_i u_j u_k, \text{ ...and}$$

these sum are analogous to the formula for the directional derivatives of the function

$f: E \subset R^n \to R$, which is differentiable at x, *i.e.*, $(D_u f)(x) = \sum_{j=1}^{n} (D_j f)(x) u_j$.

4.10.5 Theorem (Taylor's)

Let $f: E \subset R^n \to R$ be a function, where E is an open subset R^n. Suppose $f \in \xi^{(m)}$, and $a \in E$ and $x \in R^n$, further suppose that $L(a, a + x) \subset E$. Then there exist $\theta \in (0, 1)$ such that

$$f(a + x) = f(a) + \frac{(D_x f)(a)}{1!} + \frac{(D_x^2 f)(a)}{2!} + \ldots + \frac{(D_x^{m-1} f)(a)}{(m-1)!} + \frac{(D_x^m f)(a + \theta x)}{m!},$$

where $D_x = x.\nabla$.

Proof. Fix an arbitrary $a \in E$ and choose $a \in R^n$ such that $L(a, a + x) \subset E$. Define a function $g: I \to R^n$ by $g(t) = a + tx$, where I is an open interval in R containing [0, 1] as a subset. Then the function $h: I \to R$ defined by $h(t) = f(g(t))$ satisfies the conditions of Taylor's theorem for the function of one variable. Hence

$$h(1) = h(0) + \frac{h'(0)}{1!} + \frac{h''(0)}{2!} + \ldots + \frac{h^{m-1}(0)}{(m-1)!} + \frac{h^m(\theta)}{m!}, \text{ for some } \theta \in (0, 1) \ldots \ldots (i)$$

Since here h is a composite function $h(t) = f(g(t))$, where $g(t) = a + tx$. The k[th] component of p has derivative $p_k(t) = x_k$. Then by the chain rule, we have $h'(t)$ exists in I and is given by

$$h'(t) = \sum_{j=1}^{n} (D_j f)(p(t)) x_j = (D_x f)(p(t)). \text{ Again by chain rule}$$

$$h''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij} f)(p(t)) x_i x_j = (D_x^2 f)(p(t)), \ldots,$$

$$h^m(t) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} (D_{i_1 i_2 \ldots i_m} f)(p(t)) x_{i_1} x_{i_2} \ldots x_{i_m} = (D_x^m f)(p(t)). \text{ Now at } t = 1, t = \theta$$

and $t = 0$, we have $h(1) = f(g(1)) = f(a + x)$, $h(\theta) = f(g(\theta)) = f(a + \theta x)$ and $h(0) = f(g(0)) = f(a)$

$$h'(0) = \sum_{j=1}^{n} (D_j f)(a)) x_j = (D_x f)(a)), \; h''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij} f)(a) x_i x_j = (D_x^2 f)(a) \;...,$$

$$h^{(m-1)}(0) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{m-1}=1}^{n} (D_{i_1 i_2 \ldots i_{m-1}} f)(a) x_{i_1} x_{i_2} \ldots x_{i_{m-1}} = (D^{m-1}{}_x f)(a)$$

$$h^{(m)}(\theta) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} (D_{i_1 i_2 \ldots i_m} f)(a + \theta x) x_{i_1} x_{i_2} \ldots x_{i_m} = (D_x^m f)(a + \theta x). \text{ Then by } (i).$$

we have $f(a + x) = f(a) + \dfrac{(D_x f)(a)}{1!} + \dfrac{(D_x^2 f)(a)}{2!} + \ldots + \dfrac{(D_x^{m-1} f)(a)}{(m-1)!} + \dfrac{(D_x^m f)(a + \theta x)}{m!}$

$$= \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 i_2 \ldots i_k} f) x_{i_1} x_{i_2} \ldots x_{i_k} + r_m(x) = \sum_{k=0}^{m-1} \frac{(D_x^k f)(a)}{k!} + r_m(x).$$

The sum $\displaystyle\sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 i_2 \ldots i_k} f)(a) x_{i_1} x_{i_1} \ldots x_{i_k} = \sum_{k=0}^{m-1} \frac{(D_x^k f)(a)}{k!}$ is called the "***Taylor polynomial of degree m–1***" and $r_m(x)$ is called the "***Taylor remainder after m terms***" which satisfies $\displaystyle\lim_{x \to 0} \frac{r_m(x)}{|x|^{m-1}} = 0.$

Some forms. 1. Replace x by $x - a$ in (i), we have

$$f(x) = f(a) + \frac{(D f)(a)}{1!} + \frac{(D^2 f)(a)}{2!} + \ldots + \frac{(D^{m-1} f)(a)}{(m-1)!} + \frac{(D^m f)(z)}{m!}, \text{ where}$$

$D = (x - a) . \nabla$ and $z \in L(a, x)$. This identity is called ***Taylor's formula with remainder.***

If $\displaystyle\lim_{m \to \infty} \frac{D^m f(z)}{m!} = 0$, then, we have

$$f(x) = f(a) + \frac{(D f)(a)}{1!} + \frac{(D^2 f)(a)}{2!} + \cdots + \frac{(D^m f)(z)}{m!} + \cdots.$$

This is called the ***Taylor's expansion*** or ***Taylor's series*** of f at a.

4.11. STATE AND PROVE TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

Statement. *Let $f(x, y)$ and all its partial derivatives upto order n be continuous in all neighbourhoods of the point (x, y), then*

$$f(x + h, y + k) = f(x, y) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3 \frac{\partial^3 f}{\partial x^3} + 3h^2 k \frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 \frac{\partial^3 f}{\partial x \partial y^2} + k^3 \frac{\partial^3 f}{\partial y^3}\right) + \ldots\ldots$$

Proof. We know that by Taylor's Theorem for a function of a single variable x,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots$$

or
$$f(x+h) = f(x) + h \frac{d}{dx} f(x) + \frac{h^2}{2!} \frac{d^2}{dx^2} f(x) + \frac{h^3}{3!} \frac{d^3}{dx^3} f(x) + \ldots\ldots \quad \ldots(1)$$

Now treating y (and hence $y + k$ as constant), then by Taylor's Theorem for a function of single variable x, (given by equation (1) above), we have

$$f(x+h, y+k) = f(x, y+k) + h \frac{\partial}{\partial x} f(x, y+k)$$
$$+ \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x, y+k) + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} f(x, y+k) + \ldots\ldots \quad \ldots(2)$$

Now keeping x as constant and applying Taylor's Theorem for a function of a single variable y, (given by eqn. (1) above), we have

$$f(x, y+k) = f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y)$$
$$+ \frac{k^3}{3!} \frac{\partial^3}{\partial y^3} f(x, y) + \ldots\ldots \quad \ldots(3)$$

Putting this value of $f(x, y + k)$ from (3) in (2), we have

$$f(x+h, y+k) = f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!} \frac{\partial^3}{\partial y^3} f(x, y) + \ldots\ldots$$
$$+ h \frac{\partial}{\partial x} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \ldots\ldots \right]$$
$$+ \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} \left[f(x, y) + k \frac{\partial}{\partial y} f(x, y) + \ldots\ldots \right] + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} [f(x, y) + \ldots\ldots]$$

Replacing $f(x, y)$ by f (for sake of brevity) in all partial derivatives and opening brackets, we have

$$f(x+h, y+k) = f(x, y) + k \frac{\partial f}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f}{\partial y^3} + \ldots\ldots$$
$$+ h \frac{\partial f}{\partial x} + hk \frac{\partial^2 f}{\partial x \partial y} + \frac{hk^2}{2!} \frac{\partial^2 f}{\partial x \partial y^2} + \ldots\ldots + \frac{h^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{h^2 k}{2!} \frac{\partial^3 f}{\partial x^2 \partial y} + \ldots\ldots$$
$$+ \frac{h^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots\ldots$$

Now grouping terms of partial derivatives of same order in one bracket. We have

$$f(x+h, y+k) = f(x, y) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) + \left(\frac{h^2}{2!} \frac{\partial^2 f}{\partial x^2} + hk \frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2!} \frac{\partial^2 f}{\partial y^2} \right)$$
$$+ \left(\frac{h^3}{3!} \frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!} \frac{\partial^3 f}{\partial x^2 \partial y} + \frac{hk^2}{2!} \frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f}{\partial y^3} \right) + \ldots\ldots$$

or $$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \ldots\ldots \quad \ldots(4)$$

or $$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right)f$$

$$+ \frac{1}{3!}\left(h^3\frac{\partial^3}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3}{\partial x \partial y^2} + k^3\frac{\partial^3}{\partial y^3}\right)f$$

or $$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots \quad \ldots(5)$$

Cor. 1. Expansion of $f(x, y)$ in neighbourhood of the point (a, b).

We know that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$ etc. can be alternatively written as f_x, f_y, f_{xx}, f_{xy}, f_{yy} etc.

∴ Eqn. (4) can be rewritten as

$$f(x+h, y+k) = f(x, y) + (h f_x + k f_y)$$

$$+ \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \frac{1}{3!}(h^3 f_{xxx} + 3h^2k f_{xxy}$$

$$+ 3hk^2 f_{xyy} + k^3 f_{yyy}) + \ldots \quad \ldots(6)$$

Putting $x = a$ and $y = b$ in (6), we have

$$f(a+h, b+k) = f(a, b) + (h f_x(a, b) + k f_y(a, b))$$

$$+ \frac{1}{2!}(h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)) + \frac{1}{3!}(h^3 f_{xxx}(a, b)$$

$$+ 3h^2k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)) + \ldots\ldots \quad \ldots(7)$$

Cor. 2. Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ in the neighbourhood of the point (a, b).

Putting $a + h = x$ and $b + k = y$

i.e., $h = x - a$ and $k = y - b$

in eqn. (7) of Cor. 1 above, we have

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)]$$

$$+ \frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b) f_{xxy}(a, b)$$

$$+ 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \ldots\ldots \quad \ldots(8)$$

Note : Eqn. (8) above is used to expand $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ in the neighbourhood of the point (a, b).

Cor. 3. Expansion of $f(x, y)$ in powers of x and y in the neighbourhood of the point (0, 0).

NOTES

Putting $a = 0$ and $b = 0$ in eqn. (8) of Cor. 2 ; we have

$$f(x, y) = f(0, 0) + [x f_x (0, 0) + y f_y(0, 0)]$$
$$+ \frac{1}{2!} [x^2 f_{xx} (0, 0) + 2xy f_{xy} (0, 0) + y^2 f_{yy} (0, 0)]$$
$$+ \frac{1}{3!} [x^3 f_{xxx} (0, 0) + 3x^2y f_{xxy} (0, 0)$$
$$+ 3xy^2 f_{xyy} (0, 0) + y^3 f_{yyy} (0, 0)] + \ldots\ldots \quad \ldots(9)$$

Note : Equation (9) above is used to expand $f(x, y)$ in powers of x and y in the neighbourhood of the point (0, 0).

ILLUSTRATIVE EXAMPLES

Example 1. *Expand sin(x + h) (y + k) by Taylor's Theorem as far as terms of second degree.*

Sol. Here $f(x + h, y + k) = \sin(x + h) (y + k)$

Putting $h = 0$ and $k = 0$, we have

$$f(x, y) = \sin (xy)$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \sin (xy) = \cos(xy) \frac{\partial}{\partial x} (xy) = \cos (xy) \cdot y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}[y \cos (xy)] = y \frac{\partial}{\partial x} \cos (xy)$$

$$= y [- \sin (xy)] \frac{\partial}{\partial x}(xy) = - y \sin (xy) \cdot y = - y^2 \sin (xy)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (\sin (xy)) = (\cos xy) \frac{\partial}{\partial y} (xy) = (\cos xy)x = x \cos xy$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y} [x \cos (xy)] = x \cdot [- \sin xy]\, x = - x^2 \sin xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x} (x \cos xy) = x \frac{\partial}{\partial x} \cos xy + \cos xy \frac{\partial}{\partial x} x$$
$$= x[- \sin xy]y + \cos xy = - xy \sin xy + \cos xy$$

Putting these values in eqn. (4) of Art. 1 (Taylor's Expansion), namely

$$f(x+h, y+k) = f(x, y) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right) + \ldots\ldots$$

(writing upto **three** terms only as required *i.e.*, upto **second** order derivatives), we have

$$\sin (x + h) (y + k) = \sin xy + (hy \cos xy + kx \cos xy)$$
$$+ \frac{1}{2!} [- h^2 y^2 \sin xy + 2hk (- xy \sin xy + \cos xy) - k^2x^2 \sin xy] + \ldots\ldots$$

or

$$\sin (x + h) (y + k) = \sin xy + (hy + kx) \cos xy$$
$$+ \frac{1}{2!} [2hk \cos xy - (h^2y^2 + 2hkxy + k^2x^2) \sin xy] + \ldots\ldots$$

Example 2. *Expand* $\dfrac{(x+h)(y+k)}{x+h+y+k}$ *in powers of h and k upto and inclusive of the second degree terms.*

Sol. Here $f(x+h, y+k) = \dfrac{(x+h)(y+k)}{x+h+y+k}$

Putting $h = 0$ and $k = 0$, we have

$$f(x, y) = \frac{xy}{x+y}$$

Here $f(x, y)$ is a symmetric function of x and y.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{(x+y)y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x+y)^2 . 0 - y^2 . 2(x+y)}{(x+y)^4} = \frac{-2y^2}{(x+y)^3}$$

Because of symmetric nature of $f(x, y)$, therefore, we have

$$\frac{\partial f}{\partial y} = \frac{x^2}{(x+y)^2}$$

and $$\frac{\partial^2 f}{\partial y^2} = \frac{-2x^2}{(x+y)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{x^2}{(x+y)^2}\right] = \frac{(x+y)^2 . 2x - x^2 . 2(x+y)}{(x+y)^4}$$

$$= \frac{2x(x+y)[x+y-x]}{(x+y)^4} = \frac{2xy(x+y)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$$

Putting these values of partial derivatives in eqn. (4) of Art. 1 (Taylor's Expansion upto second degree terms in h and k), namely

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots\ldots$$

We have

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \left[h\frac{y^2}{(x+y)^2} + k\frac{x^2}{(x+y)^2}\right]$$

$$+ \frac{1}{2}\left[h^2 . \frac{(-2y^2)}{(x+y)^3} + 2hk . \frac{2xy}{(x+y)^3} + k^2 . \frac{-2x^2}{(x+y)^3}\right] + \ldots\ldots$$

$$= \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}h + \frac{x^2}{(x+y)^2}k - \frac{y^2}{(x+y)^3}h^2 + \frac{2xy}{(x+y)^3}hk$$

$$- \frac{x^2}{(x+y)^3}k^2 + \ldots\ldots$$

NOTES

Example 3. *Expand $e^x \sin y$ in powers of x and y as far as terms of third degree.*

Sol. Here $f(x, y) = e^x \sin y$ $\quad \therefore \quad f(0, 0) = e^0 \sin 0 = 0$

Performing partial differentiations,

$$f_x(x, y) = e^x \sin y \qquad \therefore \quad f_x(0, 0) = 0$$

$$f_y(x, y) = e^x \cos y \qquad \therefore \quad f_y(0, 0) = e^0 \cos 0 = 1$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}[f_x(x, y)] = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

$$\therefore \quad f_{xx}(0, 0) = e^0 \sin 0 = 0$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x}[f_y(x, y)] = e^x \cos y \qquad \therefore \quad f_{xy}(0, 0) = 1$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}[f_y(x, y)] = -e^x \sin y \qquad \therefore \quad f_{yy}(0, 0) = 0$$

$$f_{xxx}(x, y) = \frac{\partial}{\partial x}[f_{xx}(x, y)] = e^x \sin y \qquad \therefore \quad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = \frac{\partial}{\partial y}[f_{xx}(x, y)] = e^x \cos y \qquad \therefore \quad f_{xxy}(0, 0) = e^0 \cos 0 = 1$$

$$f_{xyy}(x, y) = \frac{\partial}{\partial x}[f_{yy}(x, y)] = -e^x \sin y \qquad \therefore \quad f_{xyy}(0, 0) = 0$$

$$f_{yyy}(x, y) = \frac{\partial}{\partial y}[f_{yy}(x, y)] = -e^x \cos y \qquad \therefore \quad f_{yyy}(0, 0) = -1$$

We know by Taylor's Theorem [Equation (9) Cor. 3 Art. 1] that

$$e^x \sin y = f(x, y) = f(0, 0) + [x\, f_x(0, 0) + y\, f_y(0, 0)]$$

$$+ \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2y\, f_{xxy}(0, 0)$$

$$+ 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots\ldots$$

Putting values of $f(0, 0)$ and partial derivatives from above,

$$e^x \sin y = 0 + [x.0 + y.1] + \frac{1}{2!}[x^2.0 + 2xy.1 + y^2.0]$$

$$+ \frac{1}{3!}[x^3.0 + 3x^2y.1 + 3xy^2(0) + y^3(-1)] + \ldots\ldots$$

or
$$e^x \sin y = y + xy + \tfrac{1}{2}x^2y - \tfrac{1}{6}y^3 + \ldots\ldots$$

Example 4. *Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of the point (1, 1).*

Sol. Here $f(x, y) = \tan^{-1}\frac{y}{x}$ and point $(a, b) = (1, 1)$ (given)

$$\therefore \qquad f(1, 1) = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$$

Performing partial differentiations,

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1 + \frac{y^2}{x^2}}\left[\frac{x(0) - y(1)}{x^2}\right]$$

$$= \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\therefore \quad f_x(1,1) = \frac{-1}{1+1} = \frac{-1}{2}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)(0)-(-y)\,2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{xx}(1,\ 1) = \frac{2\times1\times1}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$$

Again $\quad f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\tan^{-1}\dfrac{y}{x} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\,\dfrac{\partial}{\partial y}\dfrac{y}{x} = \dfrac{x^2}{x^2+y^2}\cdot\dfrac{1}{x}(1) = \dfrac{x}{x^2+y^2}$

$$\therefore \quad f_y(1,\ 1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\,0 - x\,.\,2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \quad f_{yy}(1,1) = \frac{-2}{(1+1)^2} = \frac{-2}{4} = \frac{-1}{2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2).1 - x\,.\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \quad f_{xy}(1,1) = \frac{1-1}{(1+1)^2} = 0$$

We know by Taylor's Theorem [Eqn. (8) Cor. 2, Art. 1] that

$$\tan^{-1}\frac{y}{x} = f(x, y) = f(a, b) + [(x-a)\,f_x(a, b) + (y-b)\,f_y(a, b)]$$

$$+ \frac{1}{2!}\,[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)\,f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \ldots\ldots$$

Putting $(a, b) = (1, 1)$ *i.e.*, $a = 1$, $b = 1$, and values of $f(a, b)$ and partial derivatives from above, we have

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2\,\frac{1}{2} + 2\,(x-1)(y-1)(0) + (y-1)^2\left(\frac{-1}{2}\right)\right] + \ldots\ldots$$

or $\quad \tan^{-1}\dfrac{y}{x} = \dfrac{\pi}{4} - \dfrac{1}{2}(x-1) + \dfrac{1}{2}(y-1) + \dfrac{1}{4}(x-1)^2 - \dfrac{1}{4}(y-1)^2 + \ldots\ldots$

Note : When the degree of terms upto which $f(x, y)$ is to be expanded is not mentioned, we should expand upto second degree terms.

