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Classical Mechanics

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Contents

List of Figures

UNIT 1

Newtonian Dynamics

Preparatory inputs to this unit

1. Concept of space and time.
2. Coordinate systems: Both 2-D and 3-D cartesian and polar oordinates.
3. Components of position, velocity and acceleration in cartesian and polar coordinates.
4. Basics of vector algebra.

1.1 Introduction

Newtonian dynamics provides us with a mathematical model to predict the motions of objects that we encounter in the world around us. The general principles of this model were first given by Sir Isaac Newton in the form of three laws to describe motion. The entire body of his work on dynamics was written down in a book entitled *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), first published in 1687.

Until the beginning of the 20th century, Newton's theory of motion was thought to constitute a complete description of all types of motion occurring in the Universe. The modern view is that Newton's theory is only an approximation which is valid under certain circumstances.

Newtonian dynamics attempts to connect mass, position, time, and the force for describing the motion and the causes of motion. All other propositions in regards of motion can be derived from the existing basic motional quantities by logical and mathematical analysis. The axioms with which the analysis of motion start are the Newton's laws of motion, which can only be established via experimental observations. Newton's laws, basically apply to point objects. However, they can be expanded to include extended objects by treating the later as collections of point particles. Newtonian dynamics has been found to predict results that are in excellent agreement with experimental observations.

1.2 Newton's Laws of Motion

Newton put forward three laws to provide a complete description of the motion of particles. They are known as the Newton's first, second and the third law of motion, which are stated below.

1. Newton's first law: *Everybody continues its state of rest or of motion unless an external force is applied on it.* This is also called the *law of inertia.*

2. Newton's second law: *The rate of change of linear momentum of a body is proportional to the impressed force and takes place along the direction of the force.* This is the *law of casuality.*

3. Newton's third law: *To every action there is an equal and opposite reaction.* In other words, the mutual action of any two particles are always equal and oppositely directed, along the same straight line. This law is known as the *law of reciprocity.*

Newton's first law of motion states that a point object subject to zero net external force moves in a straight line with a constant speed (*i.e.*, it does not accelerate). However, this is only true in special frames of reference called inertial frames. We can think of Newton's first law as the definition of an inertial frame. An inertial frame of reference is one in which a point object experiences zero net external force and moves in a straight line with constant speed.

The Newtons first law is probably the first instance where geometry (straight line) connects with physics (force, velocity). This connection of geometry with physics was later well established in the general theory of relativity put forward by Albert Einstein in 1915.

Newton's second law of motion essentially states that if a point object is subjected to an external force, $\vec{f}$, the resulting motion in the body can be expressed in terms of an equation, called the equation of motion, given by

$$\frac{d\vec{p}}{dt} = \vec{f},$$

where the linear momentum $\vec{p}$, is the simple product of the inertial mass of the body, m, and its velocity $\vec{v}$, *i.e.*, $\vec{p} = m\vec{v}$. If m is not a function of time then the above expression reduces to

$$m\frac{d\vec{v}}{dt} = \vec{f}.$$

Now consider a system of N mutually interacting point objects. Let the i-th object, with mass m_i, be located at a position described by the position vector $\vec{r}_i$. Let this object exert a force $\vec{f}_{ji}$ on the j-th object. The j-th object also in its turn, exerts a force $\vec{f}_{ij}$ on the i-th object. *Newton's third law of motion* essentially states that these two forces are equal and opposite; the *action* and the *reaction*, which are essentially forces, are oppositely directed irrespective of their nature. In other words,

$$\vec{f}_{ij} = -\vec{f}_{ji}.$$

One corollary of Newton's third law of motion is that an object cannot exert a force on itself. Another corollary is, all forces in the Universe have corresponding reactions. The only exceptions to this rule are the *fictitious forces* which arise in non-inertial reference frames and does not have any physical origin.

1.3 Mechanics of a single particle

1.3.1 The equation of motion

Let us consider a single point particle of mass m to be at a location $\vec{r}$, the position vector of the particle in, say, cartesian coordinates at a given moment of time t. Consider further that the particle experiences a force $\vec{f}$ at the said moment, due to which the particle gets accelerated. The acceleration can be expressed as the second order time derivative of the position vector of the particle. The linear momentum of the particle is then given by the product of its mass m and the linear velocity $\vec{v} = \frac{d\vec{r}}{dt}$. Thus, the linear momontum of the particle in motion is given by

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}. \tag{1.3.1}$$

The equation of motion, according to the Newton's second law, is

$$\vec{f} = \frac{d\vec{p}}{dt} \tag{1.3.2}$$

The momentum $\vec{p}$ can be written as the rate of change of the position vector $\vec{r}$. (equation (1.3.1)). So we can write (1.3.2) as

$$\vec{f} = m\frac{d^2\vec{r}}{dt^2} \tag{1.3.3}$$

Thus, the applied force on the moving particle can be found from the product of the mass and the acceleration induced in the particle.

The following result in regards of the Newton's second law is in order.

Statement: *Newton's first law is a special case of Newton's second law.*

Proof: In classical mechanics, the mass of a particle is considered a non-zero invariant quantity, *i.e.*, the mass m is non-zero and constant during the course of motion. Therefore,

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt}.$$

If the particle experiences an external force $\vec{f}$, we have,

$$\vec{f} = m\frac{d\vec{v}}{dt}$$

When the external impressed force $\vec{f}$ is absent, *i.e.*, if $\vec{f} = 0$, then we have

$$\begin{aligned} & m\frac{d\vec{v}}{dt} = 0 \\ \text{or,} \quad & \frac{d\vec{v}}{dt} = 0 \\ \Longrightarrow \quad & \vec{v} = \text{constant.} \end{aligned}$$

This means that the velocity $\vec{v}$ remains unchanged during motion. If $\vec{v} = 0$ initially, it will continue to remain $\vec{v} = 0$ during the entire course of motion when there is no force applied to the particle. Similarly if $\vec{v}$ is some non-zero vector initially, the vector will not change during the course of motion. In other words, a body at rest will continue to be at rest and when in motion, the velocity remains unchanged, *i.e.*, remain unaccelerated. This is the Newton's first law of motion.

Thus we can see that Newton first law is a special case of Newton's second law, under the case of no external force being applied to the particle.

1.3.2 Conservation of Linear momentum

The linear momentum of a moving paticle is the product of the mass and the velocity of the moving particle. The mass of a particle is a scalar and the velocity is a vector quantity. Since the product of a scalar and a vector quantity is always a vector, the linear momentum of a particle is a vector quantity.

If the particle does not experience any external force, we have $\vec{f} = 0$. This reduces Newoton's second law of motion to

$$\frac{d\vec{p}}{dt} = 0$$

Integrating with respect to time, we get

$$\vec{p} = \text{constant in time}$$

The linear momentum vector remains unchanged during the motion. Thus we see that if a particle is not acted on by any external force, particle conserves its linear momentum. This is the *conservation of linear momentum* for a single particle.

1.3.3 Conservation of angular momentum

Angular momentum

When a particle moves about a point, it executes curvilinear motion. Under such circumstances, the angular momentum is the relevant quantity to describe the curvilinear motion of the particle. The instantaneous distance of the particle has also role to play in describing the curvilinear motion. The angular momentum is the quantity defined for curvilinear motion to mimic the role of the linear momentum in the case of linear motion. The angular momentum of a particle is defined as the vector product of the distance vector (the vector defined from the point about which the particle executes curvilinear motion, to the particle) and the linear momentum of the particle, *i.e.*,

$$\vec{L} = \vec{d} \times \vec{p} = \vec{d} \times m\vec{v}.$$

If the origin of the coordinates coincides with the point about which a particle executes curvilinear motion, then $\vec{d} = \vec{r}$. Under such circumstances, we have

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\frac{d\vec{r}}{dt}$$

Torque

For the motion of a particle about a point, the quantity equivalent to the force in case of linear motion is the torque. The torque on a particle is defined as the vector product of the distance vector (from the point about which the particle moves, to the point of application of the force vector) and the force vector itself, *i.e.*, the torque $\vec{\tau} = \vec{d} \times \vec{f}$, where $\vec{d}$ is the distance vector and $\vec{f}$ is the force vector. Since $\vec{\tau}$ is the result of a cross product of two vectors, it is a vector quantity.

If the origin of the coordinates coincides with the point about which the particles executes curvilinear motion, we have $\vec{\tau} = \vec{r} \times \vec{f}$.

Relation between angular momentum and torque

The angular momentum of a single particle about the origin of the coordinates, is given by

$$\vec{L} = \vec{r} \times m\frac{d\vec{r}}{dt}$$

Differentiating this with respect to time, t :

$$\frac{d\vec{L}}{dt} = m\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times m\frac{d^2\vec{r}}{dt^2}$$

The first term vanishes because of the vector product between two identical vectors. Therefore,

$$\frac{d\vec{L}}{dt} = \vec{r} \times m\frac{d^2\vec{r}}{dt^2} = \vec{r} \times \vec{f} = \vec{\tau}$$

Thus,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{1.3.4}$$

Conservation of angular momentum

If the external torque on a particle is zero then $\vec{\tau} = 0$. Therefore, (1.3.4) reduces to

$$\frac{d\vec{L}}{dt} = 0, \qquad \Longrightarrow \qquad \vec{L} = \text{constant}$$

Thus, if a particle is not acted on by any external torque, the particle conserves its angular momentum during its curvilinear motion. This is the *conservation of angular momentum* for a single particle.

1.3.4 Conservation of energy

Work done on a particle and kinetic energy

According to Newton, a force causes a particle to undergo acceleration. But it is pertinent to know what happens between the application of the force and the resulting accelerating motion in the particle. We can interprete that the application of a force on a particle causes the latter to move instantaneously, or creates a situation to preserve *something* on the particle so that the latter is capable of a response *e.g.*, motion afterwards. In other words, the force can be considered to have transferred some *entity* which is either released immediately to cause motion in the particle, or is stored in the particle depending on the particle's ambience or configuration. This stored entity in the particle empowers it to be capable of action, to cause motion at later times when the entity gets opportunity to be released from the particle. This entity is known as the energy of the particle. The mechanism of transfer of energy from the force to the particle is known in simple terms as the work done by the force on the particle.

Thus we can say that energy is transferred from the source of force to the particle through some work done on the particle. When the energy is re-expressed as motion in the particle instantaneously on the application of the force, it is said to possess kinetic energy, the energy of the particle by virtue of its motion, as the moving particle can impart some action on other bodies.

The work done by a given force on a particle is defined as the dot product of the force with the infinitesimal vector distance during which an acceleration is induced on the particle.