NOTES

Example 5. *Expand* $x^2y + 3y - 2$ *in powers of* $(x - 1)$ *and* $(y + 2)$ *using Taylor's Theorem.*

Sol. Comparing $(x - 1)$ and $(y + 2) = y - (-2)$ with $x - a$ and $y - b$, we have $a = 1$ and $b = -2$.

Here $\quad f(x, y) = x^2y + 3y - 2$

$\therefore \quad f(a, b) = f(1, -2) = -2 - 6 - 2 = -10$

$\therefore \quad f_x = 2xy$ and $\therefore\ f_x(1, -2) = 2(1)(-2) = -4$

$f_{xx} = 2y$ and $\therefore\ f_{xx}(1, -2) = 2(-2) = -4$

$f_y = x^2 + 3 \quad \therefore\ f_y(1, -2) = 1^2 + 3 = 1 + 3 = 4$

$f_{yy} = 0 \quad \therefore\ f_{yy}(1, -2) = 0$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x^2 + 3) = 2x$$

$\therefore \quad f_{xy}(1, -2) = 2(1) = 2$

$f_{xxx} = 0 \quad \therefore\ f_{xxx}(1, -2) = 0$

$f_{xxy} = 2 \quad \therefore\ f_{xxy}(1, -2) = 2$

$f_{xyy} = 0 \quad \therefore\ f_{xyy}(1, -2) = 0$

$f_{yyy} = 0 \quad \therefore\ f_{yyy}(1, -2) = 0$

All higher order partial derivatives vanish (*i.e.*, become zero).

We know by Taylor's Theorem [Eqn. (8), Cor 2, Art. 1] that

$$x^2y + 3y - 2 = f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)]$$
$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)]$$
$$+ \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2 (y - b) f_{xxy}(a, b)$$
$$+ 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)].$$

Putting $a = 1$, $b = -2$ and values of $f(a, b)$ and partial derivatives from above, we have

$$x^2y + 3y - 2 = f(x, y) = -10 + [(x - 1)(-4) + (y + 2)(4)] + \frac{1}{2}[(x - 1)^2(-4)$$
$$+ 2(x - 1)(y + 2)(2) + (y + 2)^2(0)] + \frac{1}{6}[(x - 1)^3(0) + 3(x - 1)^2(y + 2)\,2$$
$$+ 3(x - 1)(y + 2)^2(0) + (y + 2)^3(0)]$$
$$= -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2).$$

4.12. TAYLOR'S THEOREM WITH REMAINDER AFTER *n* TERMS

$$f(x, y) = f(a, b) + \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}\right] f(a, b) + \frac{1}{2!}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}\right]^2 f(a, b)$$
$$+ \ldots\ldots + \frac{1}{(n - 1)!}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}\right]^{n-1} f(a, b) + \mathrm{R}n$$

where $R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f[a+(x-a)\theta,\ b+(y-b)\theta]$ $\quad (0<\theta<1)$

or $\quad f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)]$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] + \ldots\ldots + Rn \qquad \ldots(1)$$

where $R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f[a+(x-a)\Theta,\ b+(y-b)\Theta]$ $\quad (0<\Theta<1)$

...(2)

Cor. Putting $a = 0$ and $b = 0$ in equations (1) and (2), we have

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0)$$

$$+ 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots\ldots$$

$$+ \ldots\ldots + Rn \qquad \ldots(3)$$

where $Rn = \frac{1}{n!}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]^n f(x\theta, y\theta)$ ...(4) where $0<\theta<1$

Example 6. *Prove that* $\log(x-y) = x-(y+1)-\frac{x^2}{2}+x(y+1)-\frac{(y+1)^2}{2}$

$$+\frac{(x-y+1)^3}{3[\theta x+1-\theta(y+1)]^3}, \quad 0, <\theta<1.$$

Sol. We can observe from RHS that we are to expand log $(x-y)$ in powers of x and $(y+1)$ *i.e.* in powers of $x-0$ and $y-(-1)$ *i.e.* about the point $(0, -1)$.

Also it is clear from last term of RHS that we are to find remainder R_3 after second degree terms.

Let $\quad f(x, y) = \log(x-y)$

$\therefore \quad f(0, -1) = \log(0+1) = \log 1 = 0$

$f_x = \frac{1}{x-y}$ $\qquad \therefore \ f_x(0, -1) = \frac{1}{0+1} = 1$

$f_{xx} = \frac{-1}{(x-y)^2}$ $\qquad \therefore \ f_{xx}(0, -1) = \frac{-1}{(0+1)^2} = -1$

$f_y = \frac{-1}{x-y}$ $\qquad \therefore \ f_y(0, -1) = \frac{-1}{0+1} = -1$

$f_{yy} = \frac{-1}{(x-y)^2}$ $\qquad \therefore \ f_{yy}(0, -1) = \frac{-1}{(0+1)^2} = -1$

$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{1}{(x-y)^2}$ $\qquad \therefore \ f_{xy}(0, -1) = \frac{1}{(0+1)^2} = 1$

$f_{xxx} = \frac{2}{(x-y)^3}$

Again $\quad f_{xxy} = \frac{\partial}{\partial x}f_{xy} = \frac{\partial}{\partial x}\left(\frac{1}{(x-y)^2}\right) = \frac{-2}{(x-y)^3}$

$$f_{xyy} = \frac{\partial}{\partial x} f_{yy} = \frac{\partial}{\partial x}\left[\frac{-1}{(x-y)^2}\right] = \frac{2}{(x-y)^3}$$

NOTES

$$f_{yyy} = \frac{\partial}{\partial y} f_{yy} = \frac{\partial}{\partial y}\left[\frac{-1}{(x-y)^2}\right] = \frac{-2}{(x-y)^3}$$

Putting $n = 3$ in Eqn. (2) of Art. 2, we have

$$R_3 = \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f[a + (x-a)\theta,\ b + (y-b)\theta]$$

Putting $a = 0$, $b = -1$, we have

$$R_3 = \frac{1}{3!}\left[x\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right]^3 f[x\theta, -1 + (y+1)\theta]$$

$$= \frac{1}{3!}\left[x^3\frac{\partial^3}{\partial x^3} + 3x^2(y+1)\frac{\partial^3}{\partial x^2\partial y} + 3x(y+1)^2 \cdot \frac{\partial^3}{\partial x\partial y^2} + (y+1)^3\frac{\partial^3}{\partial y^3}\right] f[x\theta, -1 + (y+1)\theta]$$

$$= \frac{1}{3!}\,[x^3 f_{xxx}(x\theta, -1 + (y+1)\theta) + 3x^2(y+1) f_{xxy}(x\theta, -1 + (y+1)\theta)$$
$$+ 3x(y+1)^2 f_{xyy}(x\theta, -1 + (y+1)\theta) + (y+1)^3 f_{yyy}(x\theta, -1 + (y+1)\theta)]$$

Changing x to $x\theta$ and y to $-1 + (y+1)\theta$ in $f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}$ and putting the values in R_3 above, we have

$$R_3 = \frac{1}{3!}\left[x^3 \cdot \frac{2}{[\theta x + 1 - (y+1)\theta]^3} + 3x^2(y+1)\frac{(-2)}{[\theta x + 1 - (y+1)\theta]^3} + 3x(y+1)^2 \cdot \frac{2}{[\theta x + 1 - (y+1)\theta]^3} + (y+1)^3 \cdot \frac{(-2)}{[\theta x + 1 - (y+1)\theta]^3}\right]$$

or $$R_3 = \frac{2}{6}\left[\frac{x^3 - 3x^2(y+1) + 3x(y+1)^2 - (y+1)^3}{[\theta x + 1 - (y+1)\theta]^3}\right]$$

or $$R_3 = \frac{1}{3}\frac{[x-(y+1)]^3}{[\theta x + 1 - (y+1)\theta]^3}$$

We know by equations (1) and (2) of Art. 2 that Taylor's Theorem with remainder after $n(= 3)$ terms is

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b) f_y(a, b)]$$
$$+ \frac{1}{2!}\,[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + R_3$$

Putting $(a, b) = (0, -1)$ and values of derivatives and R_3 obtained above

$$\log(x-y) = 0 + [x(1) + (y+1)(-1)] + \frac{1}{2}\,[(x-0)^2(-1) + 2x(y+1)(1) + (y+1)^2(-1)] + \frac{1}{3}\frac{[x-(y+1)]^3}{[\theta x + 1 - (y+1)\theta]^3}$$

or $$\log(x-y) = x - (y+1) - \frac{x^2}{2} + x(y+1) - \frac{(y+1)^2}{2} + \frac{(x-(y+1))^3}{3[\theta x + 1 - (y+1)\theta]^3}.$$

NOTES

TEST YOUR KNOWLEDGE

1. Expand $\cos(x + h)(y + k)$ by Taylor's Theorem as far as terms of second degree.
2. Expand $e^x \cos y$ in powers of x and y as far as the terms of third degree.
3. Show that $e^y \log(1 + x) = x + xy - \frac{x^2}{2}$ approximately.

   [**Hint.** Find the expansion at (0, 0).]
4. (*a*) Expand $e^{ax} \sin by$ in powers of x and y as far the terms of third degree by Taylor's Theorem.

   (*b*) Prove that first four terms in the expansion of $e^{ax} \cos by$ are

   $$1 + ax + \frac{1}{2!}(a^2x^2 - b^2y^2) + \frac{1}{3!}(a^3x^3 - 3ab^2xy^2) + \ldots\ldots$$
5. Find the first six terms of the expansion of the function $e^x \log(1 + y)$ in a Taylor's series in the neighbourhood of the point (0, 0).
6. Expand e^{xy} at (1, 1).
7. (*i*) Expand $x^3 + 2x^2y + y^3$ about the point (1, 1) upto second degree only.

   (*ii*) Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ upto second degree terms.
8. Expand $x^4 + x^2y^2 - y^4$ about the point (1, 1) upto the terms of the second degree.
9. Prove that $y^x = 1 + 2(y - 1) + (x - 2)(y - 1) + (y - 1)^2 + \ldots\ldots$
10. Expand $e^x \cos y$ at $\left(1, \frac{\pi}{4}\right)$ by Taylor's Theorem.
11. Expand $(1 + x + y^2)^{1/2}$ at (1, 0).
12. Prove that $\sin x \sin y = xy - \frac{1}{6}[(x^3 + 3xy^2)\cos\theta x \sin\theta y + (y^3 + 3x^2y)\sin\theta x \cos\theta y]$, $(0 < \theta < 1]$.
13. Prove that for $0 < \theta < 1$,

    $$e^{ax} \sin by = by + abxy + \frac{1}{6}[(a^3x^3 - 3ab^2xy^2)\sin(b\theta y) + (3a^2bx^2y - b^3y^3)\cos(b\theta y)]\, e^{a\theta x}.$$

Answers

1. $\cos xy - (hy + kx)\sin xy - \frac{1}{2!}[2hk \sin xy + (hy + kx)^2 \cos xy]$
2. $1 + x + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2)$
4. (*a*) $by + ab\,xy + \frac{1}{3!}(3a^2 b\, x^2y - b^3y^3) + \ldots\ldots$
5. $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 \ldots\ldots$
6. $e[1 + (x - 1) + (y - 1) + \frac{1}{2!}((x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2) + \ldots\ldots]$
7. (*i*) $4 + [7(x - 1) + 5(y - 1) + \frac{1}{2}[10(x - 1)^2 + 6(y - 1)^2 + 8(x - 1)(y - 1)] + \ldots\ldots$

   (*ii*) $-10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + z)$.
8. $1 + [6(x - 1) - 2(y - 1)] + \frac{1}{2}[14(x - 1)^2 - 10(y - 1)^2 + 8(x - 1)(y - 1)] + \ldots\ldots$

NOTES

10. $\frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2+......\right]$

11. $\sqrt{2}\left[1+\frac{(x-1)}{4}-\frac{(x-1)^2}{32}+\frac{y^2}{4}+......\right]$.

SUMMARY

- (*a*) A nonempty set $X \subset R^n$ is said to be a vector space if $x + y \in X$ and $cx \in X$ for all $x \in X$, $y \in X$, and for all scalars c, where $x = (a_1, a_2, \ldots, a_n)$, $y = (b_1, b_2, \ldots, b_n) \in R^n$ and $a_1, a_2, \ldots, a_n$; $b_1, b_2 \ldots, b_n$ are scalars etc.

  (*b*) If $x_1, \ldots, x_k \in R^n$ and $c_1, \ldots, c_k$ are scalars, then the vector $c_1x_1 + ... + c_k x_k$ is called a linear combination of $x_1, \ldots, x_k$.

  (*c*) The set of all linear combinations of elements of a set S called a linear span of S and denoted by L(S). Note that every L(S) is a vector space.

  (*d*) A set $S = \{x_1, \ldots, x_k\} \subset R^n$ is said to be linearly independent if the relation $c_1x_1 + \ldots + c_kx_k = 0$ implies that $c_1 = \ldots = c_k = 0$. Otherwise S is said to be linearly dependent.

  (*e*) Let B is a linearly independent subset of a vector space X such that L(B) = X, then B is called a basis of X.

- If a vector space X is spanned by a set of r vectors, then $\dim X \le r$, where r is a positive integer and $\dim R^n = n$.

- Suppose X is a vector space and $\dim X = n$.

  (*a*) A set E of n vectors in X spans X if and only if E is linearly independent.

  (*b*) X has a basis and every basis consists of n vectors.

  (*c*) If $1 \le r \le n$ and $\{y_1, y_2 \ldots, y_r\}$ is an independent set in X, then X has basis containing $\{y_1, y_2 \ldots, y_r\}$.

- A mapping A of a vector space X into a vector space Y is said to be a linear transformation if $A(x_1 + x_2) = Ax_1 + Ax_2$ and $A(cx) = cA(x)$ for all $x, x_1, x_2 \in X$ and all scalars c. Linear transformation of X into χ are often called linear operators on X. A linear operator on a finite dimensional vector space is one-one or onto then it is invertible. A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X [i.e.. A(X) = X] or A is onto.

- Let X and Y be vector spaces then L(X, Y) denote the set of all linear transformations of X into Y. The set of all linear transformations of X into itself is denoted by L(X, X) or L(X).

- For $A \in L(R^n, R^m)$, the norm $\|A\|$ of A is defined by $\|A\| = \sup \{|Ax| : x \in R^n, |x| \le 1\}$. If $A \in L(R^n, R^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of R^n into R^m. If $A, B \in L(R^n, R^m)$, and c is a scalar, then $\|A + B\| \le \|A\| + \|B\|$, $\|cA\| = |c| . \|A\|$. If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$ then $\|BA\| \le \|B\| . \|A\|$.

- Suppose E is an open set in R^n, f maps E into R^m and $x \in E$. If there exists a linear transformation A of R^n into R^m such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0 \ldots \ldots \qquad \ldots(ii)$$

  then f is said to be differentiable at x, and we write $f'(x) = A$. If f is differentiate at every $x \in E$, we say that f is differentiable in E.

- **Chain rule.** Suppose E is an open set in R^n, f maps E into R^m. f is differentiate at $x_0 \in E$, g maps an open set containing $f(E)$ into R^k, and g is differentiable at $f(x_0)$. Then the mapping F of E into R^k, defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))\, f'(x_0)$.
- Let f be a function from an open set $E \subset R^n$ and for $x \in E$, $1 \le i \le m$, $1 \le j \le n$, we define

$$D_j f_i(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t}$$

provided the limit exists. Write $f_i(x) = f_i(x_1, \ldots, x_n)$, we see that $D_j f_i$ is the derivative of f_i with respect to x_j, keeping the other variables $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ fixed. The notation $\partial f_i/\partial x_j$ is often used in place of $D_j f_i$ it is called a ***partial derivative*** of f_i with respect to x_j.

- **Continuously differentiable function:** A differentiable mapping f of an open set $E \subset R^n$ into R^m is said to be continuously differentiable in E if f' is a continuous mapping of E into $L(R^n, R^m)$.
- **Taylor's Theorem.** Let $f : E \subset R^n \to R$ be a function, where E is an open subset R^n. Suppose $f \in \xi^{(m)}$, and $a \in E$ and $x \in R^n$, further suppose that $L(a, a + x) \subset E$. Then there exist $\theta \in (0, 1)$ such that

$$f(a + x) = f(a) + \frac{(D_x f)(a)}{1!} + \frac{(D_x^2 f)(a)}{2!} + \ldots + \frac{(D_x^{m-1} f)(a)}{(m-1)!} + \frac{(D_x^m f)(a + \theta x)}{m!}.$$

❑❑❑

UNIT

5

JACOBIANS

STRUCTURE

5.1. INTRODUCTION

Jacobian is a functional determinant, useful in transformation of variables from cartesian to polar, cylindrical and spherical polar co-ordinates in multiple integrals. Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851) who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

5.2. DEFINITIONS

NOTES

(*i*) If u and v are the functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the **Jacobian** or functional determinant of u and v with respect to x and y. It is written as $\frac{\partial(u, v)}{\partial(x, y)}$ or $J(u, v)$.

(*ii*) If u, v and w are the functions of three independent variables x, y and z, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is called the Jacobian of u, v and w with respect to x, y and z. It is written as

$\frac{\partial(u, v, w)}{\partial(x, y, z)}$ or $J(u, v, w)$.

(*iii*) Similarly if $u_1, u_2, ..., u_n$ are the functions of independent variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, ..., u_n$ with respect to variables $x_1, x_2, ..., x_n$. It is written as

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} \text{ or } J(u_1, u_2, ..., u_n).$$

5.3. PROPERTIES OF JACOBIANS (CHAIN RULES)

I. If u, v are functions of r, s where r, s are functions of x, y then

$$\boxed{\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}}.$$

NOTES

Proof. Since u, v are composite functions of x, y

$$\left.\begin{aligned} \therefore \quad \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} = v_r r_x + v_s s_x \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \end{aligned}\right\} \quad ...(1)$$

Now $$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

$$= \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{[Using (1)]}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}.$$

Note. If u_1, u_2, u_3 are functions of y_1, y_2, y_3 and y_1, y_2, y_3 are functions of x_1, x_2, x_3 then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)} \cdot \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$$

II. If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then

$$J_1 J_2 = 1 \text{ i.e., } \boxed{\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1.}$$

Proof. Let $u = u(x, y)$ and $v = v(x, y)$ so that u and v are functions of x, y.

Differentiating partially w.r.t. u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} = u_x x_u + u_y y_u \\ 0 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\ 0 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} = v_x x_u + v_y y_u \\ 1 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} = v_x x_v + v_y y_v \end{aligned}\right\} \quad ...(1)$$

Now $$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \quad \text{[Using (1)]}$$

Note. It can be extended to three variables as $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 1.$

NOTES

5.4. THEOREMS ON JACOBIANS

Theorem 1. *If the relations connecting u_i's and x_i's are of the form,*

$$u_1 = f_1(x_1)$$
$$u_2 = f_2(x_1, x_2)$$
$$\vdots$$
$$\vdots$$
$$u_n = f_n(x_1, x_2, \ldots, x_n)$$

then $$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

Proof. We know that

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} \qquad \ldots(1)$$

u_1 is function of x_1 only therefore

$$\frac{\partial u_1}{\partial x_2} = 0, \frac{\partial u_1}{\partial x_3} = 0, \ldots\ldots, \frac{\partial u_1}{\partial x_n} = 0$$

u_2 is function of x_1 and x_2 hence only $\frac{\partial u_2}{\partial x_1}$ and $\frac{\partial u_2}{\partial x_2}$ will exist

$$\text{the rest } \frac{\partial u_2}{\partial x_3} = 0, \frac{\partial u_2}{\partial x_4} = 0, \ldots, \frac{\partial u_2}{\partial x_n} = 0$$

u_3 is function of x_1, x_2 and x_3, therefore $\frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3}$ will exist and the rest will be zero.

u_n is function of x_1, x_2, ..., x_n therefore all $\frac{\partial u_n}{\partial x_1}, \frac{\partial u_n}{\partial x_2}, \ldots, \frac{\partial u_n}{\partial x_n}$ will exist.