Kinetic energy for a particle

The kinetic energy of a moving particle is the energy it possesses by virtue of its motion. To deduce the form of the kinetic energy in a single particle, we consider that a force $\vec{f}$ is applied for an infintesimal duration dt to the particle moving with a velocity $\vec{v}$. This force displaces the particle by an infinitesimal amount $d\vec{r}$, and hence an acceleration, *i.e.*, change in the velocity over time dt in the particle. Thus

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

This acceleration is related to the force by the Newton's second law of motion, *viz.*,

$$\vec{f} = m\frac{d\vec{v}}{dt}$$

The infinitesimal amount of work done by the force to the said displacement is then,

$$dW = \vec{f}.d\vec{r}$$

Since the work done on the particle is re-expressed as the kinetic energy of motion of the particle, the incremental change in the kinetic energy is given by

$$dT = dW = \vec{f}.d\vec{r} = m\frac{d\vec{v}}{dt}.\frac{d\vec{r}}{dt}dt = m\vec{v}.\frac{d\vec{v}}{dt}dt = m\vec{v}.d\vec{v} = mvdv$$

where v is the magnitude of the velocity vector $\vec{v}$. Here it is obvious the the vectors $\vec{v}$ and $d\vec{v}$ have the identical direction.

If the particle starts from rest and accelerate under the given force, the total work W done by the force in raising the velocity of the particle from 0 to $\vec{v}$ can be found by integrating the above expression, for velocities from 0 to v, *i.e.*,

$$T = W = \int_0^v dW = \int_0^v mvdv = \frac{1}{2}mv^2 \qquad (1.3.5)$$

Thus the kinetic energy of a particle of mass m and moving with a velocity $\vec{v}$ is, $T = \frac{1}{2}mv^2$.

Conservative force

If the work done by a force on a particle for a displacement, does not depend on the actual path of motion of the particle, the force is called a conservative force. In such a case the work done does depend only on the initial and final location of the particle. Thus, for a conservative force $\vec{F}$ acting on a particle in displacing it from location A to B, the amount of work done along the path ACB is the same as that along ADB.

In the figure (1.1) we see that the work done by the conservative force along the paths ACB and ADB are equal, so the work done on the particles is path independent, *i.e.*,

$$W = \int_{ACB} \vec{F}.d\vec{r} = \int_{ADB} \vec{F}.d\vec{r}$$

Figure 1.1: Path independence of the work done for a conservative force field.

But as the work done through the reverse path is negative, we have

$$\int_{ADB} \vec{F}.d\vec{r} = -\int_{BDA} \vec{F}.d\vec{r}$$

$$\Longrightarrow \quad \int_{ACB} \vec{F}.d\vec{r} + \int_{BDA} \vec{F}.d\vec{r} = 0,$$

$$\Longrightarrow \quad \int_{ACBDA} \vec{F}.d\vec{r} = 0$$

$$\text{or,} \quad \boxed{\oint \vec{F}.d\vec{r} = 0}$$

i.e., the work done by a conservative force round a closed curve $ACBDA$ vanishes. In terms of vector integration, we say that the contour integration of a conservative force vanishes.

Conservative force field and potential energy

We have already seen that for the conservative forces, the work done round a closed path is zero, *i.e.*, $\oint \vec{F}.d\vec{r} = 0$. Using Stokes' theorem of Vector Calculus, this integral can be converted to the surface integral over a surface whose boundary is defined by the contour as above. Thus we write,

$$\oint \vec{F}.d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right).d\vec{S} = 0$$

Since the elemental surface $d\vec{S}$ is arbitrary, the integrand must equal zero, *i.e.*,

$$\boxed{\vec{\nabla} \times \vec{F} = 0.} \tag{1.3.6}$$

This is the necessary and sufficient condition for $\vec{F}$ to be a conservative force.

Again, from the Vector Calculus we know that for a scalar function ϕ the following identity holds:

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \tag{1.3.7}$$

Considering (1.3.6) and (1.3.7) we can see that a consvervation force vector $\vec{F}$ can be expressed as a gradient of a scalar, to an arbitrary constant c as

$$\vec{F} = c\vec{\nabla}\phi$$

In order to match with the experimental results of newtonian dynamics, the value of c is set as $c = -1$. So

$$\vec{F} = -\vec{\nabla}\phi$$

Thus, a conservative force function can always be expressed as a negative gradient of a scalar function. This scalar function is known as the *potential energy function* associated with the particle that is in motion under a conservative force field. Physically, the potential energy function of a particle at a given point within a conservative force field is the total amount of work done on a particle in bringing it from infinity, *i.e.*, from outside the conservative force field, to the given point, *i.e.*,

$$\phi(r) = \int_{\infty}^{r} \vec{F} \cdot d\vec{r}$$

The potential energy of a particle at a given point in a conservative force field is thus the potential of the point multiplied by the mass of the particle.

Conservation of total energy under conservative force field

Let a particle moving under a conservative force field $\vec{F}(r)$ with a velocity, say v_A, located at a given point A be brought to point B, where the velocity of the particle becomes v_B. Let the potential energy function at point A be ϕ_A and at B it is ϕ_B under the conservative force field. The work done in bringing the particle from the point A to point B by the conservative force is then

$$W = \int_A^B \vec{F}.d\vec{r} = \int_A^B -\vec{\nabla}\phi.d\vec{r} = -\int_A^B d\phi = \phi_A - \phi_B \tag{1.3.8}$$

This amount of work done W on the particle serves to change the velocity of the particle from v_A to v_B which changes the kinetic energy of the particle from T_A to T_B. The change in the kinetic energy is thus

$$T = T_B - T_A = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \tag{1.3.9}$$

Since $T = W$, (refer (1.3.5)) we combine (1.3.9) and (1.3.8) to write

$$\phi_A - \phi_B = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2,$$

or,

$$\phi_A + \frac{1}{2}mv_A^2 = \phi_B + \frac{1}{2}mv_B^2. \quad (1.3.10)$$

The sum total of potential energy and the kinetic energy at position A is the same as that at position B. This sum total is called the total energy of the particle. In other words, the total energy of the particle remains constant, *i.e.*, conserved within a conservative force field. This is the *conservation of total energy* in the single particle dynamics.

Examples and numerical problems

Example 1.3.1 *A particle of mass m is thrown vertically upwards under gravity with an initial velocity $v_0 = 20$ m/sec. and descends back on the earth's surface after attaining a height h. Calculate*

(a) the maximum height the particle attained

(b) the velocity at half the maximum height, both during ascending and descending. of the maximum height.

Solution: The equation of motion of the given particle is

$$\frac{d^2x}{dt^2} = -g$$

$$\frac{dx}{dt} = -gt + A,$$

$$\implies \quad x = -\frac{1}{2}gt^2 + At + B$$

where A and B are constants of integration.
In this case, the boundary conditions here are:

(i) at $x = 0$, the time $t = 0$ and the velocity $v_0 = 20$ m/sec

(ii) at the maximum height h, the velocity $v = 0$ m/sec

Now,

(a) Let the time taken to reach the height be T.

Now, at $x = 0$ we have

$$v_0 = 0 + A, \quad \implies A = v_0 = 20\ m/sec$$

$$0 = 0 + 0 + B, \quad \implies B = 0$$

And at $x = h$,

$$0 = -gT + A, \quad \implies T = \frac{A}{g}$$

$$h = -\frac{1}{2}g\left(\frac{A}{g}\right)^2 + \frac{A^2}{g} + 0, \quad \Longrightarrow h = \frac{A^2}{2g}$$

Hence,

$$h = \frac{400}{2 \times 9.81} \approx 20.39 \; metres$$

(b) At half the maximum height the time taken is, say, t' seconds and velocity be v'. Then

$$v' = -gt' + A$$

and

$$\frac{h}{2} = -\frac{1}{2}gt'^2 + At'$$

Combining,

$$(v' - A)^2 + 2A(v - A) + gh = 0 \qquad \Longrightarrow v' = \pm\sqrt{A^2 - gh}$$

Putting $A = v_0 = 20 \; m/sec,$ $g = 9.81 \; m/sec^2$ and $h = 20.39 \; m$, we get

$$v' = 14.14 \; m/sec, \; -14.14 \; m/sec$$

So these are the two moments of time and hence two values of velocity corresponding to a given height-one on ascending and the other during descending. Interestingly, the magnitudes of the velocities are equal but the directions are opposite. The positive velocity corresponds to velocity while ascending and the negaive is during the descending time, both at the same height.

Example 1.3.2 *A ball is falling freely from a height h under the action of gravity of the earth. Assuming that the atmosphere offers a resistive force proportional to the velocity,*

1. *Find the expression of the velocity of the ball as a function of time.*

2. *The terminal velocity of a body is that uniform veloicty which the body assumes after the moment when the downward acceleration is exactly counteracted by the resistive force of the atmosphere. Find the terminal velocity for the ball.*

Solution: Consider the body to be falling vertically down along the positive z-direction. Further let the mass of the ball be m, the acceleration due to gravity g and the instantaneous velocity be $v = \frac{dz}{dt}$. The resistive force of the atmosphere is taken as $f = -km\vec{v}$. The negative sign here indicates that the force acts along the direction opposite to that of velocity.

1. Newton's second law of motion for this case yields,

$$m\frac{dv}{dt} = mg - kmv, \qquad k \text{ is the constant of proportionality, characteristics of the atmosphere.}$$

$$\text{or,} \qquad \frac{dv}{dt} + kv = g,$$

which is a first order ordinary differential equation in v and can be solved by standard method, *i.e.*,

$$\frac{1}{k}\frac{d(kv-g)}{(kv-g)} = -dt$$

Integrating,

$$kv - g = Ce^{-kt},$$

C is the constant of integration. For the free fall from a height, the initial velocity is zero, *i.e.*, at $t = 0$, we have $v = 0$. This gives $C = -g$. The solution is then given by

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

which is the required expression of velocity as a function of time.

2. To find the velocity after long enough time we have $t \longrightarrow \infty$. Putting this into the expression for velocity above,

$$v \longrightarrow \frac{g}{k} = constant$$

i.e., the body on descending through the resistive atmosphere assumes a constant velocity after sufficiently long time. This is the terminal velocity. So, the terminal velocity for the given situation is

$$v_T = \frac{g}{k}$$

We see that the terminal velocity is governed by the acceleration due to gravity g and the characteristics of the atmosphere, k.

Example 1.3.3 *A bob of mass m is tied at one end of an inextendible and weightless string of length l which is suspended from the other end at a point so that the bob can swing in one plane. When the amplitude of swing is kept small, the system is called a* simple pendulum. *Write down the differential equation for a simple pendulum.*

Solution: When undisturbed, let the bob be at A, hung vertically along OA. This position is the equilibrium position for the bob, because the weight (downward force due to earth's gravity) of the bob is counterbalanced by the upward tension force in the string. When the bob is displaced to B, it makes at the point O an angle θ, say, with the vertical line. Two forces acting on the bob are the downward pull of gravity with magnitude mg and the Tension force T along BO in the string. The force of gravity acting on the bob is resolved as follows:

1. $mg\cos\theta$ along OB, which counterbalances the tension force T in the string, and

2. $mg\sin\theta$ which creates a restoring torque $\vec{\tau} = -mgl\sin\theta\ \hat{\theta}$ trying to rotate the bob back towards the equilibrium position, A. Here $\hat{\theta}$ is the unit vector, taken to be positive for counterclockwise rotation.