Putting these values in (1), we get

$$\frac{\partial(u_1, u_2, u_3, \ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Expanding the determinant in terms of first row, we get

$$\boxed{\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.}$$

NOTES

Theorem 2. *If functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.*

Proof. It is given that u, v and w are not independent variables, then there will be a relation $F(u, v, w) = 0$, which will connect these independent variables.

Differentiating this relation with respect to x, y and z, we get

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x} = 0 \quad \ldots(1)$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial y} = 0 \quad \ldots(2)$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial z} = 0 \quad \ldots(3)$$

Eliminating $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ from (1), (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

or

$$\boxed{\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.}$$

5.5. JACOBIAN OF IMPLICIT FUNCTIONS

If u_1, u_2 and u_3 are the implicit functions of x_1, x_2, x_3 i.e.,

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

then,

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \left[\frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}\right].$$

Proof. Differentiating F_1, F_2 and F_3 with respect to x_1, x_2 and x_3, we get

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0 \Rightarrow \sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial F_1}{\partial x_1}$$

$$\frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_3} = 0 \Rightarrow \sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial F_1}{\partial x_2}$$

$$\frac{\partial F_1}{\partial x_3}+\frac{\partial F_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_3}+\frac{\partial F_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_3}+\frac{\partial F_1}{\partial u_3}\cdot\frac{\partial u_3}{\partial x_3}=0 \quad\Rightarrow\quad \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_1}{\partial x_3}.$$

NOTES

Similarly, $\displaystyle\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1}=-\frac{\partial F_2}{\partial x_1},\ \sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2}=-\frac{\partial F_2}{\partial x_2},\ \sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_2}{\partial x_3}$

Also, $\displaystyle\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1}=-\frac{\partial F_3}{\partial x_1},\ \sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2}=-\frac{\partial F_3}{\partial x_2}$ and $\displaystyle\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_3}{\partial x_3}$

Now, $$\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}\times\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}\frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3}\\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3}\\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3}\end{vmatrix}\times\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{vmatrix}$$

$$=\begin{vmatrix}\Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\\ \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\\ \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\end{vmatrix}$$

$$=\begin{vmatrix}-\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & -\frac{\partial F_1}{\partial x_3}\\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_3}\\ -\frac{\partial F_3}{\partial x_1} & -\frac{\partial F_3}{\partial x_2} & -\frac{\partial F_3}{\partial x_3}\end{vmatrix}\quad \left|\ \text{Putting values of each summation}\right.$$

$$=(-1)^3\begin{vmatrix}\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3}\\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3}\\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3}\end{vmatrix}=(-1)^3\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}$$

Hence, $$\boxed{\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}=(-1)^3\left[\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}\Big/\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}\right]}$$

5.6. FUNCTIONAL RELATIONSHIP

"Let $u_1, u_2, u_3, \ldots, u_n$ be functions of $x_1, x_2, x_3, \ldots, x_n$. Then the necessary condition for the existence of a relation of the form $F(u_1, u_2, \ldots, u_n) = 0$ is that the Jacobian $\dfrac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}$ should vanish identically."

NOTES

Proof. Let there exist a relation $F(u_1, u_2, ..., u_n) = 0$. ...(1)

We are to show that $\dfrac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = 0$.

Differentiating (1) partially with respect to $x_1, x_2, ..., x_n$, we get

with respect to x_1,

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1}+....+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_1}=0$$

with respect to x_2,

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}+....+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_2}=0$$

....

....

with respect to x_n,

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_n}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_n}+....+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_n}=0$$

Since, we cannot have at the same time

$$\frac{\partial F}{\partial u_1}=\frac{\partial F}{\partial u_2}=\frac{\partial F}{\partial u_3}=....=\frac{\partial F}{\partial u_n}=0$$

otherwise relation (1) will reduce to a trivial identity, hence eliminating $\dfrac{\partial F}{\partial u_1}, \dfrac{\partial F}{\partial u_2}, ..., \dfrac{\partial F}{\partial u_n}$ from these above equations, we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \frac{\partial u_3}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

or $$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = 0.$$

5.7 INVERSE FUNCTION THEOREM

The inverse function theorem states, roughly speaking,that a continuously differentiable mapping f is invertible in neighbourhood of any point x at which the linear transformation $f'(x)$ is invertible:

5.7.1 Theorem (The Inverse Function)

Suppose f is a ζ'-mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then, *(a) there exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one-to-one on U, and $f(U) = V(b)$ If g is the inverse of f, defined in V by $g(f(x)) = x (x \in U)$, then $g \in \zeta'(V)$. In component form the conclusion of this theorem can be stated as:*

Suppose $y = f(x)$ is represented in component form as $y_i = f_i(x_1, x_2, ..., x_n)$ $(1 \le i \le n)$. Then this system of equations can be solved for $x_1, x_2, ..., x_n$ in terms of $y_1, y_2, ..., y_n$ if we restrict x and y in sufficiently small neighbourhood of a and b respectively. Moreover, the solutions are unique and continuously differentiable.

Proof. (*a*) Put $f'(a) = A$, and choose λ so that $2\lambda \| A^{-1} \| = 1$ ...(*i*)

Since $f'(a)$ is a ζ'-mapping *i.e.*, f is continuous on E. In particular f' is continuous at a in E. Then there is an open ball $U \subset E$, with center at a such that

$\| f'(x) - f'(a) \| < \lambda \quad (x \in U)\ \lambda$ *i.e.*, $\| f'(x) - A \| < \lambda \quad (x \in U)$ ...(*ii*)

We associate to each $y \in R^n$ a function ϕ, defined by

$$\phi(x) = x + A^{-1}(y - f(x)) \qquad (x \in U) \qquad ...(iii)$$

Note that $f(x) = y$ then $\phi(x) = x$ and if $\phi(x) = x$, then $x = x + A^{-1}(y - f(x))$

Or, $0 = A^{-1}(y - f(x))$ which implies that $y - f(x) = 0$ or $y = f(x)$. Hence $y = f(x)$ iff x is a fixed point of ϕ. Now since $\phi'(x) = I - A^{-1}(f'(x)) = A^{-1}A - A^{-1}(f'(x)) = A^{-1}(A - f'(x))$. Then (*i*) and (*ii*), imply that

$\| \phi'(x) \| = \| A^{-1}(A - f'(x)) \| \le \| A^{-1} \| A - f'(x) \| < (1/2\lambda) \cdot \lambda = 1/2 \ (x \in U)$ ...(*iv*)

Hence, we have $| \phi(x_1) - \phi(x_2) | \le 1/2 \ | x_1 - x_2 |$ for $x_1, x_2 \in U$ ...(*v*)

It follows that ϕ is a contraction mapping, hence by banach contraction principle ϕ has unique fixed point in U, so that $f(x) = y$ for at most one $x \in U$. Thus f is 1-1 in U.

Next, let $V = f(U)$, and choose an arbitrary $y_0 \in V$, then $y_0 = f(x_0)$ for some $x_0 \in U$. Let $B_r(x_0)$ be an open ball with center at x_0 and radius $r > 0$, so small that its closure $\overline{B_r(x_0)}$ lies in U, it is possible since U is an open set. Now our aim is to prove that V is open, to prove it we shall prove that $y \in V$ whenever $| y - y_0 | < \lambda r$. Fix y such that $| y - y_0 | < \lambda r$, now by definition of ϕ, we see that

$| \phi(x_0) - x_0 | = | A^{-1}(y - f(x_0)) = | A^{-1}(y - y_0) | \le \| A^{-1} \| \, | y - y_0 | < (1/2\lambda)\lambda r = r/2$.

If $x \in \overline{B_r(x_0)}$, then from (5), we have

$$\begin{aligned} | \phi(x) - x_0 | &= | \phi(x) - \phi(x_0) + \phi(x_0) - x_0 | \\ &\le | \phi(x) - \phi(x_0) | + | \phi(x_0) - x_0 | \\ &< (1/2) \, | x - x_0 | + r/2 < (r/2) + (r/2) = r. \end{aligned}$$

Hence $\phi(x) \in B_r(x_0)$. Since as per our choice $\overline{B_r(x_0)}$ lies in U and so (*v*) holds if $x_1, x_2 \in \overline{B_r(x_0)}$. Thus ϕ is a contraction of $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$. Since $\overline{B_r(x_0)}$ is closed subset of R^n so that $\overline{B_r(x_0)}$ is complete. Therefore, by Banach contraction principle ϕ has a fixed point $x \in \overline{B_r(x_0)}$ and for this x, $f(x) = y$. Thus $y \in f\left(\overline{B_r(x_0)}\right) \subset f(U) = V$. This proves part (*a*) of the theorem.

(*b*) Choose $y, y + k \in V$ and since $f(U) = V$, there exist $x, x + h \in U$, such that $y = f(x)$, $y + k = f(x + h)$. Now by definition of ϕ as in (*iii*), we have

$$\begin{aligned} \phi(x + h) - \phi(x) &= [x + h + A^{-1}(y - f(x + h))] - [x + A^{-1}(y - f(x))] \\ &= h + A^{-1}[f(x) - f(x + h)] = h + A^{-1}[y - y - k] = h - A^{-1}k. \end{aligned}$$

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Then by (5), we have $|h - A^{-1}k| \le |\phi(x+h) - \phi(x)| \le (1/2)|x+h-x| = (1/2)|h|$, which implies that $|A^{-1}k| = |h - (h - A^{-1}k)| \ge |h| - |h - A^{-1}k| \ge |h| - (1/2)h| = 1/2|h|$, therefore

$$|h| \le 2|A^{-1}k| \le 2\|A^{-1}\|\,|k| = \lambda^{-1}|k| \qquad \ldots(vi)$$

Now by (*i*), (*ii*) and $f'(x)$ has an inverse, say T. Since,

$$\begin{aligned} g(y+k) - g(y) - Tk &= g(f(x+h) - g(f(x)) - Tk = x + h - x - Tk = h - Tk \\ &= Tf'(x)h - T(y + k - y) = Tf'(x)h - T[f(x+h) - f(x)] \\ &= -T[f(x+h) - f(x) - f'(x)h]. \end{aligned}$$

Therefore by (*vi*), we have

$$\begin{aligned} \frac{|g(y+k) - g(y) - Tk|}{|k|} &= \frac{|T[f(x+h) - f(x) - f'(x)h|}{|k|} \\ &\le \frac{\|T\|}{\lambda} \cdot \frac{|f(x+h) - f(x) - f'(x)|}{|h|} \to 0 \end{aligned}$$

as $k \to 0$ (since by (*vi*) $h \to 0$ as $k \to 0$).

Hence $g'(y) = T = \{f'(x)\}^{-1} = \{f'(g(y))\}^{-1}$ $(y \in V)$ ...(*vii*)

Finally, note that g is a continuous mapping of V onto U (since g is differentiable), that f' is a continuous mapping of U into the set Ω of all invertible elements of $L(R^n)$, and that inversion is a continuous mapping of Ω onto Ω. If we combine these facts with (*i*), we see that $g \in \zeta'(V)$. This completes the proof.

The following is an immediate consequence of part (*a*) of the inverse function theorem.

5.7.2 Theorem

If f is a ζ'-mapping of an open set $E \subset R^n$ into R^n and if $f'(x)$ is invertible for every $x \in E$, then $f(W)$ is an open subset of R^n for every open set $W \subset E$. In other words, f is an open mapping of E into R^n.

5.8. THE IMPLICIT FUNCTION THEOREM

If f is a continuously differentiable real function in the plane, then the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighbourhood of any point (a, b) at which $f(a, b) = 0$ and $\partial f / \partial y \ne 0$. Likewise, one can solve for x in terms of y near (a, b) if $\partial f / \partial x \ne 0$ at (a, b). For example consider $f(x, y) = x^2 + y^2 - 1$ and a point (0, 1) so that

(*i*) $f(0, 1) = 0$ and (*ii*) $\partial f / \partial y \ne 0$ at (0, 1). Here the possible solutions are $y = +\sqrt{(1-x^2)}$ is the implicit function in a neighbourhood of (0, 1), where $|x| < 1$, $y > 0$ and $y = -\sqrt{(1-x^2)}$ is the implicit function in a neighbourhood of (0, 1) where $|x| < 1$, $y < 0$. Our next result is the very informal statement is the simplest case (the case $m = n = 1$ of ***"implicit function theorem"***. Its proof makes strong use of the fact that continuously differentiable transformation behave locally very much like their derivatives. Accordingly, we first prove the linear version of implicit function theorem as:

Notation: In our further discussion we shall make the use of the following notations:

If $x = (x_1, ..., x_n) \in R^n$ and $y = (y_1, ..., y_m) \in R^m$, then we write (x, y) for the point (or vector) as $(x, y) = (x_1, ..., x_n, y_1, ..., y_m) \in R^{n+m}$. Then an arbitrary $A \in L(R^{n+m}, \in R^n)$ can be split into two linear transformations A_x and A_y defined by $A_x h = A(h, 0)$ and $A_y k = A(0, k)$ for all $h \in R^n$, $k \in R^m$. Then clearly $A_x \in L(R^n)$ and $A_y \in L(R^m, R^n)$ and $A(h, k) = A_x h + A_y k$.

5.8.1 Theorem (Implicit function theorem, linear version)

If $A \in L(R^{n+m}, R^n)$ and if A_x is invertible then there corresponds to every $k \in R^m$ such that $A(h, k) = 0$, and h can be computed from k by the formula $h = -(A_x)^{-1} A_y k$.

Proof. Since by $A(h, k) = A_x h + A_y k$, then $A(h, k) = 0 \Leftrightarrow A_x h + A_y k = 0$. Now since A_x is invertible then, we get $h = -(A_x)^{-1} A_y k$.

The conclusion of above Theorem can be restated as: "The equation $A(h, k) = 0$ can be solved (uniquely) for h if k is given and that the solution h is a linear function of k."

5.8.2 Theorem (Implicit function theorem, non-linear version)

Let f be a ζ'-mapping of an open set $E \subset R^{n+m}$ into R^n, such that $f(a, b) = 0$ for some point $(a, b) \in E$. Put $A = f'(a, b)$ and assume that A_x is invertible, then there exist open sets $U \subset R^{n+m}$ and $W \subset R^m$, with $(a, b) \in U$ and $b \in W$, having the following property:

To every $y \in W$ corresponds a unique x such that $(x, y) \in U$ and $f(x, y) = 0$ ...(*i*)

If this x is defined to be $g(y)$, then g is a ζ'-mapping of W into R^n, $g(b) = a$,

$$f(g(y), y) = 0 \qquad (y \in W) \qquad ...(ii)$$

and

$$g'(b) = -(A_x)^{-1} A_y \qquad ...(iii)$$

The function g is "implicitly" defined by (*ii*), hence the name of the theorem.

Note: The equation $f(x, y) = 0$ can be written as a system of n equations in $n + m$ variables as:

$$\left.\begin{array}{l} f_1(x_1, ..., x_n, y_1, ..., y_m) = 0 \\ f_2(x_1, ..., x_n, y_1, ..., y_m) = 0 \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(x_1, ..., x_n, y_1, ..., y_m) = 0. \end{array}\right\} \qquad ...(iv)$$

The assumption that A_x is invertible means that the n by n matrix.

$$\begin{pmatrix} D_1 f_1 & \cdots & \cdots & D_n f_n \\ \cdots & \cdots & \cdots & \cdots \\ D_1 f_n & \cdots & \cdots & D_n f_n \end{pmatrix}$$

evaluated at (a, b) defines an invertible linear operator in R^n : in other words, its column vectors should be independent, or equivalently, its determinant should be $\neq 0$. Also if (*iv*) holds when $x = a$ and $y = b$, then the conclusion of the theorem is that (*iv*) can be solved for $x_1, ..., x_n$ in terms of $y_1, ..., y_m$, for every y near b, and that these solutions are continuously differentiable functions of y.

Proof. Define F by $F(x, y) = (f(x, y), y) \; ((x, y) \in E)$ ...(*iv*)

Since f is a ζ'-mapping of $E \subset R^{n+m}$ into R^n and F is a mapping of $E \subset R^{n+m}$ into R^{n+m}

Hence by (*v*) F is a ζ'-mapping of E into R^{n+m}.

Since $f(a, b) = 0$ for some $(a, b) \in E$ and $A = f'(a, b)$, we then have

$$f(a+h, b+k) = f(a+h, b+k) - f(a, b) = A(h, k) + r(h, k),$$

where r is the remainder that occurs in the definition of $f'(a, b)$. Now since by definition of F,

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$$\begin{aligned} F(a+h, b+k) - F(a, b) &= (f(a+h, b+k), b+k) - (f(a, b), b), \\ &= (f(a+h, b+k), b+k) - (0, b) = (f(a+h, b+k), k) \\ &= (A(h, k) + r(h, k), k+0) \\ &= (A(h, k), k) + (r(h, k), 0) \end{aligned}$$

It follows that $F'(a, b)$ is the linear operator on R^{n+m} that maps (h, k) to $(A(h, k), k)$, *i.e.*, $F'(a, b) \in L(R^{n+m})$ in the sense that $F'(a, b) : (h, k) \to (A(h, k), k)$ for each $(h, k) \in R^{n+m}$. If the image vector $(A(h, k), k) = 0 \in R^{n+m}$, then $A(h, k) = 0 \in R^n$, and $k = 0 \in R^m$. Hence $A(h, 0) = 0$, *i.e.*, $A_x h = 0$ and the preceding theorem implies that $h = 0$. Thus by properties of linear transformation $F'(a, b)$ one-one and hence invertible. Therefore by inverse function theorem there exists open sets U and V in R^{n+m} with $(a, b) \in U$, $(0, b) \in V$, such that F is one-one mapping of U onto V. If we suppose W be the set of all $y \in R^m$, such that $(0, y) \in V$, *i.e.*, $W = \{y \in R^m: (0, y) \in V\}$. We see that $b \in W$, as $(0, b) \in V$, which is an open set, it follows that W is also open. If $y \in W$, then $(0, y) = F(x, y)$ for some $(x, y) \in U$. Then by (v), we have $f(x, y) = 0$ for this x. We now prove that for the $y \in W$ there exists a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. Suppose on the contrary that it is possible that for this $y \in W$ there exists another $(x', y) \in U$ and $f(x', y) = 0$, then $F(x', y) = (f(x', y), y) = (0, y) = (f(x, y), y) = F(x, y)$, and since F is injective so $x = x'$.

Further define $g(y)$ for $y \in W$ such that $(g(y), y) \in U$ and suppose that (ii) hold *i.e.*, $(g(y), y) = 0$. Then $F(g(y), (y)) = (f,(g(y), y) = (0, y)$ $(y \in W)$ ...(vi)

Let G be a mapping from V onto U *i.e.*, $F^{-1} = G$. Since F is a ζ'-mapping of an open set E into R^{n+m}. $F'(a, b)$ is invertible for some $(a, b) \in E$, $(0, b) = (f(a, b), b) = F(a, b)$ $(b \in W)$, keeping in view the equation (vi), and using the inverse function theorem, we have $G \in \zeta'(E)$. Further by (vi), we have $(g(y), y) = F^{-1}(0, y) = G(0, y)$ $(y \in W)$. Since $G \in \zeta'(E)$ gives $g \in \zeta'(E)$.