The angular momentum of the bob is

$$\vec{L} = ml\frac{d\theta}{dt}\,\hat{\theta}.$$

Hence the equation of motion for the bob, in terms of magnitude, is

$$\frac{dL}{dt} = \tau$$

$$\text{or,} \qquad \frac{d}{dt}\left(ml\frac{d\theta}{dt}\right) = -mgl\sin\theta$$

$$\Longrightarrow \qquad \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

For sufficiently small θ, we can write $\sin\theta \approx \theta$. So,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{1.3.11}$$

$$\text{or,} \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \tag{1.3.12}$$

This is the standard ordinary differential equation of an oscillator

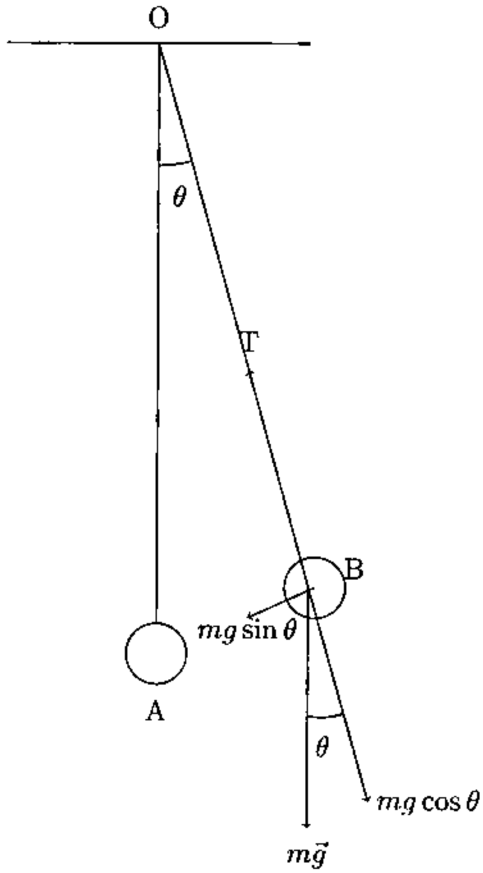
$$\frac{d^2y}{dt^2} = -\omega^2 y$$

where the variable y, (here θ, *i.e.*, the angle the string makes with the vertical line), oscillates with respect to time t and the angular frequency of oscillation is given as $\omega = \frac{2\pi}{\mathsf{T}}$ (T being the period of oscillation), is the coefficient of the variable $-y$ on the right hand side. So, here we find

$$\omega = \frac{2\pi}{\mathsf{T}} = \sqrt{\frac{g}{l}},$$

from which we can calculate the time period of oscillation T for the pendulum as

$$\mathsf{T} = 2\pi\sqrt{\frac{l}{g}}$$



Example 1.3.4 *Explain the law of conservation of total energy in the case of an oscillating simple pendulum.*

Solution: In the case of a simple pendulum when the bob is at the extreme left, it has the maximum potential energy, as it is raised with respect to the mean position. But its kinetic energy is zero as the bob stops oscillating for a fraction of a second before moving towards the right. When the bob reaches the mean position, it has a zero potential energy but maximum kinetic energy.

Similarly when the bob of the pendulum swings to extreme right, it has the maximum potential energy but zero kinetic energy. Thus law of conservation of energy holds clearly in this case of an oscillating pendulum.

1.4 A system of Multiple Particles in Newtonian Dynamics

In the earlier section we discussed the dynamics and conservation properties of a single particle. But in real world, we need to deal with material objects consisting of large number of particles. Thus the development of a formalism in the dynamics of a system consisting of many particles is in order. Here we deduce the equations for n particles in Newtonian mechanics and find related conserved quantities.

1.4.1 Mechanics of a system of particles

Let us consider a system consisting of n particles with masses $m_1, m_2, \ldots, m_n$ and the corresponding position vectors $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ respectively. Since the system is composed of more than one particle, the possibility of any interaction among the particles themselves cannot be denied. So, the forces that act on particles include external forces acting on the particles from outside as well as the internal forces arising from the interactions among the constituent particles.

Suppose the i-th particle of the system experiences external force $\vec{F}_i^e$ and internal force $\vec{F}_{ji}$ due to the j-th particle of the system. Then the total force $\vec{F}_i$ acting on the i-th particles is simply

$$\vec{F}_i = \vec{F}_i^e + \sum_{\substack{j \\ i \neq j}} \vec{F}_{ji}$$

If the linear momentum of the i-th particle is $\vec{p_i}(= m_i\vec{v_i} = m_i\dot{\vec{r_i}})$, then according to Newton's second law, the rate of change of the linear momentum of the i-th particle equals the sum of the external forces and all the internal forces acting on the i-th particle, *i.e.*,

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i^e + \sum_{i=1}^{n} \vec{F}_{ji}$$

The total force, external and internal, acting on the system is given by taking the sum of the expression above, *i.e.*,

$$\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{n} \vec{F}_i^e + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ji} \tag{1.4.1}$$

The term $\sum_{i=1}^{n} \vec{F}_i^e$ represents the total external force $\vec{F}^e$ applied on the system. Since $\vec{F}_{ji}$ represents the internal force acting on i-th particle due to j-th particle and $\vec{F}_{ij}$ being the internal force on the j-th particle due to i-th particle, their vector sum cancels because of the Newton's third law of motion, one representing the action and the other the reaction. So, $\vec{F}_{ji} + \vec{F}_{ij} = 0$. Hence,

$$\begin{aligned}\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} &= \sum_{i=1}^{n} \frac{d}{dt}\left(m_i\dot{\vec{r}}_i\right) \\ &= \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i\vec{r}_i\right)\end{aligned}$$

Here the mass of each individual particle of te system is considered constant. We define a vector $\vec{R}$ as

$$\vec{R} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i\vec{r}_i}{M},$$

where M is the total mass of the system, is again a constant. The vector $\vec{R}$ defines a point known as the *Center of Mass* of the system. The above equation then reduces to

$$\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i\vec{r}_i\right) = \frac{d^2}{dt^2}\left(M\vec{R}\right) = M\frac{d^2\vec{R}}{dt^2}$$

$$\text{Therefore,} \quad \vec{F}^e = M\frac{d^2\vec{R}}{dt^2} \tag{1.4.2}$$

Linear momentum for a system of particles

The total linear momentum of the system of n particles is given by the vector sum of the linear momenta corresponding to the constituent particles of the system, *i.e.*,

$$\vec{P} = \sum_{i=1}^{n} m_i\vec{v}_i = \sum_{i=1}^{n} m_i\dot{\vec{r}}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i\dot{\vec{r}}_i\right) = M\frac{d\vec{R}}{dt}$$

or,

$$\frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2} \tag{1.4.3}$$

From equations ((1.4.2)) and ((1.4.3)) we have

$$\vec{F}^e = \frac{d\vec{P}}{dt} \tag{1.4.4}$$

Conservation theorem for linear momentum of a system of particles

If the total external force acting on a system of particles is zero, *i.e.*, if $\vec{F}^e = 0$, we can write equation (1.4.4) as

$$\frac{d\vec{P}}{dt} = 0$$

$$\Longrightarrow \quad \vec{P} = \text{constant}$$

This means, the total linear momentum of the system will not change and remain constant throughout the motion of the system so long as there is no external force acting on the system. In short, we infer that

For an isolated system, i.e., if the system is not acted on by any external force, the total linear momentum of the system is conserved.

This is known as the *conservation of linear momentum* for a system of many particles.

Angular Momentum for a System of Particles

Let $\vec{L}_i$ be the angular momentum, $\vec{r}_i$ the position vector and $\vec{p}_i$ be the linear momentum of i-th particle of the system.

$\therefore$ Angular momentum $= \vec{L}_i = (\vec{r}_i \times \vec{p}_i)$

Let total angular momentum be $\vec{L}$, which is the sum of all the individual angular momenta of the constituent particles. Then

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} (\vec{r}_i \times \vec{p}_i)$$

Now,

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \frac{d}{dt}\left\{\sum_{i=1}^{n}(\vec{r}_i \times \vec{p}_i)\right\} \\
&= \sum_{i=1}^{n}\left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i\right) + \sum_{i=1}^{n}\left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right) \\
&= 0 + \sum_{i=1}^{n}\left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right) \qquad \left[\because \quad \frac{d\vec{r}_i}{dt} \times \vec{p}_i = \vec{v}_i \times m_i\vec{v}_i = m_i(\vec{v}_i \times \vec{v}_i) = 0\right] \\
&= \sum_{i=1}^{n}\left[\vec{r}_i \times \left(\vec{F}_i^e + \sum_{j=1}^{n}\vec{F}_{ji}\right)\right] \\
&= \sum_{i=1}^{n}\left[\vec{r}_i \times \left(\vec{F}_i^e + \sum_{\substack{j=1 \\ j\neq i}}^{n}\vec{F}_{ji}\right)\right] \\
\therefore \quad \frac{d\vec{L}}{dt} &= \sum_{i=1}^{n}(\vec{r}_i \times \vec{F}_i^e) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}(\vec{r}_i \times \vec{F}_{ji}) \qquad (1.4.5)
\end{aligned}$$

The second term can be considered as a sum of the pairs *i.e.*,

$$\begin{aligned}
\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} &= \vec{r}_i \times \vec{F}_{ji} - \vec{r}_j \times \vec{F}_{ji} \\
&= (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \\
&= \vec{r}_{ij} \times \vec{F}_{ji} \qquad [\vec{r}_i - \vec{r}_j = \vec{r}_{ij}, \text{ the distance vector from } j\text{-th to } i\text{-th particl} \\
&= 0 \qquad [\because \vec{r}_{ij} \text{ and } \vec{F}_{ij} \text{ are two collinear vectors.}] \\
\therefore \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(\vec{r}_i \times \vec{F}_{ji}) &= 0 \qquad (1.4.
\end{aligned}$$

Using equation (1.4.6), the equation (1.4.5) reduces to

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}(\vec{r}_i \times \vec{F}_i^e) = \sum_{i=1}^{n} \vec{\tau}_i^{\,e} = \vec{\tau}^{\,e} \tag{1.4.7}$$

where $\vec{\tau}_i^{\,e}$ is the torque on i-th particle and $\vec{\tau}^{\,e}$ is the total torque applied to the system of particles.

Conservation of Angular Momentum for a system of particles

If the total applied torque on a system of multiple particles is zero, *i. e.* , if $\vec{\tau}^{\,e} = 0$, then

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \vec{\tau}^{\,e} = 0 \\ i.e., \quad \vec{L} &= \text{constant in time} \end{aligned}$$

We see that the total angular momentum of a system remains constant with the passage of time provided the system is not acted on by any external torque. That is, a system of particles conserves its total angular momentum provided no external torque acts on the system. This is known as the *conservation of total angular momentum* for a system of particles.