Now put $(g(y), y) = \phi(y)$, then

$$\begin{aligned} \phi'(y)k &= \lim_{t\to0}\frac{\phi(y+tk)-\phi(y)}{t} = \lim_{t\to0}\frac{(g(y+tk), y+tk)-(g(y), y)}{t} \\ &= \lim_{t\to0}\frac{(g(y+tk)-g(y), tk)}{t} = \left(\lim_{t\to0}\frac{(g(y+tk)-g(y)}{t}, k\right) \\ &= (g'(y)k, k). \end{aligned}$$

Thus we have $\phi'(y)k = (g'(y)k, k)$ $(y \in W, k \in R^m)$ and by (ii) $f(\phi(y)) = f(g(y), y)$ $(y \in W)$. Thus by chain rule, we have $f'(\phi(y))\, \phi'(y) = 0$ also when $y = b$, we have,

$\phi(y) = (g(b), b) = (a, b)$ and $f'(\phi(y)) = f'(a, b) = A$. Then $A\phi'(b) = 0$ also

$A(h, k) = A_x h + A_y k$, $A_x g'(b)k + A_y k = A(g'(b)k, k) = A\phi'(b)k = 0$, for all $k \in R^m$.

Hence $A_x g'(b) + A_y = 0$ or $g'(b) = -(A_x)^{-1}A_y$.

5.9 DIFFERENTIATION OF INTEGRALS

Suppose ϕ is a function of two variables x and t, which can be integrated with respect to x and which can be differentiated with respect to t. There arise a natural question that, under what conditions will the result be the same if these two limit processes are carried out in the opposite order? More precisely: Under what conditions on ϕ can one prove that the equation

$$\frac{d}{dt}\int_a^b \phi(x,t)\,dx = \int_a^b \frac{\partial\phi(x,t)}{\partial t}dx \qquad \ldots(i)$$

is true ? For the sake of convenience we use the notation $\phi'(x) = \phi(x, t)$ ...(ii)

Thus ϕ' is, for each t, a function of one variable.

5.9.1 Theorem

Suppose

(i) $f(x, t)$ is defined for $a \le x \le b$, $c \le t \le d$;

(ii) α is an increasing function on $[a, b]$;

(iii) $\phi' \in R(\alpha)$ for every $t \in [c, d]$;

(iv) $c < s < d$, and to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that

$|(D_2\phi)(x, t) - (D_2\phi)(x, s)| < \varepsilon$ for all $x \in [a, b]$ and for all $t \in (s - \delta, s + \delta)$. Define

$$f(t) = \int_a^b \phi(x,t)\,d\alpha(x) \quad (c \le t \le d) \qquad \ldots(iii)$$

Then $(D_2\phi)^s \in R(\alpha$, $f'(s)$ exists, and

$$f'(s) = \int_a^b (D_2\phi,(x,s))\,d\alpha(x). \qquad \ldots(iv)$$

Note that (c) simply asserts the existence of the integrals (iii) for all $t \in [c, d]$ and (d) certainly holds whenever $D_2\phi$ is continuous on the rectangle on which ϕ is defined.

Proof. For $0 < |t - s| < \delta$, consider the difference quotients

$$\psi(x, t) = \frac{\phi(x,t)-\phi(x,s)}{t-s} \qquad \ldots(iv)$$

Now by mean value theorem there corresponds to each (x, t) there exists a number u between s and t such that

$$\psi(x, t) = (D_2\phi)(x, u).$$

Hence (d) implies that

$$|\psi(x, t) - (D_2\phi)(x, s)| < \varepsilon \ (a \le x \le b,\ 0 < |t - s| < \delta) \qquad \ldots(vi)$$

Note that by (v), we have

$$\frac{f(t)-f(s)}{t-s} = \frac{1}{t-s}\left[\int_a^b \phi(x,t)d\alpha(x) - \int_a^b \phi(x,s)d\alpha(x)\right]$$

$$= \int_a^b \psi(x,t)d\alpha(x) \qquad \ldots(vii)$$

By (vi), $\psi^t \to (D_2\phi)^s$, uniformly on $[a, b]$, as $t \to s$. Since each $\psi^t \in R(\alpha)$, it follows that $(D_2\phi)^s \in R(a)$. Now from (vii), we have

$$f'(s) = \lim_{t\to s}\frac{f(t)-f(s)}{t-s} = \lim_{t\to s}\int_a^b \psi(x,t)\,d\alpha(x)$$

$$= \int_a^b \lim_{t\to s}\psi(x,t)\,d\alpha(x) = \int_a^b \psi(x,s)\,d\alpha(x) = \int_a^b \psi\,d\alpha(x) \qquad \ldots(viii)$$

Since $\psi^s \in R(\alpha)$ for each $s \in [c, d]$, which implies that the integral on the RHS of ($viii$) exists and so $f(s)$ exists for all $s \in [c, d]$. Further from (v), we have

$$\psi^s = \psi(x, t) = \lim_{t \to s} \frac{\phi(x,t) - \phi(x,s)}{t - s} = (D_2\phi)(x, s)$$

Hence by (*viii*), we have

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$$f'(s) = \int_b^a (D_2\phi)(x, s). \text{ This completes the proof.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $x = r\cos\theta$, $y = r\sin\theta$, *find* $\frac{\partial(x, y)}{\partial(r, \theta)}$ *and* $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Sol.
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}. \qquad \Big|\because \frac{\partial(x, y)}{\partial(r, \theta)} \times \frac{\partial(r, \theta)}{\partial(x, y)} = 1 \text{ using II property}$$

Example 2. *If* $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, *show that:*

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta \text{ and find } \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}.$$

Sol.
$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \left\{\cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right\}$$

$$= r^2\sin\theta\,[\cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

$$\therefore \quad \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)} = \frac{1}{r^2\sin\theta} \qquad | \text{ Using II property}$$

Example 3. *If* $x = a\cosh\alpha\cos\beta$, $y = a\sinh\alpha\sin\beta$, *then show that*

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \frac{a^2}{2}[\cosh 2\alpha - \cos 2\beta].$$

Sol. $$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{vmatrix} \quad ...(1)$$

Differentiating partially with respect to α and β the equations

$$x = a \cosh \alpha \cos \beta \quad ...(2)$$

and $$y = a \sinh \alpha \sin \beta \quad ...(3)$$

we get: $$\frac{\partial x}{\partial \alpha} = a \sinh \alpha \cos \beta, \; \frac{\partial x}{\partial \beta} = - a \cosh \alpha \sin \beta$$

$$\frac{\partial y}{\partial \alpha} = a \cosh \alpha \sin \beta \text{ and } \frac{\partial y}{\partial \beta} = a \sinh \alpha \cos \beta$$

Putting in (1)

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \begin{vmatrix} a \sinh \alpha \cos \beta & - a \cosh \alpha \sin \beta \\ a \cosh \alpha \sin \beta & a \sinh \alpha \cos \beta \end{vmatrix}$$

$$= a^2 (\sinh^2 \alpha \cos^2 \beta + \cosh^2 \alpha \sin^2 \beta)$$

$$= a^2 [(\cosh^2 \alpha - 1) \cos^2 \beta + \cosh^2 \alpha (1 - \cos^2 \beta)$$

$$= a^2 [\cosh^2 \alpha - \cos^2 \beta] = \frac{a^2}{2} [\cosh 2\alpha + 1 - 1 - \cos 2\beta]$$

$$= \frac{a^2}{2} [\cosh 2\alpha - \cos 2\beta].$$

Example 4. (*i*) *Calculate the Jacobian* $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ *of the following:*

$$u = x + 2y + z, \; v = x + 2y + 3z, \; w = 2x + 3y + 5z$$

(*ii*) *If* $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, *then compute the Jacobian* $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Sol. (*i*) $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 2$$

(*ii*) $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} yz & z(x-y) & y(x-z) \\ y+z & x-y & x-z \\ 1 & 0 & 0 \end{vmatrix}$$

| By $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$= \begin{vmatrix} z(x-y) & y(x-z) \\ x-y & x-z \end{vmatrix} = (x-y)(x-z) \begin{vmatrix} z & y \\ 1 & 1 \end{vmatrix}$$

$$= (x-y)(x-z)(z-y) = (x-y)(y-z)(z-x)$$

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Example 5. *If* $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, *then show that* $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$.

Sol. $y_1 = \frac{x_2 x_3}{x_1}$, then $\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}, \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$y_2 = \frac{x_1 x_3}{x_2}$, then $\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \frac{\partial y_2}{\partial x_2} = \frac{-x_1 x_3}{x_2^2}, \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$

$y_3 = \frac{x_1 x_2}{x_3}$, then $\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}, \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}, \frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$.

$$\therefore \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

Example 6. (*i*) *Verify the chain rule for Jacobians if* $x = u$, $y = u \tan v$, $z = w$.

(*ii*) *If* $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ *and* $u = r \sin\theta \cos\phi$, $v = r \sin\theta \sin\phi$, $w = r\cos\theta$ *then calculate the Jacobian* $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

Sol. (*i*) $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u \sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v$$

Solving for u, v, w interms of x, y, z; we have

$$u = x, v = \tan^{-1} \frac{y}{x} \quad \text{and} \quad w = z$$

$$\therefore \quad J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2 + y^2} = \frac{1}{x\left[1 + \left(\frac{y}{x}\right)^2\right]} = \frac{1}{u \sec^2 v}$$

Therefore, $JJ' = 1$

Hence chain rule is verified.

(ii) $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\sqrt{\frac{w}{v}}\sqrt{\frac{u}{w}}\sqrt{\frac{v}{u}} + \frac{1}{8}\sqrt{\frac{v}{w}}\sqrt{\frac{w}{u}}\sqrt{\frac{u}{v}} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$ | From Ex. 2.

Therefore, by the property of Jacobians, we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} = \frac{1}{4} r^2 \sin \theta.$$

Example 7. (i) *If $y_1 = 1 - x_1$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1 x_2 (1 - x_3)$, ..., $y_n = x_1 x_2 x_3 \ldots x_{n-1} (1 - x_n)$, then show that*

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

(ii) *If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$, then show that*

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Sol. (i) We know that

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_n}$$
$$= (-1) \cdot (-x_1)(-x_1 x_2) \ldots\ldots (-x_1 x_2 \ldots\ldots x_{n-1})$$
$$= (-1)^n x_1^{n-1} x_2^{n-2} \ldots\ldots x_{n-1}.$$

(ii) $$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3}$$ | Theorem 1, Art. 5.4

$$= (-\sin x_1)(-\sin x_1 \sin x_2)(-\sin x_1 \sin x_2 \sin x_3)$$
$$= -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Example 8. (i) *If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find the Jacobian $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.*

(ii) *If $x + y + z = u$, $y + z = uv$, $z = uvw$, then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$.*

Sol. (i) $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} yz & zx & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

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$$= 2\begin{vmatrix} y(z-x) & x(z-y) & xy \\ x-z & y-z & z \\ 0 & 0 & 1 \end{vmatrix}$$

| Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$= 2(z-x)(y-z)\begin{vmatrix} y & -x & xy \\ -1 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(y-z)(z-x)(y-x) = -2(x-y)(y-z)(z-x)$$

Since $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 1$ | By chain Rule

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}.$$

(*ii*) $x = u - uv = u(1-v)$

$y = uv - uvw = uv(1-w)$

$z = uvw$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

| Applying $R_1 \to R_1 + (R_2 + R_3)$

$$= u^2v(1-w) + u^2vw = u^2v.$$

Example 9. *If* $u = x(1-r^2)^{-1/2}$, $v = y(1-r^2)^{-1/2}$, $w = z(1-r^2)^{-1/2}$, *where* $r^2 = x^2 + y^2 + z^2$, *then show that* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-5/2}$.

Sol. $$u = \frac{x}{\sqrt{1-x^2-y^2-z^2}} \quad \ldots(1) \quad | \text{ Given}$$

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial u}{\partial x} = \frac{\sqrt{1-x^2-y^2-z^2} - x\left[\frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2-y^2-z^2}}\right]}{(1-x^2-y^2-z^2)}$$

$$= \frac{1-x^2-y^2-z^2+x^2}{(1-x^2-y^2-z^2)^{3/2}} = \frac{1-y^2-z^2}{(1-x^2-y^2-z^2)^{3/2}}$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = -\frac{x}{2}\cdot\frac{(-2y)}{(1-x^2-y^2-z^2)^{3/2}} = \frac{xy}{(1-x^2-y^2-z^2)^{3/2}}$$

Similarly, $$\frac{\partial u}{\partial z} = \frac{xz}{(1-x^2-y^2-z^2)^{3/2}}$$

In the same way, other partial derivatives can be found.

Now, $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{1}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} 1-y^2-z^2 & xy & xz \\ yx & 1-z^2-x^2 & yz \\ zx & zy & 1-x^2-y^2 \end{vmatrix}$$

$$= \frac{xyz}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} \frac{1-y^2-z^2}{x} & y & z \\ x & \frac{1-z^2-x^2}{y} & z \\ x & y & \frac{1-x^2-y^2}{z} \end{vmatrix}$$

$$= \frac{1}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} 1-y^2-z^2 & y^2 & z^2 \\ x^2 & 1-z^2-x^2 & z^2 \\ x^2 & y^2 & 1-x^2-y^2 \end{vmatrix}$$

$$= \frac{1}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} 1-x^2-y^2-z^2 & 0 & -(1-x^2-y^2-z^2) \\ 0 & 1-x^2-y^2-z^2 & -(1-x^2-y^2-z^2) \\ x^2 & y^2 & 1-x^2-y^2 \end{vmatrix}$$

$$= (1-x^2-y^2-z^2)^{-5/2} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ x^2 & y^2 & 1-x^2-y^2 \end{vmatrix}$$

$$= (1-x^2-y^2-z^2)^{-5/2} \{(1-x^2-y^2+y^2)+x^2\} = (1-r^2)^{-5/2}$$

Example 10. *If* $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$ *and* $u+v+w = x^2+y^2+z^2$, *then show that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$.

Sol. Let $F_1 \equiv (u^3+v^3+w^3-x-y-z)$,

$F_2 \equiv (u^2+v^2+w^2-x^3-y^3-z^3)$

and $F_3 \equiv (u+v+w-x^2-y^2-z^2)$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \left[\frac{\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}}\right] \quad ...(1)$$

Now consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} \quad ...(2)$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

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$$= \begin{vmatrix} 0 & 0 & -1 \\ 3(z^2 - x^2) & 3(z^2 - y^2) & -3z^2 \\ 2(z - x) & 2(z - y) & -2z \end{vmatrix} \quad \left| \begin{array}{l} \text{Applying} \\ C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3 \end{array} \right.$$

$$= 6\ (z - x)\ (z - y)\ (-1) \begin{vmatrix} z + x & z + y \\ 1 & 1 \end{vmatrix}$$

$$= -6\ (z - x)\ (z - y)\ (x - y) = 6\ (x - y)\ (y - z)\ (z - x)$$

Now consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} \quad ...(3)$$

$$= \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6 \begin{vmatrix} u^2 - w^2 & v^2 - w^2 & 3w^2 \\ u - w & v - w & 2w \\ 0 & 0 & 1 \end{vmatrix} \quad \left| \begin{array}{l} \text{Applying} \\ C_1 \to C_1 - C_3, C_2 \to C_2 - C_3 \end{array} \right.$$

$$= 6\ (u - w)\ (v - w) \begin{vmatrix} u + w & v + w \\ 1 & 1 \end{vmatrix} = 6\ (u - w)\ (v - w)\ (u - v)$$

$$\therefore \qquad \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3\ \frac{6(x - y)\ (y - z)\ (z - x)}{6(u - v)\ (v - w)\ (u - w)} = \frac{(x - y)\ (y - z)\ (z - x)}{(u - v)\ (v - w)\ (w - u)}.$$

Example 11. *(i) If u, v, w are the roots of the cubic $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, then find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$*

(ii) If u, v, w are the roots of the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$, then find $\frac{\partial(u, v, w)}{\partial(a, b, c)}$.

Sol. (*i*) If u, v, w are the roots of the equation

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

i.e., $$3\lambda^3 - 3\lambda^2(x + y + z) + 3\lambda(x^2 + y^2 + z^2) - (x^3 + y^3 + z^3) = 0$$

then, $$S_1 = u + v + w = \frac{3(x + y + z)}{3}$$

$$S_2 = uv + vw + wu = \frac{3(x^2 + y^2 + z^2)}{3}$$

$$S_3 = uvw = \frac{x^3 + y^3 + z^3}{3}$$

Let $\qquad F_1 \equiv u + v + w - x - y - z = 0$

$$F_2 \equiv uv + vw + wu - (x^2 + y^2 + z^2) = 0$$

$$F_3 \equiv uvw - \frac{1}{3}(x^3 + y^3 + z^3) = 0$$

$$\therefore \qquad \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \left[\frac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}} \right] \qquad ...(1)$$

Now,
$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^2 & y^2 - x^2 & z^2 - x^2 \end{vmatrix} \qquad \left| \begin{array}{l} \text{By } C_2 \to C_2 - C_1, \\ C_3 \to C_3 - C_1 \end{array} \right.$$

$$= -2(y - x)(z - x)(z + x - y - x)$$

$$= -2(x - y)(y - z)(z - x)$$

Also,
$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} & \dfrac{\partial F_1}{\partial w} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} & \dfrac{\partial F_2}{\partial w} \\ \dfrac{\partial F_3}{\partial u} & \dfrac{\partial F_3}{\partial v} & \dfrac{\partial F_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v + w & w + u & u + v \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v + w & u - v & u - w \\ vw & w(u - v) & v(u - w) \end{vmatrix} \qquad \left| \begin{array}{l} \text{By } C_2 \to C_2 - C_1, \\ C_3 \to C_3 - C_1 \end{array} \right.$$

$$= \begin{vmatrix} u - v & u - w \\ w(u - v) & v(u - w) \end{vmatrix} = (u - v)(u - w) \begin{vmatrix} 1 & 1 \\ w & v \end{vmatrix}$$

$$= -(u - v)(v - w)(w - u)$$

Hence,
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \left[\frac{-2(x - y)(y - z)(z - x)}{-(u - v)(v - w)(w - u)} \right] \qquad | \text{ Using (1)}$$

$$\Rightarrow \qquad J(u, v, w) = -\frac{2(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}.$$

(*ii*) If u, v, w are the roots of the equation

$$(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$$

i.e.,
$$3x^3 - 3x^2(a + b + c) + 3x(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3) = 0$$

then,
$$S_1 = u + v + w = \frac{3(a + b + c)}{3}$$

$$S_2 = uv + vw + wu = \frac{3(a^2 + b^2 + c^2)}{3}$$

$$S_3 = uvw = \frac{a^3 + b^3 + c^3}{3}$$

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Let
$$F_1 \equiv u + v + w - a - b - c = 0$$
$$F_2 \equiv uv + vw + wu - a^2 - b^2 - c^2 = 0$$
$$F_3 \equiv uvw - \frac{1}{3}(a^3 + b^3 + c^3) = 0$$

$$\frac{\partial(u, v, w)}{\partial(a, b, c)} = (-1)^3 \left[\frac{\partial(F_1, F_2, F_3)}{\partial(a, b, c)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}\right] \quad ...(1)$$

Now,
$$\frac{\partial(F_1, F_2, F_3)}{\partial(a, b, c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = -2(a-b)(b-c)(c-a)$$

Also,
$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix} = -(u-v)(v-w)(w-u)$$

Hence from (1),
$$\frac{\partial(u, v, w)}{\partial(a, b, c)} = (-1)^3 \left[\frac{-2(a-b)(b-c)(c-a)}{-(u-v)(v-w)(w-u)}\right]$$
$$= -\left[\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}\right].$$

Example 12. *Determine functional dependence and find relation between*

(*i*) $u = \dfrac{x+y}{x-y},\ v = \dfrac{xy}{(x-y)^2}$ (*ii*) $u = \dfrac{x-y}{x+y},\ v = \dfrac{x+y}{x}$.