1.4.2 Kinetic and Potential Energies for a System of Particles

Kinetic Energy for a system of particles

The amount of work dW_i done by the force $\vec{F}_i$ acting on the i-th particle for its infinitesimal vector displacement by $d\vec{r}_i$ is given by

$$dW_i = \vec{F}_i \cdot d\vec{r}_i$$

The total amount of work done W_i by the same force in bringing the particle from say, location P_1 to location P_2 can then be found by integrating dW_i from P_1 to P_2, *i.e.*,

$$W_i = \int_{P_1}^{P_2} dW_i = \int_{P_1}^{P_2} \vec{F}_i.d\vec{r}_i$$

As the force acting on the i-th particle is the sum of the external and internal forces, *i.e.*,

$$\vec{F}_i = \vec{F}_i^e + \sum_{\substack{j \\ i\neq j}} \vec{F}_{ji};$$

we can write the expression for the work done on i-th particle as the sum of the work done by the external force and that by the internal forces:

$$W_i = \int_{P_1}^{P_2} \vec{F}_i^e.d\vec{r}_i + \sum_{\substack{j \\ j\neq i}} \int_{P_1}^{P_2} \vec{F}_{ji}.d\vec{r}_i. \tag{1.4.8}$$

We have already seen in the case of a single particle dynamics that the work done is related to the kinetic energy of motion. If T_{1i} and T_{2i} are the kinetic energies of the i-th particle at P_1 and P_2, then the difference of the kinetic energies at these two positions for the i-th particle is given by

$$\begin{aligned} W_i = \int_{P_1}^{P_2} \vec{F}_i.d\vec{r}_i &= \int_{V_{1i}}^{V_{2i}} m_i \frac{d\vec{v}_i}{dt}.\frac{d\vec{r}_i}{dt}dt = \int_{V_{1i}}^{V_{2i}} m_i \vec{v}_i.\frac{d\vec{v}_i}{dt}dt \\ &= \int_{V_{1i}}^{V_{2i}} m_i \vec{v}_i.d\vec{v}_i = \int_{V_{1i}}^{V_{2i}} m_i v_i dv_i \\ &= \frac{1}{2}m_i V_{2i}^2 - \frac{1}{2}m_i V_{1i}^2 = T_{2i} - T_{1i} \end{aligned}$$

where V_{1i} and V_{2i} are the magnitudes of the velocities of i-th particle at locations P_1 and P_2 respectively.

The kinetic energy difference of the entire N-particle system is simply the summation of the above expressions for all the particles, *i.e.*,

$$\begin{aligned} W_{12} = \sum_{i=1}^{N} W_i &= \sum_{i=1}^{N} \int_{P_1}^{P_2} \vec{F}_i.d\vec{r}_i = \sum_{i=1}^{N}(T_{2i} - T_{1i}) \\ \Longrightarrow \qquad W_{12} &= T_2 - T_1 \end{aligned} \tag{1.4.9}$$

where W_{12} is the total work done by the external force and $T_1 = \sum_{i=1}^{N} T_{1i}$ and $T_2 = \sum_{i=1}^{N} T_{2i}$ are respectively the initial and final values of total kinetic energy of the system.

Thus we see that the Work W_{12} done on the system of N particles between two locations, *viz.*, P_1 and P_2 by external forces $\vec{F} = \sum_{i=1}^{N} \vec{F}_i$ is the difference of the total kinetic energies of the system at the two locations. This is known as the *Work-Energy Theorem* for a system of particles.

Potential energy of a system of particles

For a conservative system, the external force $\vec{F}_i^c$ acting on i-th particle can be expressed as the gradient of some scalar function, *i.e.*,

$$\vec{F}_i^c = -\vec{\nabla}_i \Phi_i,$$

where $\vec{\nabla}_i$ and Φ_i are the gradient operator and the potential energy respectively, corresponding to the i-th particle.

As the total external force $\vec{F}^e$ is the vector sum of the external forces $\sum_{i=1}^{N} \vec{F}_i^e$ acting on the system, we can relate the total potential energy of the system and the total external force as

$$\vec{F}^e = \sum_i \vec{F}_i^e = -\sum_i \vec{\nabla}_i \Phi_i^e$$

so that the work done by the external forces in bringing the system from configuration 1 to configuration 2 is given by

$$\begin{aligned} W_{12}^e &= \sum_i \int_1^2 \vec{F}_i^e \cdot d\vec{r}_i \\ &= -\sum_i \int_1^2 \vec{\nabla}_i \Phi_i^e \cdot d\vec{r}_i \\ &= -\sum_i \int_1^2 d\Phi_i^e = \sum_i \Phi_{1i}^e - \sum_i \Phi_{2i}^e \\ \therefore \quad W_{12}^e &= \Phi_1^e - \Phi_2^e \end{aligned} \tag{1.4.10}$$

If we consider the mutual internal forces between the i-th and j-th particles to be conservative, then the internal forces $\vec{F}_{ij}$ and $\vec{F}_{ji}$ are expressible in term of gradient of a potential energy Φ_{ji}^{int} which is a function of the relative distance between the interacting particles, *i.e.*,

$$\Phi_{ji}^{int} \equiv \Phi_{ji}^{int}(|\vec{r}_j - \vec{r}_i|) = \Phi_{ji}^{int}(|\vec{r}_{ji}|),$$

is given by

$$\Phi^{int} = \sum_i{}' \sum_j \Phi_{ji}^{int}(|\vec{r}_j - \vec{r}_i|) = \sum_i{}' \sum_j \Phi_{ji}^{int}(|\vec{r}_{ji}|)$$

where $\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$ such that $\Phi_{ji}^{int} = \Phi_{ij}^{int}$. The prime over the summation here sigifies exclusion of self-interacting pairs from the discussion, *i.e.*, the cases with indices $i = j$ are excluded.