Sol. (*i*)
$$\frac{\partial u}{\partial x} = \frac{-2y}{(x-y)^2},\ \frac{\partial u}{\partial y} = \frac{2x}{(x-y)^2}$$
$$\frac{\partial v}{\partial x} = \frac{-y(x+y)}{(x-y)^3},\ \frac{\partial v}{\partial y} = \frac{x(x+y)}{(x-y)^3}$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{-2y}{(x-y)^2} & \dfrac{2x}{(x-y)^2} \\ \dfrac{-y(x+y)}{(x-y)^3} & \dfrac{x(x+y)}{(x-y)^3} \end{vmatrix} = \frac{2(x+y)}{(x-y)^5}\begin{vmatrix} -y & x \\ -y & x \end{vmatrix} = 0$$

Hence the functional relationship exists between u and v.

Now,
$$u = \frac{x+y}{x-y} \Rightarrow x = y\left(\frac{u+1}{u-1}\right) \quad ...(1)$$

We have,
$$v = \frac{xy}{(x-y)^2} = \frac{y^2\left(\dfrac{u+1}{u-1}\right)}{y^2\left(\dfrac{u+1}{u-1} - 1\right)^2} \quad | \text{ Using (1)}$$
$$= \left(\frac{u+1}{u-1}\right) \cdot \frac{(u-1)^2}{4} = \frac{u^2-1}{4}$$

$\Rightarrow$ $u^2 - 4v = 1$

(*ii*)
$$\frac{\partial u}{\partial x} = \frac{2y}{(x+y)^2},\ \frac{\partial u}{\partial y} = \frac{-2x}{(x+y)^2},\ \frac{\partial v}{\partial x} = \frac{-y}{x^2},\ \frac{\partial v}{\partial y} = \frac{1}{x}$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2}{x^2(x+y)^2}\begin{vmatrix} y & -x \\ -y & x \end{vmatrix} = 0$$

Hence the functional relationship exists between u and v.

Now, $$u = \frac{x-y}{x+y} \;\Rightarrow\; x = y\left(\frac{1+u}{1-u}\right) \quad \ldots(1)$$

We have, $$v = \frac{x+y}{x} = 1+\frac{y}{x} = 1+\frac{1-u}{1+u} \quad \text{| Using (1)}$$

$$v = \frac{2}{1+u}.$$

Example 13. (*i*) *If* $u = x + 2y + z$, $v = x - 2y + 3z$ *and* $w = 2xy - xz + 4yz - 2z^2$, *show that they are not independent. Find the relation between* u, v *and* w.

(*ii*) *If* $u = \frac{x+y}{z}$, $v = \frac{y+z}{x}$, $w = \frac{y(x+y+z)}{xz}$, *then show that* u, v, w *are not independent and find the relation between them.*

Sol. u, v and w will not be independent if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

(*i*) $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x-4y+6z & -x+2y-3z \end{vmatrix} \quad \begin{array}{|l} \text{By } C_2 \to C_2 - 2C_1, \\ \quad C_3 \to C_3 - C_1 \end{array}$$

$$= -4(-x+2y-3z) - 2(2x-4y+6z)$$

$$= 4x - 8y + 12z - 4x + 8y - 12z = 0$$

Hence u, v and w are not independent.

Now, $$u + v = 2x + 4z \quad \text{and} \quad u - v = 4y - 2z$$

Multiplying these, we get

$$(u+v)(u-v) = (2x+4z)(4y-2z)$$

$$u^2 - v^2 = 4(2xy + 4yz - zx - 2z^2)$$

$$u^2 - v^2 = 4w.$$

(*ii*) $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & \frac{-(x+y)}{z^2} \\ \frac{-(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ \frac{-y^2-yz}{x^2z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

$$= \frac{1}{x^4z^4}\begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -yz(y+z) & xz(x+2y+z) & -xy(x+y) \end{vmatrix}$$

$$= \frac{1}{x^4z^4}\begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & x+y+z & x+y+z \\ -yz(y+z) & z(x+y)(x+y+z) & y(z-x)(x+y+z) \end{vmatrix}$$

| By $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

NOTES

$$= \frac{(x+y+z)^2}{x^4z^4}\begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & y(z-x) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4z^4}\begin{vmatrix} 0 & 0 & -1 \\ -y & 1 & 1 \\ -yz(y+x) & z(x+y) & y(z-x) \end{vmatrix}$$

| By $C_1 \to C_1 + zC_3$

$$= -\frac{(x+y+z)^2}{x^4z^4}\,[-yz(x+y)+yz(x+y)] = 0$$

Hence u, v and w are not independent.

Now, $$uv = \frac{(x+y)(y+z)}{zx} = \frac{xy+y^2+yz+zx}{zx} = \frac{y(x+y+z)}{zx} + 1 = w+1.$$

Example 14. *If $u = \sin^{-1} x + \sin^{-1} y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, find $\frac{\partial(u, v)}{\partial(x, y)}$. Is u, v functionally related? If so, find the relationship.*

Sol. $$u = \sin^{-1} x + \sin^{-1} y \qquad \ldots(1) \quad | \text{ Given}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}} \text{ and } \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

Also, $$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} \qquad \ldots(2) \quad | \text{ Given}$$

$$\frac{\partial v}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}},\ \frac{\partial v}{\partial y} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Now, $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{-xy}{\sqrt{(1-x^2)(1-y^2)}} + 1 - 1 + \frac{xy}{\sqrt{(1-x^2)(1-y^2)}} = 0$$

Hence u and v are not independent. They are functionally related.

We have from (1), $$u = \sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$= \sin^{-1} v \qquad | \text{ From (2)}$$
$$\Rightarrow \quad v = \sin u$$

Example 15. *If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\frac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related? If so, find the relationship.*

Sol. $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

NOTES

Hence u, v are functionally related.

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \qquad v = \tan^{-1} u$$

Example 16. *Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless*

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H}.$$

Sol. Let $\qquad u = ax^2 + 2hxy + by^2,\ v = Ax^2 + 2Hxy + By^2$

u and v will not be independent if there exists a relationship between them and in that case $\dfrac{\partial(u, v)}{\partial(x, y)}$ should vanish identically.

i.e.,
$$\begin{vmatrix} 2(ax+hy) & 2(hx+by) \\ 2(Ax+Hy) & 2(Hx+By) \end{vmatrix} = 0$$

or, $\qquad (ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$

or, $\qquad (aH - Ah)\,x^2 + (aB - Ab)\,xy + (Bh - bH)\,y^2 = 0$

Now, the variables x and y are independent and as such the co-efficients of x^2 and y^2 should vanish separately.

$\therefore \qquad aH - Ah = 0 \qquad$ or $\qquad \dfrac{a}{A} = \dfrac{h}{H}$

and $\qquad Bh - bH = 0 \qquad$ or $\qquad \dfrac{h}{H} = \dfrac{b}{B}$

Hence $\dfrac{a}{A} = \dfrac{h}{H} = \dfrac{b}{B}$ and these conditions also make the co-efficient of xy zero.

Hence $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{h}{H}$ are the required conditions.

TEST YOUR KNOWLEDGE

1. If $x = u(1+v)$, $y = v(1+u)$, then find the value of $\dfrac{\partial(x, y)}{\partial(u, v)}$.
2. If $x = uv$, $y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.
3. If $u = x(1-y)$, $v = xy$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.
4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.
5. If $y_1 = 1 - x_1$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1x_2(1 - x_3)$, find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.
6. If $u = \dfrac{y-x}{1+xy}$ and $v = \tan^{-1} y - \tan^{-1} x$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

NOTES

7. If $u = x^2 - y^2$, $v = 2xy$, find $\frac{\partial(x, y)}{\partial(u, v)}$.

8. If $x = e^v \sec u$, $y = e^v \tan u$, then evaluate $\frac{\partial(x, y)}{\partial(u, v)}$.

9. If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_3 + x_4$, $u_1u_2u_3u_4 = x_4$ then, show that $\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u^3{}_1 u^2{}_2 u_3$.

10. If $u = \frac{x}{y - z}$, $v = \frac{y}{z - x}$, $w = \frac{z}{x - y}$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

11. If $u = x^2 - y^2$, $v = 2xy$ and $x = r \cos \theta$, $y = r \sin \theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 4r^3$.

12. If $u_1 = x_1 + x_2 + x_3$, $u_1{}^2 u_2 = x_2 + x_3$ and $u_1{}^3\, u_3 = x_3$, find the value of Jacobian $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$.

13. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, then show that $\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u - v)}$.

14. If
$$u^3 + v + w = x + y^2 + z^2$$
$$u + v^3 + w = x^2 + y + z^2$$
$$u + v + w^3 = x^2 + y^2 + z$$
show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$.

15. If λ, μ, ν be the roots of the equation $\frac{x}{a + k} + \frac{y}{b + k} + \frac{z}{c + k} = 1$ in k; prove that
$$\frac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)} = \frac{-(\nu - \lambda)(\lambda - \mu)(\mu - \nu)}{(a - b)(b - c)(c - a)}.$$

16. Use the Jacobian to prove that the functions $u = x + y - z$, $v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Find the relation between them.

17. Use the Jacobian to prove that the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$ are not independent of one another. Find the relation between them.

18. If $y_1(x_1 - x_2) = 0$, $y_2(x_1{}^2 + x_1x_2 + x_2{}^2) = 0$ show that, $\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = 3y_1y_2\left(\frac{x_1 + x_2}{x_1^3 - x_2^3}\right)$.

19. (*i*) Determine functional dependence and find relation between $u = \frac{x - y}{x + y}$, $v = \frac{xy}{(x + y)^2}$.

(*ii*) Are the functions: $u = \frac{x - y}{x + z}$, $v = \frac{x + z}{y + z}$

functionally dependent? If so, find the relation between them.

20. Find the Jacobian of the functions
$$y_1 = (x_1 - x_2)(x_2 + x_3)$$
$$y_2 = (x_1 + x_2)(x_2 - x_3)$$
$$y_3 = x_2(x_1 - x_3)$$
hence show that the functions are not independent. Find the relation between them.

21. Prove that the functions

(*i*) $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x(x + 2y - z)$ are not independent. Find the relation between them.

(*ii*) $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent. Find the relation between them.

NOTES

22. If $u = x + y + z + t$, $v = x^2 + y^2 + z^2 + t^2$, $w = x^3 + y^3 + z^3 + t^3$, $r = xyz + yzt + zxt + txy$ show that u, v, w, r are not independent and find a relation between them.

23. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$, prove that u, v, w are not independent and hence find the relation between them.

24. If $u = x^3 + x^2y + x^2z - z^2(x + y + z)$, $v = x + z$ and $w = x^2 - z^2 + xy - zy$ prove that u, v and w are connected by a functional relation.

Answers

1. $1 + u + v$	**2.** $\dfrac{(u-v)^2}{4uv}$	**3.** x
4. r	**5.** $-x_1^2x_2$	**6.** 0
7. $\dfrac{1}{4(x^2+y^2)}$	**8.** $-e^{2v}\sec u$	**10.** 0
12. u_1^{-5}	**16.** $(u + v)^2 + (u - v)^2 = 4w$	**17.** $u^2 = v + 2u$
19. (*i*) $4v = 1 - u^2$	(*ii*) Yes, $v = \dfrac{1}{1-u}$	**20.** $y_1 + y_2 - 2y_3 = 0$
21. (*i*) $u^2 - v^2 = 8w$	(*ii*) $u^2 + v^2 = 2w$	**22.** $u^3 = 3uv - 2w + 6r$
23. $2w = u(3v - u^2)$	**24.** $u = wv$.	

EXTREMUM PROBLEMS WITH CONSTRAINTS

5.10. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function $f(x, y)$ is said to have a **maximum value at $x = a$, $y = b$ if**

$f(a, b) > f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

A function $f(x, y)$ is said to have a **minimum value at $x = a$, $y = b$** if

$f(a, b) < f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

Thus, $f(x, y)$ has a maximum or minimum value at a point (a, b) according as

$$\Delta f = f(a + h, b + k) - f(a, b) < \text{or} > 0.$$

Geometrically, the surface $z = f(x, y)$ has a maximum at the point (a, b) if $f(a, b)$ is the greatest in the small neighbourhood of the point (a, b). As such, the surface descends in all directions at this point. The point of maxima may well be compared with the highest point of a dome. The surface has a minimum at the point (a, b) if $f(a, b)$ is the smallest in the small neighbourhood of the point (a, b). As such, the surface ascends in all directions at this point. The point of minima may well be compared with the lowest point of a bowl. However, sometimes, at the point (a, b), the tangent plane is horizontal and the surface descends in certain directions and ascends in other directions. Such a point is called a saddle point.

A maximum or a minimum value of a function is called its **extreme value.**

5.11. CONDITIONS FOR F(X, Y) TO BE MAXIMUM OR MINIMUM

NOTES

By Taylor's theorem, we have

$$\Delta f = f(a+h, b+k) - f(a, b)$$

$$= [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots \quad \ldots(1)$$

For small values of h and k, the second and higher order terms are still smaller and may be neglected. Thus, sign of Δf = sign of $[hf_x(a, b) + kf_y(a, b)]$.

Taking $h = 0$, the sign of Δf changes with the sign of k. Similarly taking $k = 0$, the sign of Δf changes with the sign of h. Since Δf changes sign with h and k, $f(x, y)$ cannot have a maximum or a minimum value at (a, b) unless $f_x(a, b) = 0 = f_y(a, b)$.

Hence the necessary conditions for (x, y) to have a maximum or a minimum value at (a, b) are

$$f_x(a, b) = 0, \; f_y(a, b) = 0.$$

If these conditions are satisfied, then for small values of h and k, we have, from (1),

$$\Delta f = \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

$$= \frac{1}{2!}[h^2 r + 2hks + k^2 t] + \ldots \text{ where } r = f_{xx}(a, b), \quad s = f_{xy}(a, b), \quad t = f_{yy}(a, b)$$

or $$\Delta f = \frac{1}{2r}[h^2 r^2 + 2hkrs + k^2 rt] + \ldots = \frac{1}{2r}[(h^2 r^2 + 2hkrs + k^2 s^2) + k^2 rt - k^2 s^2)] + \ldots$$

$$= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)] + \ldots \quad \ldots(2)$$

Now $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δf will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ has a maximum or a minimum at (a, b) according as $r < 0$ or $r > 0$.

If $rt - s^2 < 0$, then Δf changes sign with h and k. Hence there is neither a maximum nor a minimum value at (a, b). The point (a, b) is a saddle point in this case.

If $rt - s^2 = 0$ then from (2), we have the quadratic expression $rh^2 + 2shk + th^2$ $= \frac{1}{r}(rh + ks)^2$ and therefore of the same sign as r. Hence in this case the minimum or maximum value of the function will depend on the sign of r.

If $rt - s^2 = 0$ and $\frac{h}{k} = -\frac{s}{r} = \alpha$ (say) then second degree terms of RHS of (2) vanish and we shall have to consider higher degree terms on the R.H.S. of (2). Now the third degree terms must collectively vanish when $\frac{h}{k} = \alpha$ otherwise by changing the sign of both h and k, the sign of the expression $f(a+h, b+k) - f(a, b)$ can be changed. And in this case, the fourth degree terms must collectively be of the same sign as r and t when $\left|\frac{h}{k}\right| = \alpha$.

Note. The point (a, b) is called a *stationary point* if $f_x(a, b) = 0$, $f_y(a, b) = 0$. The value $f(a, b)$ is called a *stationary value*. Thus every extreme value is a stationary value but the converse may not be true.

5.12. RULE TO FIND THE EXTREME VALUES OF A FUNCTION Z = F(X, Y)

(*i*) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(*ii*) Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ simultaneously.

Let (a, b) ; (c, d) ...... be the solutions of these equations.

(*iii*) For each solution in step (*ii*), find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

(*iv*) (*a*) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (a, b) of step (*ii*), then z has a maximum value at (a, b).

(*b*) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (a, b) of step (*ii*), then z has a minimum value at (a, b).

(*c*) If $rt - s^2 < 0$ for a particular solution (a, b) of step (*ii*), then z has no extreme value at (a, b).

(*d*) If $rt - s^2 = 0$, the case is doubtful and requires further investigation.

5.13. CONDITIONS FOR F(X, Y, Z) TO BE MAXIMUM OR MINIMUM

Working Rule

Let the function be $u = f(x, y, z)$.

(*i*) Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ and equate them to zero.

(*ii*) Find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial z \partial x}, \frac{\partial^2 u}{\partial x \partial y}$. They are denoted by A, B, C, F, G, H respectively.

(*iii*) Also find $AB - H^2 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = D_1$ (say) and find $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = D_2$ (say).

The given function u will have a minimum if $A > 0$, $D_1 > 0$, $D_2 > 0$

and will have a maximum if $A < 0$, $D_1 > 0$, $D_2 < 0$.

(*iv*) If these above conditions are not satisfied, we have neither maximum nor minimum.

NOTES

ILLUSTRATIVE EXAMPLES

Example 1. *Find the extreme values of function $x^3 + y^3 - 3axy$.*

Sol. Here $f(x, y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax,\ r = f_{xx} = 6x,\ s = f_{xy} = -3a,\ t = f_{yy} = 6y$$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $x^2 - ay = 0$ ...(1) and $y^2 - ax = 0$ ...(2)

From (1), $y = \dfrac{x^2}{a}$

$\therefore$ From (2), $\dfrac{x^4}{a^2} - ax = 0$ or $x(x^3 - a^3) = 0$ or $x = 0, a$

when $x = 0$, $y = 0$; when $x = a$, $y = a$.

$\therefore$ There are two stationary points (0, 0) and (a, a).

Now, $rt - s^2 = 36xy - 9a^2$

At (0, 0) $rt - s^2 = -9a^2 < 0$

$\Rightarrow$ There is no extreme value at (0, 0).

At (a, a) $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\Rightarrow$ $f(x, y)$ has extreme value at (a, a)

Now, $r = 6a$

If $a > 0$, $r > 0$ so that $f(x, y)$ has a **minimum** value at (a, a).

Minimum value $= a^3 + a^3 - 3a^3 = -a^3$.

If $a < 0$, $r < 0$ so that $f(x, y)$ has a **maximum** value at (a, a).

Maximum value $= -a^3 - a^3 + 3a^3 = a^3$.

Example 2. *Test the function $f(x, y) = x^3y^2(6 - x - y)$ for maxima and minima for points not at the origin.*

Sol. Here $f(x, y) = x^3y^2 (6 - x - y) = 6x^3y^2 - x^4y^2 - x^3y^3$

$$\therefore \quad f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2$$

$$r = f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(6 - 2x - y)$$

$$s = f_{xy} = 36x^2y - 8x^3y - 9x^2y^2 = x^2y(36 - 8x - 9y)$$

$$t = f_{yy} = 12x^3 - 2x^4 - 6x^3y = x^3(12 - 2x - 6y)$$

Now $f_x = 0$

$\Rightarrow$ $x^2y^2(18 - 4x - 3y) = 0$ ...(1)

and $f_y = 0$

$\Rightarrow$ $x^3y(12 - 2x - 3y) = 0$ ...(2)

From (1) and (2),

$$4x + 3y = 18$$

$$2x + 3y = 12 \quad \text{and} \quad x = 0 = y$$

Solving, we get $x = 3$, $y = 2$ and $x = 0 = y$. Leaving $x = 0 = y$, we get $x = 3$, $y = 2$.
Hence (3, 2) is the only stationary point under consideration.
Now,

$$rt - s^2 = 6x^4y^2(6 - 2x - y)(12 - 2x - 6y) - x^4y^2(36 - 8x - 9y)^2$$

At (3, 2) $\quad rt - s^2 = (+)$ ve (> 0)

Also, $\quad r = 6(3)(4)(6 - 6 - 4) = (-)$ ve (< 0)

$\therefore$ $f(x, y)$ has a maximum value at (3, 2).

Example 3. *Examine for extreme values:*

(*i*) $x^2 + y^2 + 6x + 12$ $\qquad$ (*ii*) $x^3 + y^3 - 63(x + y) + 12xy$.

Sol. (*i*) Let $f(x, y) = x^2 + y^2 + 6x + 12$ ...(1)

$\therefore$ $\quad f_x = 2x + 6,\ f_y = 2y,\ f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$

For maxima and minima,

$$f_x = 0 \quad \text{and} \quad f_y = 0$$
$$\Rightarrow \quad 2x + 6 = 0 \quad \text{and} \quad 2y = 0$$
$$\Rightarrow \quad x = -3 \quad \text{and} \quad y = 0$$

Hence $(-3, 0)$ is the stationary point.

At $(-3, 0)$, $rt - s^2 = 4 (> 0)$

Also, $\quad r = 2 (> 0)$

Hence $f(x, y)$ is minimum when $x = -3$ and $y = 0$.

Minimum value $= (-3)^2 + 6(-3) + 12 = 3$.

(*ii*) Let $\quad f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$ ...(1)

$\therefore$ $\quad f_x = 3x^2 - 63 + 12y, \quad f_y = 3y^2 - 63 + 12x$

$\quad f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = 12$

For extremum, $\quad f_x = 0$ and $f_y = 0$

$\Rightarrow$ $\quad 3x^2 - 63 + 12y = 0$ and $3y^2 - 63 + 12x = 0$

$\Rightarrow$ $\quad x^2 + 4y = 21$ ...(2)

and $\quad y^2 + 4x = 21$ ...(3)

Solving (2) and (3), we get

$$x^2 - y^2 - 4(x - y) = 0$$
$$\Rightarrow \quad (x - y)(x + y - 4) = 0$$
$$\Rightarrow \quad x = y \quad \text{and} \quad x + y = 4 \qquad \text{...(4)}$$

If $x = y$, from (2), $x^2 + 4x - 21 = 0$

$$\Rightarrow \quad (x + 7)(x - 3) = 0$$
$$\Rightarrow \quad x = -7, 3$$
$$\therefore \quad y = -7, 3$$

Hence stationary points are $(-7, -7)$, $(3, 3)$

If $x + y = 4$, from (2),

$$x^2 + 4(4 - x) = 21$$
$$\Rightarrow \quad x^2 - 4x - 5 = 0$$
$$\Rightarrow \quad x = -1, 5$$
$$\therefore \quad y = 5, -1$$

Hence other stationary points are $(-1, 5)$, $(5, -1)$.

NOTES

At (– 7, – 7),

$$rt - s^2 = 36(-7)(-7) - 144 = (+)\text{ ve } (> 0)$$

Also, $r = 6(-7) = -42 (< 0)$

Hence $f(x, y)$ is maximum at $(-7, -7)$.

$$\text{Maximum value} = (-7)^3 + (-7)^3 - 63(-14) + 12(-7)(-7) = 784$$

At (3, 3),

$$rt - s^2 = 36(3)(3) - 144 = 180 (> 0)$$

Also, $r = 6(3) = 18 (> 0)$

Hence $f(x, y)$ is minimum at (3, 3).

$$\text{Minimum value} = (3)^3 + (3)^3 - 63(3 + 3) + 12(9) = -216$$

At (– 1, 5),

$$rt - s^2 = 36(-1)(5) - 144 = -324 (< 0)$$

Hence $f(x, y)$ has neither maximum nor minimum at (– 1, 5). (– 1, 5) is a saddle point.

At (5, – 1),

$$rt - s^2 = 36(5)(-1) - 144 = -324 (< 0)$$

Hence $f(x, y)$ has neither maximum nor minimum also at (5, – 1). (5, – 1) is a saddle point.

Example 4. *Examine for minimum and maximum values: sin x + sin y + sin (x + y).*

Sol. Here $f(x, y) = \sin x + \sin y + \sin (x + y)$

$$f_x = \cos x + \cos (x + y)$$
$$f_y = \cos y + \cos (x + y)$$
$$r = f_{xx} = -\sin x - \sin (x + y)$$
$$s = f_{xy} = -\sin (x + y)$$
$$t = f_{yy} = -\sin y - \sin (x + y)$$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow \quad \cos x + \cos (x + y) = 0 \quad ...(1) \quad$ and $\quad \cos y + \cos (x + y) = 0 \quad ...(2)$

Subtracting equation (2) from (1),

$$\cos x - \cos y = 0 \text{ or } \cos x = \cos y \quad \therefore \quad x = y$$

From (1), $\cos x + \cos 2x = 0$

or $$\cos 2x = -\cos x = \cos (\pi - x)$$

or $$2x = \pi - x \quad \therefore \quad x = \frac{\pi}{3}$$

$\therefore$ $$x = y = \frac{\pi}{3} \text{ is a stationary point.}$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $\quad r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3},\ s = \frac{\sqrt{3}}{2},\ t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore$ $$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

Also $r < 0$

$\therefore$ $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.

Example 5. *In a plane triangle ABC, find the maximum value of cos A cos B cos C.*

Sol. $\cos A \cos B \cos C = \cos A \cos B \cos[\pi - (A + B)]$ $\quad | \because \quad A + B + C = \pi$

$$= -\cos A \cos B \cos (A + B) = f(A, B)$$

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A \cos (A + B) - \cos A \sin (A + B)]$$

$$= \cos B \sin [A + (A + B)] = \cos B \sin (2A + B)$$

$$\frac{\partial f}{\partial B} = -\cos A[-\sin B \cos (A + B) - \cos B \sin (A + B)]$$

$$= \cos A \sin [B + (A + B)] = \cos A \sin (A + 2B)$$

$$r = 2 \cos B \cos (2A + B)$$

$$s = -\sin A \sin (A + 2B) + \cos A \cos (A + 2B)$$

$$= \cos [A + (A + 2B)] = \cos (2A + 2B)$$

$$t = 2 \cos A \cos (A + 2B)$$

Now $\quad \frac{\partial f}{\partial A} = 0 \quad$ and $\quad \frac{\partial f}{\partial B} = 0$

$\Rightarrow \quad \cos B \sin (2A + B) = 0 \quad$ ...(1)

$\quad \cos A \sin (A + 2B) = 0 \quad$ ...(2)

If $\cos B = 0$, then $\quad B = \frac{\pi}{2}$

From (2), $\cos A \sin (A + \pi) = 0$ or $\cos A(-\sin A) = 0$

$\Rightarrow$ either $\cos A = 0$ *i.e.*, $A = \frac{\pi}{2}$ which is not possible

$(\because \quad A + B + C = \pi \quad \Rightarrow \quad C = 0)$

or $\quad \sin A = 0$ *i.e.*, $A = 0$ or π which is not possible.

$\therefore \quad \cos B \neq 0$. Similarly $\cos A \neq 0$

$\therefore$ From (1), $\quad \sin (2A + B) = 0 \quad$ or $\quad 2A + B = \pi$

From (2), $\quad \sin (A + 2B) = 0 \quad$ or $\quad A + 2B = \pi$

Solving these equations, $\quad A = B = \frac{\pi}{3}$

At $\quad A = B = \frac{\pi}{3}$,

$$r = 2 \cos\frac{\pi}{3} \cos \pi = -1,\ s = \cos\frac{4\pi}{3} = -\frac{1}{2}, \quad t = 2 \cos\frac{\pi}{3} \cos \pi = -1$$

$$rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4}\ (> 0). \text{ Also } \quad r = -1\ (< 0)$$

$\Rightarrow \quad f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} \cos\frac{\pi}{3} \cos\frac{2\pi}{3} = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}$.

NOTES

Example 6. *A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Sol. Let x, y and z be the length, breadth and height respectively. Let V be the given capacity and S, the surface.

V is given $\Rightarrow$ V is constant

$$V = xyz \quad \text{or} \quad z = \frac{V}{xy} \qquad \ldots(1)$$

$$S = xy + 2xz + 2yz = xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y) \qquad \text{[Using (1)]}$$

$$f_x = y - \frac{2V}{x^2},\ f_y = x - \frac{2V}{y^2}$$

$$r = f_{xx} = \frac{4V}{x^3},\ s = f_{xy} = 1,\ t = f_{yy} = \frac{4V}{y^3}$$

Now $\qquad f_x = 0 \quad \text{and} \quad f_y = 0$

$$\Rightarrow \quad y - \frac{2V}{x^2} = 0 \quad \ldots(1) \qquad x - \frac{2V}{y^2} = 0 \quad \ldots(2)$$

From (1), $\qquad y = \dfrac{2V}{x^2}$

$\therefore$ From (2), $\quad x - 2V.\dfrac{x^4}{4V^2} = 0$

or $$x\left(1 - \frac{x^3}{2V}\right) = 0 \quad \text{or} \quad x = (2V)^{1/3} \qquad [\because \ x \neq 0$$

and $$y = \frac{2V}{x^2} = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$$

$\therefore$ $x = y = (2V)^{1/3}$ is a stationary point.

At this point, $\qquad r = \dfrac{4V}{2V} = 2 > 0,\ s = 1,\ t = \dfrac{4V}{2V} = 2$

so that $\qquad rt - s^2 = 4 - 1 = 3 > 0$ and $r > 0$

$\Rightarrow$ S is minimum when $x = y = (2V)^{1/3}$

Also $$z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}} = \frac{(2V)^{1/3}}{2} = \frac{y}{2}$$

Hence S is minimum when $x = y = (2V)^{1/3}$.

Example 7. *Find the shortest distance between the lines*

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \textit{and} \quad \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

Sol. Any point P on the first line is $(\lambda + 3, 5 - 2\lambda, 7 + \lambda)$. Similarly, any point Q on the second line is $(7\mu - 1, -1 - 6\mu, \mu - 1)$

Distance between these two points is

$$PQ = \sqrt{(\lambda + 3 - 7\mu + 1)^2 + (5 - 2\lambda + 1 + 6\mu)^2 + (7 + \lambda - \mu + 1)^2}$$

$$= \sqrt{6\lambda^2 + 86\mu^2 - 40\lambda\mu + 116}$$

If the distance is max. or min. so will be the square of the distance.

Let $$u = (PQ)^2 = 6\lambda^2 + 86\,\mu^2 - 40\,\lambda\mu + 116 \quad \text{...(1)}$$

$$\frac{\partial u}{\partial \lambda} = 12\lambda - 40\mu, \quad \frac{\partial u}{\partial \mu} = 172\mu - 40\lambda$$

For stationary points,

$$\frac{\partial u}{\partial \lambda} = 0 \;\Rightarrow\; 12\lambda - 40\mu = 0 \quad \text{...(2)}$$

and $$\frac{\partial u}{\partial \mu} = 0 \;\Rightarrow\; 172\mu - 40\lambda = 0 \quad \text{...(3)}$$

Solving (2) and (3), we get $\lambda = 0, \mu = 0$ $\therefore$ (0, 0) is stationary point.

Now, $$\frac{\partial^2 u}{\partial \lambda^2} = 12,\; \frac{\partial^2 u}{\partial \mu^2} = 172,\; \frac{\partial^2 u}{\partial \lambda\,\partial \mu} = -40$$

$$rt - s^2 = (12)\,(172) - (-40)^2 > 0$$

Also, $r > 0$

Hence u is minimum at (0, 0). Shortest distance is given by $PQ = \sqrt{116} = 2\sqrt{29}$.

Example 8. (*i*) *Given* $f(x, y, z) = \dfrac{5xyz}{x + 2y + 4z}$. *Find the values of* x, y, z *for which* $f(x, y, z)$ *is a maximum, subject to the condition* $xyz = 8$.

(*ii*) *Examine for extreme values:* $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Sol. (*i*) $$u = \frac{5xyz}{x + 2y + 4z} \quad \text{...(1)}$$

$$xyz = 8 \quad \text{...(2)}$$

From (1) and (2), $$u = \frac{40}{x + 2y + 4z}$$

$\therefore$ $$du = -\frac{40}{(x + 2y + 4z)^2}\,(dx + 2dy + 4dz)$$

For max. and min. of u, $du = 0$

$\Rightarrow$ $$dx + 2dy + 4dz = 0 \quad \text{...(3)}$$

From (2), $\log x + \log y + \log z = \log 8$

Differentiation gives. $$\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0 \quad \text{...(4)}$$

From (3) and (4), after introducing λ such that

$$dx\left(1 + \frac{\lambda}{x}\right) + dy\left(2 + \frac{\lambda}{y}\right) + dz\left(4 + \frac{\lambda}{z}\right) = 0$$

$\Rightarrow$ $$1 + \frac{\lambda}{x} = 0 \quad \Rightarrow \quad x = -\lambda$$

$$2 + \frac{\lambda}{y} = 0 \quad \Rightarrow \quad y = -\frac{\lambda}{2}$$

$$4 + \frac{\lambda}{z} = 0 \quad \Rightarrow \quad z = -\frac{\lambda}{4}$$

From (2), $-\dfrac{\lambda^3}{8} = 8 \quad \Rightarrow \quad \lambda = -4$

$\therefore$ u is stationary at the point $x = 4, y = 2, z = 1$.

NOTES

Now, $$u = \frac{40}{x + 2y + 4z}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-40}{(x + 2y + 4z)^2} \cdot \left(1 + 4\frac{\partial z}{\partial x}\right)$$

From (2), $\log x + \log y + \log z = \log 8$

Differentiating w.r.t. x partially, we get

$$\frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-z}{x}$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{-40}{(x + 2y + 4z)^2} \cdot \left(1 - \frac{4z}{x}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{80}{(x + 2y + 4z)^3}\left(1 + 4\frac{\partial z}{\partial x}\right)\left(1 - \frac{4z}{x}\right) - \frac{40}{(x + 2y + 4z)^2}\left(\frac{4z}{x^2} - \frac{4}{x}\frac{\partial z}{\partial x}\right)$$

$$= \frac{80}{(x + 2y + 4z)^3}\left(1 - \frac{4z}{x}\right)^2 - \frac{40}{(x + 2y + 4z)^2}\left(\frac{8z}{x^2}\right)$$

At the stationary point,

$$r = \frac{80}{(12)^3}(1 - 1)^2 - \frac{40}{144}\left(\frac{1}{4} + \frac{1}{4}\right) < 0$$

$\therefore$ u is maximum at the point (4, 2, 1).

(*ii*) We have.

$$u = x^2 + y^2 + z^2 - xy + x - 2z \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} = 2x - y + 1, \ \frac{\partial u}{\partial y} = 2y - x, \ \frac{\partial u}{\partial z} = 2z - 2$$

For extreme values,

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u}{\partial z} = 0$$

$$\therefore \quad 2x - y + 1 = 0,\ 2y - x = 0 \quad \text{and} \quad 2z - 2 = 0$$

Solving above equations, we get

$$x = -\frac{2}{3},\ y = -\frac{1}{3},\ z = 1$$

Hence $\left(-\dfrac{2}{3}, -\dfrac{1}{3}, 1\right)$ is the stationary point.

Now, $$A = \frac{\partial^2 u}{\partial x^2} = 2(> 0), \quad B = \frac{\partial^2 u}{\partial y^2} = 2, \qquad C = \frac{\partial^2 u}{\partial z^2} = 2$$

$$F = \frac{\partial^2 u}{\partial y\,\partial z} = 0, \qquad G = \frac{\partial^2 u}{\partial z\,\partial x} = 0, \quad H = \frac{\partial^2 u}{\partial x\,\partial y} = -1$$

$\therefore \qquad D_1 = AB - H^2 = 4 - 1 = 3(> 0)$

$$D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6(> 0)$$

Since $A > 0$, $D_1 > 0$ and $D_2 > 0$,

hence the given function u will have a minimum at $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$

and minimum value $= \left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + 1 - \frac{2}{9} - \frac{2}{3} - 2 = -\frac{4}{3}$.

5.14. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

To find the maximum or minimum values of a function of three (or more) variables, when the variables are not independent but are connected by some given relation, we try to convert the given function to the one, having least number of independent variables with the help of the given relation.

When this procedure is not practicable, we use Lagrange's method.

Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value.

Let the variables x, y, z be connected by the relation $\phi(x, y, z) = 0$ ...(1)

For $f(x, y, z)$ to have a maximum or minimum value, the necessary condition is

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

$$\therefore \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad ...(2)$$

Also, from (1), taking differentials, we get $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$ ...(3)

Multiplying (3) by a parameter λ and adding to (2), we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will hold good if $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$, $\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$, $\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$.

These equations together with equation (1), give the values of x, y, z and λ for a maximum or minimum.

NOTES

Lagrange's method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical considerations of the problem.

Note. The above equations can be easily obtained by considering Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

and considering the stationary values of $F(x, y, z)$. For stationary values of $F(x, y, z)$, $dF = 0$.

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0.$$

Example 9. *A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found using Lagrange's method of multipliers.*

Sol. Let x and y be two parts into which the given wire is cut so that

$$x + y = b \qquad ...(1)$$

Let the piece of wire of length x be bent into a square so that each side $= \frac{x}{4}$

$$\therefore \quad \text{Area of square} = \frac{x^2}{16} \qquad ...(2)$$

Similarly, the piece of wire of length y is bent into a circle with perimeter y.

$$\therefore \quad \text{Area of circle} = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi} \qquad ...(3)$$

Consider Lagrange's function

$$F(x, y) = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x + y - b) \qquad ...(4)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \quad \frac{x}{8} + \lambda = 0 \qquad ...(5)$$

$$\frac{y}{2\pi} + \lambda = 0 \qquad ...(6)$$

$$\Rightarrow \quad x = -8\lambda \quad \text{and} \quad y = -2\pi\lambda$$

Substituting these values in (1), we get

$$-8\lambda - 2\pi\lambda = b$$

$$\lambda = \frac{-b}{8 + 2\pi} \qquad ...(7)$$

$$\therefore \quad x = -8\lambda = \frac{8b}{8 + 2\pi} \quad \text{and} \quad y = -2\pi\lambda = \frac{2\pi b}{8 + 2\pi}$$

The least value of the sum of areas

$$= \frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{1}{16}\cdot\frac{64b^2}{(8 + 2\pi)^2} + \frac{1}{4\pi}\cdot\frac{4\pi^2 b^2}{(8 + 2\pi)^2}$$

$$= \frac{b^2(\pi + 4)}{4(\pi + 4)^2} = \frac{b^2}{4(\pi + 4)}.$$

Example 10. *The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.*

Sol. Let x, y, z be the three positive numbers.

$$x + y + z = k \quad \text{(say)} \qquad \ldots(1)$$

Let $$u = xyz \qquad \ldots(2)$$

Consider Lagrange's function

$$F(x, y, z) = xyz + \lambda(x + y + z - k)$$

For stationary values,

$$dF = 0$$

$\Rightarrow \quad (yz + \lambda)\, dx + (zx + \lambda)\, dy + (xy + \lambda)\, dz = 0$

$$\Rightarrow \quad yz + \lambda = 0 \qquad \ldots(3)$$

$$zx + \lambda = 0 \qquad \ldots(4)$$

$$xy + \lambda = 0 \qquad \ldots(5)$$

Multiplying (3), (4) and (5) by x, y and z respectively, and adding, we get

$$3xyz + \lambda(x + y + z) = 0$$

$$\Rightarrow \quad 3u + \lambda k = 0$$

$$\Rightarrow \quad \lambda = -\frac{3u}{k}$$

From (3), (4) and (5),
$$yz - \frac{3u}{k} = 0$$

$$zx - \frac{3u}{k} = 0$$

and
$$xy - \frac{3u}{k} = 0$$

$$\Rightarrow \quad xyz - \frac{3ux}{k} = 0 \quad \Rightarrow \quad u - \frac{3ux}{k} = 0$$

$$xyz - \frac{3uy}{k} = 0 \quad \Rightarrow \quad u - \frac{3uy}{k} = 0$$

$$xyz - \frac{3uz}{k} = 0 \quad \Rightarrow \quad u - \frac{3uz}{k} = 0$$

$$\Rightarrow \quad x = \frac{k}{3},\ y = \frac{k}{3},\ z = \frac{k}{3}.$$

Hence $\left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}\right)$ is the stationary point. Now to find whether u is maximum or minimum. Let z be a function of x and y.

$$u = xy\,(k - x - y) = kxy - x^2y - xy^2$$

$$\frac{\partial u}{\partial x} = ky - 2xy - y^2,\ \frac{\partial u}{\partial y} = kx - x^2 - 2xy$$

Now,
$$r = \frac{\partial^2 u}{\partial x^2} = -2y,\ s = \frac{\partial^2 u}{\partial x \partial y} = k - 2x - 2y,\ t = \frac{\partial^2 u}{\partial y^2} = -2x$$

$$\therefore \quad rt - s^2 = 4xy - (k - 2x - 2y)^2$$

$$= 4\frac{k}{3}.\frac{k}{3} - \left(k - \frac{2k}{3} - \frac{2k}{3}\right)^2 = \frac{4k^2}{9} - \frac{k^2}{9} = \frac{k^2}{3} > 0$$

NOTES

Also, $r = -2y = -2\frac{k}{3} < 0$

Hence u is maximum at $x = y = z = \frac{k}{3}$ and maximum value $= \frac{k}{3}.\frac{k}{3}.\frac{k}{3} = \frac{k^3}{27}$.

Example 11. *Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.*

Sol. Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid and let R be the radius of the sphere.

Then, $x^2 + y^2 + z^2 = R^2$ ...(1)

Volume $V = 8xyz$ ...(2)

Consider Lagrange's function

$$F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$$

For stationary values,

$$dF = 0$$

$\Rightarrow$ $\{8yz + \lambda(2x)\}\, dx + \{8xz + \lambda(2y)\}\, dy + \{8xy + \lambda(2z)\}\, dz = 0$

$\Rightarrow$ $8yz + 2\lambda x = 0$ ...(3)

$8zx + 2\lambda y = 0$ ...(4)

$8xy + 2\lambda z = 0$ ...(5)

From (3), $2\lambda x^2 = -8xyz$

From (4), $2\lambda y^2 = -8xyz$

From (5), $2\lambda z^2 = -8xyz$

$\therefore$ $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

or $x^2 = y^2 = z^2$ or $x = y = z$.

Hence rectangular solid is a cube.

Example 12. *Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol. Let (x, y, z) be a vertex of the parallelopiped then it lies on the given ellipsoid

$\therefore$ $\phi(x, y, z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ ...(1)

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid.

$\therefore$ Volume $V = 8xyz$ ...(2)

Consider Lagrange's function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \quad \left\{8yz + \lambda\left(\frac{2x}{a^2}\right)\right\} dx + \left\{8zx + \lambda\left(\frac{2y}{b^2}\right)\right\} dy + \left\{8xy + \lambda\left(\frac{2z}{c^2}\right)\right\} dz = 0$$

NOTES

$$\Rightarrow \quad 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \qquad ...(3)$$

$$8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \qquad ...(4)$$

$$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \qquad ...(5)$$

Multiplying (3), (4), (5) by x, y, z respectively and adding, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\Rightarrow \quad 24xyz + 2\lambda = 0 \quad \Rightarrow \quad \lambda = -12xyz$$

From (3), $8yz - 12xyz\left(\frac{2x}{a^2}\right) = 0$

$$\Rightarrow \quad 1 - \frac{3x^2}{a^2} = 0 \quad \Rightarrow \quad x = \frac{a}{\sqrt{3}}$$

Similarly from (4) and (5), we get $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$.

$\therefore$ Volume of the largest rectangular parallelopiped $= 8xyz = \frac{8abc}{3\sqrt{3}}$.

Example 13. *The temperature T at any point (x, y, z) in space is $T = 400\, xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.*

Sol. Consider Lagrange's function

$$F(x, y, z) = 400\, xyz^2 + \lambda(x^2 + y^2 + z^2 - 1) \qquad ...(1)$$

For stationary values.

$$dF = 0$$

$$\Rightarrow \quad [400yz^2 + \lambda(2x)]\, dx + [400\, xz^2 + \lambda(2y)]\, dy + [800\, xyz + \lambda(2z)]\, dz = 0$$

$$\Rightarrow \quad 400yz^2 + 2\lambda x = 0 \qquad ...(2)$$

$$400xz^2 + 2\lambda y = 0 \qquad ...(3)$$

$$800xyz + 2\lambda z = 0 \qquad ...(4)$$

Multiplying (2) by x, (3) by y and (4) by z and adding, we get

$$1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0$$

$$\Rightarrow \quad \lambda = -800xyz^2 \qquad ...(5) \quad | \because \; x^2 + y^2 + z^2 = 1$$

From (2), $\quad 400yz^2 - 1600x^2yz^2 = 0$ | Using (5)

$$\Rightarrow \quad x = \pm\frac{1}{2}$$

Similarly, $\quad y = \pm\frac{1}{2}$

From (4) $\quad 800xyz - 1600xyz^3 = 0$

$$\Rightarrow \quad 1 - 2z^2 = 0$$

$$\Rightarrow \quad z = \pm\frac{1}{\sqrt{2}}$$

Putting values of x, y, z in T, we get

$$T = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 50.$$

NOTES

Example 14. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere* $x^2 + y^2 + z^2 = 1$.

Sol. Let (x, y, z) be any point on the sphere.

Distance of the point A(3, 4, 12) from (x, y, z) is given by

$\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$.

If the distance is maximum or minimum, so will be the square of the distance.

Let $$f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 \quad ...(1)$$

subject to the condition that $\phi(x, y, z) \equiv x^2 + y^2 + z^2 - 1 = 0$ ...(2)

Consider Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$
$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

For stationary values, $dF = 0$.

$\Rightarrow \quad [2(x-3) + 2\lambda x]dx + [2(y-4) + 2\lambda y]dy + [2(z-12) + 2\lambda z]dz = 0$

$$\Rightarrow \quad 2(x-3) + 2\lambda x = 0 \quad ...(3)$$
$$2(y-4) + 2\lambda y = 0 \quad ...(4)$$
$$2(z-12) + 2\lambda z = 0 \quad ...(5)$$

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) - 6x - 8y - 24z + 2\lambda(x^2 + y^2 + z^2) = 0$$

or $$2 - 6x - 8y - 24z + 2\lambda = 0 \quad | \text{ Using (2)}$$

or $$3x + 4y + 12z = 1 + \lambda \quad ...(6)$$

From (3), (4) and (5), $\quad x = \frac{3}{1+\lambda}, y = \frac{4}{1+\lambda}, z = \frac{12}{1+\lambda}$.

Putting these values of x, y, z in (6), we have

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1 + \lambda$$

or $$(1+\lambda)^2 = 169 \quad \text{or} \quad 1 + \lambda = \pm 13$$

$\therefore$ $$\lambda = 12 \quad \text{or} \quad -14$$

When $\lambda = 12$, $\quad x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$

When $\lambda = -14$, $\quad x = -\frac{3}{13}, y = -\frac{4}{13}, z = -\frac{12}{13}$

Thus, we get two points $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ on the sphere which are at a *maximum or minimum* distance from the given point A.

Now, $$AP = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2} = 12$$

$$AQ = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2} = 14$$

$\therefore$ $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ is at a minimum distance from A and the minimum distance = 12.

$Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ is at a maximum distance from A and the maximum distance = 14.

Example 15. (*i*) *Find the minimum distance from the point (1, 2, 0) to the cone* $z^2 = x^2 + y^2$.

(*ii*) *Find a point on the paraboloid* $z = x^2 + y^2$ *nearest to the point (3, – 6, 4).*

Sol. (*i*) Let (x, y, z) be any point on the cone then distance from the point (1, 2, 0) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

If the distance is maximum or minimum, so will be the square of the distance.

Let $\qquad u = D^2 = (x-1)^2 + (y-2)^2 + z^2 \qquad ...(1)$

Also. $\qquad \phi \equiv x^2 + y^2 - z^2 = 0 \qquad ...(2)$

Consider Lagrange's function

$$F(x, y, z) = u + \lambda\phi = (x-1)^2 + (y-2)^2 + z^2 + \lambda(x^2 + y^2 - z^2) \qquad ...(3)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \{2(x-1) + \lambda(2x)\}\,dx + \{2(y-2) + \lambda(2y)\}\,dy + \{2z + \lambda(-2z)\}\,dz = 0$$

$$\Rightarrow \quad 2(x-1) + \lambda(2x) = 0 \qquad ...(4)$$

$$2(y-2) + \lambda(2y) = 0 \qquad ...(5)$$

$$2z + \lambda(-2z) = 0 \qquad ...(6)$$

$$\Rightarrow \quad x = \frac{1}{1+\lambda},\ y = \frac{2}{1+\lambda},\ \lambda = 1$$

$$\therefore \quad x = \frac{1}{2},\ y = 1$$

Putting the values of x and y in eqn. (2), we get

$$\frac{1}{4} + 1 - z^2 = 0 \quad \Rightarrow \quad z^2 = \frac{5}{4} \quad \Rightarrow \quad z = \pm\frac{\sqrt{5}}{2}$$

Hence, minimum distance from the point (1, 2, 0) is

$$u = \left(\frac{1}{2} - 1\right)^2 + (1-2)^2 + \frac{5}{4} = \frac{5}{2} = D^2$$

$$\therefore \quad D = \sqrt{\frac{5}{2}}$$

(*ii*) Let (x, y, z) be any point on paraboloid nearest to the point (3, – 6, 4).

$$\therefore \quad D = \sqrt{(x-3)^2 + (y+6)^2 + (z-4)^2}$$

Let $\qquad u = D^2 = (x-3)^2 + (y+6)^2 + (z-4)^2 \qquad ...(1)$

Also, $\qquad \phi \equiv x^2 + y^2 - z = 0 \qquad ...(2)$

Consider Lagrange's function

$$F(x, y, z) = (x-3)^2 + (y+6)^2 + (z-4)^2 + \lambda(x^2 + y^2 - z) \qquad ...(3)$$

For stationary values,

$$dF = 0$$

$$\therefore \quad 2(x-3) + \lambda(2x) = 0 \quad \text{...(4)}$$

$$2(y+6) + \lambda(2y) = 0 \quad \text{...(5)}$$

$$2(z-4) + \lambda(-1) = 0 \quad \text{...(6)}$$

From (4), (5) and (6),

$$x = \frac{3}{1+\lambda},\ y = \frac{-6}{1+\lambda},\ z = \frac{\lambda+8}{2}$$

Putting values of x, y, z in (2), we get

$$\left(\frac{3}{1+\lambda}\right)^2 + \left(\frac{-6}{1+\lambda}\right)^2 = \frac{\lambda+8}{2}$$

$$\Rightarrow \quad \lambda^3 + 10\lambda^2 + 17\lambda - 82 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Hence, $\quad x = 1, \quad y = -2, z = 5$

Example 16. *Find the maximum and minimum distances from the origin to the curve*

$$x^2 + 4xy + 6y^2 = 140.$$

Sol. Let (x, y) be any point on the curve. Distance of the point A(0, 0) from (x, y) is given by $\sqrt{(x-0)^2 + (y-0)^2}$.

If the distance is maximum or minimum, so will be the square of the distance.

Let $\quad f(x, y) = x^2 + y^2 \quad$ ...(1)

subject to the condition

$$\phi(x, y) \equiv x^2 + 4xy + 6y^2 - 140 = 0 \quad \text{...(2)}$$

Consider Lagrange's function

$$F(x, y) = f(x, y) + \lambda\phi(x, y) = x^2 + y^2 + \lambda(x^2 + 4xy + 6y^2 - 140)$$

For stationary values.

$$dF = 0$$

$$\Rightarrow \quad [2x + \lambda(2x + 4y)]\,dx + [2y + \lambda(4x + 12y)]\,dy = 0$$

$$\Rightarrow \quad 2x + \lambda(2x + 4y) = 0 \quad \text{...(3)}$$

$$2y + \lambda(4x + 12y) = 0 \quad \text{...(4)}$$

Multiplying (3) by x and (4) by y and adding,

$$2x^2 + 2y^2 + \lambda(2x^2 + 8xy + 12y^2) = 0$$

$$\Rightarrow \quad \lambda = -\left(\frac{x^2+y^2}{140}\right) = -\frac{f}{140} \quad \Big|\ \because x^2 + 4xy + 6y^2 = 140$$

From (3) and (4),

$$2x - \frac{f}{70}(x + 2y) = 0$$

and

$$2y - \frac{f}{35}(x + 3y) = 0$$

$$\Rightarrow \quad (140 - f)x - 2fy = 0 \quad \text{...(5)}$$

and

$$-fx + (70 - 3f)y = 0 \quad \text{...(6)}$$

Solving (5) and (6), we get

$$f^2 - 490f + 9800 = 0$$

$$\Rightarrow \quad f = \frac{490 \pm \sqrt{(490)^2 - 4(9800)}}{2}$$

$$= 245 \pm 35\sqrt{41} = 469.1093,\ 20.8906$$

Hence maximum distance $= \sqrt{469.1093} = 21.6589$

minimum distance $= \sqrt{20.8906} = 4.5706.$

Example 17. *Find the minimum value of* $x^2 + y^2 + z^2$*, given that* $ax + by + cz = p$.

Sol. Let $\quad u = x^2 + y^2 + z^2 \quad$ ...(1)

where $\quad \phi(x, y, z) = ax + by + cz - p = 0 \quad$ ...(2)

Consider Lagrange's function, $\quad F(x, y, z) = (x^2 + y^2 + z^2) + \lambda(ax + by + cz - p)$

For stationary values, $dF = 0$

$$\Rightarrow \quad (2x + \lambda a)dx + (2y + \lambda b)dy + (2z + \lambda c)dz = 0$$

$\Rightarrow \quad 2x + \lambda a = 0 \quad$ ...(3)

$2y + \lambda b = 0 \quad$ ...(4)

$2z + \lambda c = 0 \quad$ ...(5)

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

or $\quad 2u + \lambda p = 0 \quad$ | Using (1) and (2)

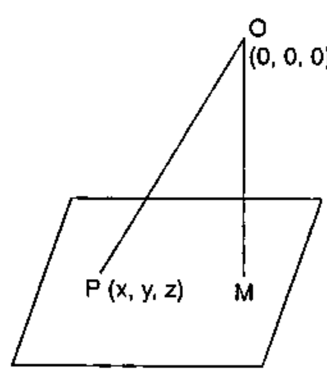
$$\therefore \quad \lambda = -\frac{2u}{p}$$

From (3), (4) and (5), $\quad x = \frac{au}{p}, y = \frac{bu}{p}, z = \frac{cu}{p}$

$$\therefore \quad \text{From (1),} \quad u = \frac{(a^2 + b^2 + c^2)u^2}{p^2}$$

$$\text{or} \quad u = \frac{p^2}{a^2 + b^2 + c^2}.$$

This is the **maximum or minimum** value of u. Now u is the square of the distance of any point P(x, y, z) on the plane (2) from the origin. Also, the length of perpendicular from O on the plane is $\frac{p}{\sqrt{a^2 + b^2 + c^2}}$. Clearly, OP is least when P coincides with M, the foot of the perpendicular from O on the plane.



Hence the minimum value of $u = \frac{p^2}{a^2 + b^2 + c^2}$.

Example 18. *Find the dimensions of a rectangular box of maximum capacity whose surface area is given when*

(*i*) *box is open at the top* $\qquad$ (*ii*) *box is closed.*

Sol. Let x, y, z be the dimensions of the rectangular box so that its volume is

$$V = xyz \quad \text{...(1)}$$

NOTES

Total surface area of the box is

$$S = nxy + 2yz + 2zx = \text{given constant} \qquad ...(2)$$

Here, $n = 1$ if the box is open at the top and $n = 2$ if the box is closed on all sides.

Consider Lagrange's function

$$F(x, y, z) \equiv xyz + \lambda (nxy + 2yz + 2zx - S) \qquad ...(3)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \quad \{yz + \lambda (ny + 2z)\} dx + \{zx + \lambda (nx + 2z)\} dy + \{xy + \lambda (2y + 2x)\} = 0$$

$$\Rightarrow \quad yz + \lambda (ny + 2x) = 0 \qquad ...(4)$$

$$zx + \lambda (nx + 2z) = 0 \qquad ...(5)$$

$$xy + \lambda (2y + 2x) = 0 \qquad ...(6)$$

Multiplying eqns. (4), (5) and (6) by x, y and z respectively and adding, we get

$$3xyz + \lambda \{2 (nxy + 2yz + 2zx)\} = 0$$

$$\Rightarrow \quad 3V + \lambda (2S) = 0$$

$$\Rightarrow \quad \lambda = -\frac{3V}{2S} \qquad ...(7)$$

Substituting value of λ from (7) in (4), (5) and (6), we get

$$yz - \frac{3V}{2S}(ny + 2z) = 0$$

$$\Rightarrow \quad yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

$$nxy + 2xz = \frac{2S}{3} \qquad ...(8)$$

Similarly, $$nxy + 2yz = \frac{2S}{3} \qquad ...(9)$$

$$2yz + 2zx = \frac{2S}{3} \qquad ...(10)$$

Subtracting (9) from (8), we get

$$2z (x - y) = 0 \quad \Rightarrow \quad x = y \qquad ...(11)$$

Subtracting (10) from (9), we get

$$nxy - 2zx = 0$$

$$\Rightarrow \quad x (ny - 2z) = 0 \quad \Rightarrow \quad ny = 2z \qquad ...(12)$$

Substituting (11) and (12) in (2), we get

$$nx \,.\, x + 4x \,.\, \frac{nx}{2} = S$$

$$\Rightarrow \quad 3nx^2 = S$$

$$\Rightarrow \quad x^2 = \frac{S}{3n}$$

(*i*) When box is open at the top, $n = 1$

$$\therefore \quad x^2 = \frac{S}{3} \quad \Rightarrow \quad x = \sqrt{\frac{S}{3}}$$

The dimensions of the box are:

$$x = y = \sqrt{\frac{S}{3}}, \; z = \frac{1}{2}\sqrt{\frac{S}{3}} \qquad | \text{ Using (12)}$$

(*ii*) When box is closed, $n = 2$

$$\therefore \qquad x^2 = \frac{S}{6} \quad \Rightarrow \quad x = \sqrt{\frac{S}{6}}$$

The dimensions are

$$x = y = z = \sqrt{\frac{S}{6}}. \qquad | \text{ Using (12)}$$

Example 19. *Prove that the stationary values of* $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ *where* $lx + my + nz = 0$ *and* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *are the roots of the equation*

$$\frac{l^2a^4}{1 - a^2u} + \frac{m^2b^4}{1 - b^2u} + \frac{n^2c^4}{1 - c^2u} = 0.$$

Sol. Consider Lagrange's function,

$$F(x, y, z) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda\,(lx + my + nz) + \mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

For stationary values, $dF = 0$

$$\Rightarrow \quad \left(\frac{2x}{a^4} + \lambda l + \frac{2\mu x}{a^2}\right) dx + \left(\frac{2y}{b^4} + \lambda m + \frac{2\mu y}{b^2}\right) dy + \left(\frac{2z}{c^4} + \lambda n + \frac{2\mu z}{c^2}\right) dz = 0$$

$$\Rightarrow \quad \frac{2x}{a^4} + \lambda l + \frac{2\mu x}{a^2} = 0 \qquad ...(1)$$

$$\frac{2y}{b^4} + \lambda m + \frac{2\mu y}{b^2} = 0 \qquad ...(2)$$

$$\frac{2z}{c^4} + \lambda n + \frac{2\mu z}{c^2} = 0 \qquad ...(3)$$

Multiplying (1), (2), (3) by x, y, z respectively and adding, we get

$$2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda\,(lx + my + nz) + 2\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\Rightarrow \quad 2u + \lambda(0) + 2\mu\,(1) = 0 \qquad | \text{ from given relations}$$

$$\Rightarrow \quad \mu = -u$$

$\therefore$ Equation (1) becomes $\frac{2x}{a^4} + \lambda l - \frac{2ux}{a^2} = 0$

or $$\frac{2x}{a^4}\,(1 - a^2u) = -\lambda l \text{ or } x = -\frac{\lambda l a^4}{2(1 - a^2u)}$$

Similarly, $$y = -\frac{\lambda m b^4}{2(1 - b^2u)},\; z = -\frac{\lambda n c^4}{2(1 - c^2u)}$$

To eliminate λ between them, multiply these values of x, y, z by l, m, n respectively and add. Then

$$lx + my + nz = -\frac{\lambda}{2}\left[\frac{l^2a^4}{1 - a^2u} + \frac{m^2b^4}{1 - b^2u} + \frac{n^2c^4}{1 - c^2u}\right]$$

$$\text{Since } lx + my + nz = 0, \text{ we have } \frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$$

which is a quadratic in u and gives two stationary values of u.

NOTES

Example 20. (*i*) *Use Lagrange's method of undetermined multipliers to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$, $xyz + 1 = 0$.*

(*ii*) *If x and y satisfy the relation $ax^2 + by^2 = ab$, prove that the extreme values of the function $u = x^2 + xy + y^2$ are given by the roots of the equation $4(u - a)(u - b) = ab$.*

Sol. (*i*) Here $\quad u = x^2 + y^2 + z^2$

subject to $\quad x + y + z = 1 \quad \text{and} \quad xyz + 1 = 0.$

For a minimum u,

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0 \qquad \text{...(1)}$$

Also, $$dx + dy + dz = 0 \qquad \text{...(2)}$$

and $$yz\,dx + zx\,dy + xy\,dz = 0 \qquad \text{...(3)}$$

Multiplying (1) by 1, (2) by λ and (3) by μ and adding, we get

$$x + \lambda + \mu\,yz = 0 \qquad \text{...(4)}$$

$$y + \lambda + \mu\,zx = 0 \qquad \text{...(5)}$$

$$z + \lambda + \mu\,xy = 0 \qquad \text{...(6)}$$

Subtracting (5) from (4), we get

$$x - y + \mu\,z(y - x) = 0$$

$$\Rightarrow \quad (x - y)(1 - \mu z) = 0$$

$$\Rightarrow \quad x = y \quad \text{or} \quad \mu z = 1$$

Similarly, $\quad y = z \quad \text{or} \quad \mu x = 1 \qquad$ | From (5) and (6)

From these, we choose the solution,

$$x = y \quad \text{and} \quad \mu = \frac{1}{x} \qquad \text{...(7)}$$

Multiplying (4) and (5) by x and y respectively and subtracting (5) from (4), we get

$$x^2 - y^2 + \lambda\,(x - y) = 0$$

$$\Rightarrow \quad (x - y)(x + y + \lambda) = 0$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x + y + \lambda = 0$$

Similarly, $\quad y = z \quad \text{or} \quad y + z + \lambda = 0 \qquad$ ...(8) | From (5) and (6)

From (4), (7) and (8), $\quad x - (y + z) + \dfrac{1}{x}(yz) = 0$

$$\Rightarrow \quad x - (1 - x) + \frac{1}{x}\left(-\frac{1}{x}\right) = 0$$

$$\Rightarrow \quad 2x^3 - x^2 - 1 = 0$$

$$\Rightarrow \quad (1 - x)(2x^2 + x + 1) = 0$$

Second factor gives imaginary roots. A real solution is

$$x = 1,\ y = x = 1,\ z = \frac{xyz}{xy} = -1$$

Hence the minimum $u = 1 + 1 + (-1)^2 = 3$.

Since u can increase indefinitely for numerically large x, y, z therefore the above value is a minimum.

NOTES

(*ii*) Here $$u = x^2 + xy + y^2$$

Consider Lagrange's function

$$F(x, y) = x^2 + xy + y^2 + \lambda\left(\frac{x^2}{b} + \frac{y^2}{a} - 1\right) \quad \ldots(1)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \quad \left[(2x+y) + \lambda\left(\frac{2x}{b}\right)\right]dx + \left[x + 2y + \lambda\left(\frac{2y}{a}\right)\right]dy = 0$$

$$\Rightarrow \quad 2x + y + \frac{2\lambda x}{b} = 0 \quad \ldots(2)$$

$$x + 2y + \frac{2\lambda y}{a} = 0 \quad \ldots(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$2(x^2 + xy + y^2) + 2\lambda\left(\frac{x^2}{b} + \frac{y^2}{a}\right) = 0$$

$$\Rightarrow \quad 2u + 2\lambda(1) = 0 \quad \left|\ \because \frac{x^2}{b} + \frac{y^2}{a} = 1\right.$$

$$\Rightarrow \quad \lambda = -u \quad \ldots(4)$$

From (2), $$2x + y - \frac{2ux}{b} = 0 \quad | \text{ Using (4)}$$

$$\Rightarrow \quad 2\left(1 - \frac{u}{b}\right)x + y = 0 \quad \ldots(5)$$

From (3), $$x + 2y - \frac{2uy}{a} = 0 \quad | \text{ Using (4)}$$

$$\Rightarrow \quad x + 2\left(1 - \frac{u}{a}\right)y = 0 \quad \ldots(6)$$

From (5), $$\frac{x}{y} = \frac{-1}{2\left(1 - \frac{u}{b}\right)}$$

From (6), $$\frac{x}{y} = -2\left(1 - \frac{u}{a}\right)$$

Equating values of $\frac{x}{y}$, we get $$\frac{-1}{2\left(1 - \frac{u}{b}\right)} = -2\left(1 - \frac{u}{a}\right)$$

$$\Rightarrow \quad 4\left(\frac{u}{a} - 1\right)\left(\frac{u}{b} - 1\right) = 1$$

$$\Rightarrow \quad 4(u - a)(u - b) = ab$$

NOTES

Example 21. *Find the maxima and minima of $u = x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically.*

Sol. Let us define a function F such that

$$F = x^2 + y^2 + z^2 + \lambda_1(ax^2 + by^2 + cz^2 - 1) + \lambda_2 (lx + my + nz) \quad \ldots(1)$$

For maxima and minima of F, we have

$$\frac{\partial F}{\partial x} = 2x + 2ax\lambda_1 + l\lambda_2 = 0 \quad \ldots(2)$$

$$\frac{\partial F}{\partial y} = 2y + 2by\lambda_1 + m\lambda_2 = 0 \quad \ldots(3)$$

$$\frac{\partial F}{\partial z} = 2z + 2cz\lambda_1 + n\lambda_2 = 0 \quad \ldots(4)$$

Multiplying eqns. (2), (3), (4) by x, y, z respectively and adding, we get

$$2(x^2 + y^2 + z^2) + 2\lambda_1(ax^2 + by^2 + cz^2) + \lambda_2(lx + my + nz) = 0$$

$\Rightarrow \quad 2u + 2\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = -u$

$\therefore$ From (2), $\quad 2x - 2axu + l\lambda_2 = 0$

$$\Rightarrow \quad x = \frac{l\lambda_2}{2(au - 1)}$$

Similarly, $\quad y = \dfrac{m\lambda_2}{2(bu - 1)}, \quad z = \dfrac{n\lambda_2}{2(cu - 1)}$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$l \cdot \frac{l\lambda_2}{2(au - 1)} + m \cdot \frac{m\lambda_2}{2(bu - 1)} + n \cdot \frac{n\lambda_2}{2(cu - 1)} = 0$$

$$\Rightarrow \quad \frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0$$

which gives the reqd. max. and min. values of u.

Geometrical interpretation. $x^2 + y^2 + z^2$ is the square of the distance of any point (x, y, z) from the origin (0, 0, 0). Hence in this problem, we have found the maximum and minimum values of the square of distance of the origin from the point of intersection of the central conicoid $ax^2 + by^2 + cz^2 = 1$ by the central plane $lx + my + nz = 0$.

TEST YOUR KNOWLEDGE

1. Examine for extreme values:

(*i*) $x^3 + y^3 - 3xy$

(*ii*) $3x^2 - y^2 + x^3$

(*iii*) $x^2y^2 - 5x^2 - 8xy - 5y^2$

(*iv*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

(*v*) $x^2 - xy + y^2 + 3x - 2y + 1$

(*vi*) $xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$

(*vii*) $\sin x \sin y \sin (x + y)$

(*viii*) $\cos x + \cos y + \cos (x + y)$

(*ix*) $x^3y^2(1 - x - y)$

(*x*) $2(x - y)^2 - x^4 - y^4$

(*xi*) $x^2 - xy + y^2 - 2x + y$

(*xii*) $x^4 + y^4 - x^2 - y^2 - 1$

(*xiii*) $x^3 + y^3 - 12x - 3y + 20$

(*xiv*) $x^2 + xy + y^2 + \dfrac{1}{x} + \dfrac{1}{y}$.

NOTES

2. (*i*) Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.
   (*ii*) Divide a number, say 120 into three parts so that the sum of their product taken two at a time shall be maximum.
3. (*i*) Find the maximum and minimum values of $ax + by$ when $xy = c^2$.
   (*ii*) Find the minimum value of $x^2 + y^2$ when $ax + by = c$.
4. Find the stationary points of the function $z = x^3y^2(12 - x - y)$ satisfying the condition $x > 0$, $y > 0$ and examine their nature.
5. Find the minimum values of $x^2 + y^2 + z^2$ when
   (*i*) $x + y + z = 3a$ (*ii*) $xyz = a^3$.
6. Find the minimum value of the function $x + y + z$ subject to the condition $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.
7. Find the minimum value of $ax + by + cz$ subject to the condition $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 1$.
8. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition
   (*i*) $ax + by + cz = a + b + c$ (*ii*) $xy + yz + zx = 3a^2$.
9. (*i*) Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
   (*ii*) Find the shortest distance from origin to the surface $xyz^2 = 2$.
10. (*i*) Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.
    (*ii*) Use Lagrange's method to determine the minimum distance from the origin to the plane $3x + 2y + z = 12$.
11. Show that the maximum and minimum values of $r^2 = x^2 + y^2$ where $ax^2 + 2hxy + by^2 = 1$ are given by the roots of the quadratic equation $\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2$.
12. Find the extreme value of $x^2 + y^2 + z^2$ subject to the condition $xy + yz + zx = p$.
13. (*i*) Find the shortest and the longest distances from the point (1, 2, – 1) to the sphere
    $$x^2 + y^2 + z^2 = 24.$$
    (*ii*) Find the maximum and minimum distances from the origin to the curve
    $$5x^2 + 6xy + 5y^2 - 8 = 0.$$
14. Find the maxima and minima of $u = x^2 + y^2 + z^2$ where $ax^2 + by^2 + cz^2 = 1$.
15. The temperature 'T' at any point (x, y, z) in space is $T(x, y, z) = Kxyz^2$ where K is constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
16. If $u = a^3x^2 + b^3y^2 + c^3z^2$, where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by
    $$x = \frac{\Sigma a}{a}, y = \frac{\Sigma a}{b}, z = \frac{\Sigma a}{c}.$$
17. (*i*) Find the maximum value of $x^m y^n z^p$, when $x + y + z = a$.
    (*ii*) Find the maximum value of $x^p y^q z^r$ when the variables x, y, z are subject to the condition
    $$ax + by + cz = p + q + r.$$
18. (*i*) A rectangular box, which is open at the top, has a capacity of 32 c.c. Determine, using Lagrange's method of multipliers, the dimensions of the box such that the least material is required for the construction of the box.
    (*ii*) A rectangular box, which is open at the top, has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box. Use Lagrange's method of multipliers to obtain the solution.

NOTES

19. If $u = ax^2 + by^2 + cz^2$ where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$, prove that stationary values of u satisfy the equation $\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0$.

20. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1) using the method of Lagrange's multipliers.

Answers

1. (*i*) Min. value = – 1 at (1, 1) (*ii*) Max. value = 4 at (– 2, 0)
(*iii*) Max. value = 0 at (0, 0)
(*iv*) Max. value = 112 at (4, 0) ; Min. value = 108 at (6, 0)
(*v*) Min. value $= -\frac{4}{3}$ at $\left(-\frac{4}{3}, \frac{1}{3}\right)$ (*vi*) Min. value $= 3a^2$ at (a, a)
(*vii*) Max. value $= \frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$; Min. value $= -\frac{3\sqrt{3}}{8}$ at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$
(*viii*) Min. value $= -\frac{3}{2}$ at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ (*ix*) Max. value $= \frac{1}{432}$ at $\left(\frac{1}{2}, \frac{1}{3}\right)$
(*x*) Max. value = 8 at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.
(*xi*) Min. at (1, 0), Min. value = – 1
(*xii*) Max. at (0, 0), Max. value = 1
Min. at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Min. value $= \frac{1}{2}$
Saddle pts. at $\left(0, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$
(*xiii*) Min. at (2, 1), Min. value = 2
Max. at (– 2, – 1), Max. value = 38
Saddle pts. at (– 2, 1) and (2, – 1).
(*xiv*) Min. at $\left(\frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right)$, Min value $= 3^{4/3}$

2. (*i*) 4, 8, 12 (*ii*) 40, 40, 40

3. (*i*) Min. value $= 2c\sqrt{ab}$, Max. value $= -2c\sqrt{ab}$ (*ii*) $\frac{c^2}{a^2+b^2}$

4. Maximum at (6, 4) **5.** (*i*) $3a^2$ (*ii*) $3a^2$

6. $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$ at $\{\sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c}), \sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c}), \sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c})\}$

7. $(a^{3/2} + b^{3/2} + c^{3/2})^2$

8. (*i*) $\frac{(a+b+c)^2}{a^2+b^2+c^2}$ (*ii*) $3a^2$ at (a, a, a) and $(-a, -a, -a)$

9. (*i*) (0, 0, ± 1) (*ii*) 2

10. (*i*) 3 (*ii*) 3.2071

12. p **13.** (*i*) $\sqrt{6}, 3\sqrt{6}$ (*ii*) 4, 1

14. Maximum and minimum values of u are the roots of $\left(u-\frac{1}{a}\right)\left(u-\frac{1}{b}\right)\left(u-\frac{1}{c}\right)=0$

15. $\frac{ka^4}{8}$

17. (i) $\frac{m^m n^n p^p \, a^{m+n+p}}{(m+n+p)^{m+n+p}}$. (ii) $\left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r$

NOTES

18. (i) $x = 4, y = 4, z = 2$ (ii) $x = 8, y = 8, z = 4$

20. $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$

SUMMARY

- If u and v are the functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the **Jacobian** or functional determinant of u and v with respect to x and y. It is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u, v)$.

- If u, v and w are the functions of three independent variables x, y and z, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is called the Jacobian of u, v and w with respect to x, y and z. It is written as

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ or } J(u, v, w).$$

- Similarly if $u_1, u_2, ..., u_n$ are the functions of independent variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, ..., u_n$ with respect to variables $x_1, x_2, ..., x_n$. It is written as

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} \quad \text{or} \quad J(u_1, u_2, ..., u_n).$$

NOTES

- If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}.$$

- If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then

$$J_1J_2 = 1 \text{ i.e., } \quad \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)} = 1.$$

- If functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

Jacobian of Implicit Functions:

- If u_1, u_2 and u_3 are the implicit functions of x_1, x_2, x_3 *i.e.*,

  $F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

  $F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

  $F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

  then, $$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}=(-1)^3\left[\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)}\Big/\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}\right].$$

- ***Functional Relationship:*** "Let $u_1, u_2, u_3, \ldots, u_n$ be functions of $x_1, x_2, x_3, \ldots, x_n$. Then the necessary condition for the existence of a relation of the form $F(u_1, u_2, \ldots, u_n) = 0$ is that the Jacobian $\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}$ should vanish identically."

- The inverse function theorem states, roughly speaking,that a continuously differentiable mapping f is invertible in neighbourhood of any point x at which the linear transformation $f'(x)$ is invertible:

- If f is a continuously differentiable real function in the plane, then the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighbourhood of any point (a, b) at which $f(a, b) = 0$ and $\partial f/\partial y \neq 0$. Likewise, one can solve for x in terms of y near (a, b) if $\partial f/\partial x \neq 0$ at (a, b). For example consider $f(x, y) = x^2 + y^2 - 1$ and a point $(0, 1)$ so that (*i*) $f(0, 1) = 0$ and (*ii*) $\partial f/\partial y \neq 0$ at $(0, 1)$. Here the possible solutions are $y_f = +\sqrt{(1-x^2)}$ is the implicit function in a neighbourhood of $(0, 1)$, where $|x| < 1, y > 0$ and $y = -\sqrt{(1-x^2)}$ is the implicit function in a neighbourhood of $(0, 1)$ where $|x| < 1, y < 0$. Our next result is the very informal statement is the simplest case (the case $m = n = 1$ of ***"implicit function theorem"***.

- A function $f(x, y)$ is said to have a **maximum value at $x = a, y = b$ if**

  $f(a, b) > f(a + h, b + k)$, for small and independent values of h and k, positive or negative. A function $f(x, y)$ is said to have a **minimum value at $x = a$, $y = b$** if

  $f(a, b) < f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

- Lagrange's method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical considerations of the problem.

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