The form of the potential energy term Φ_{ji} also ensures Newton's third law of motion, that the force $\vec{F}_{ji}$ on the i-th particle by the j-th particle is equal and opposite to the force $\vec{F}_{ij}$ on j-th particle by the i-th particle, *i.e.*, $\vec{F}_{ji} = -\vec{F}_{ij}$, as can be seen below.

Define $\vec{\nabla}_l$ around the location $\vec{r}_l = \hat{i}x_l + \hat{j}y_l + \hat{k}z_l$ in Cartesian coordinates as

$$\vec{\nabla}_l = \hat{i}\frac{\partial}{\partial x_l} + \hat{j}\frac{\partial}{\partial y_l} + \hat{k}\frac{\partial}{\partial z_l}$$

so that

$$\begin{aligned} \vec{\nabla}_i &= \hat{i}\frac{\partial}{\partial x_i} + \hat{j}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i} \\ \vec{\nabla}_j &= \hat{i}\frac{\partial}{\partial x_j} + \hat{j}\frac{\partial}{\partial y_j} + \hat{k}\frac{\partial}{\partial z_j} \end{aligned}$$

and

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j; \qquad x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j \quad \text{and} \quad z_{ij} = z_i - z_j.$$

The gradient operator

$$\begin{aligned}\vec{\nabla}_i &= \hat{i}\frac{\partial}{\partial x_i} + \hat{j}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i}\\ &= \hat{i}\frac{\partial x_{ij}}{\partial x_i}\frac{\partial}{\partial x_{ij}} + \hat{j}\frac{\partial y_{ij}}{\partial y_i}\frac{\partial}{\partial y_{ij}} + \hat{k}\frac{\partial z_{ij}}{\partial z_i}\frac{\partial}{\partial z_{ij}}\\ &= \hat{i}\frac{\partial}{\partial x_{ij}} + \hat{j}\frac{\partial}{\partial y_{ij}} + \hat{k}\frac{\partial}{\partial z_{ij}}\\ &= \vec{\nabla}_{ij}\end{aligned}$$

and

$$\begin{aligned}\vec{\nabla}_j &= \hat{i}\frac{\partial}{\partial x_j} + \hat{j}\frac{\partial}{\partial y_j} + \hat{k}\frac{\partial}{\partial z_j}\\ &= \hat{i}\frac{\partial x_{ij}}{\partial x_j}\frac{\partial}{\partial x_{ij}} + \hat{j}\frac{\partial y_{ij}}{\partial y_j}\frac{\partial}{\partial y_{ij}} + \hat{k}\frac{\partial z_{ij}}{\partial z_j}\frac{\partial}{\partial z_{ij}}\\ &= -\hat{i}\frac{\partial}{\partial x_{ij}} - \hat{j}\frac{\partial}{\partial y_{ij}} - \hat{k}\frac{\partial}{\partial z_{ij}}\\ &= -\vec{\nabla}_{ij} = -\vec{\nabla}_i,\end{aligned}$$

Further,

$$\begin{aligned}\vec{\nabla}_{ji} &= \hat{i}\frac{\partial}{\partial x_{ji}} + \hat{j}\frac{\partial}{\partial y_{ji}} + \hat{k}\frac{\partial}{\partial z_{ji}} = -\hat{i}\frac{\partial}{\partial x_{ij}} - \hat{j}\frac{\partial}{\partial y_{ij}} - \hat{k}\frac{\partial}{\partial z_{ij}}\\ &= -\vec{\nabla}_{ij},\end{aligned}$$

where

$$\vec{\nabla}_{ij} = \hat{i}\frac{\partial}{\partial x_{ij}} + \hat{j}\frac{\partial}{\partial y_{ij}} + \hat{k}\frac{\partial}{\partial z_{ij}}$$

such that $\vec{\nabla}_{ij} = -\vec{\nabla}_{ji}$.

With the help of the results above, we can write the internal forces as

$$\vec{F}_{ji} = -\vec{\nabla}_i \Phi_{ij}^{int} = \frac{\partial \Phi_{ij}^{int}}{\partial}$$

and

$$\vec{F}_{ij} = -\vec{\nabla}_j \Phi_{ji}^{int}$$

$$= +\vec{\nabla}_j \Phi_{ij}^{int} = +\vec{\nabla}_j \Phi_{ji}^{int} = -\vec{F}_{ij}$$

Thus when the internal forces are conservative, the work done by the internal forces on the system can be expressed as a sum over the pairs of particles. We first write the work done due to one pair of say i-th and the j-th particles as

$$W_{12}^{int}(i,j) = \int_1^2 (\vec{F}_{ji}\cdot d\vec{r}_i + \vec{F}_{ij}\cdot d\vec{r}_j)$$
$$= -\int_1^2 (\vec{\nabla}_i\Phi_{ij}^{int}\cdot d\vec{r}_i + \vec{\nabla}_j\Phi_{ij}^{int}\cdot d\vec{r}_j). \quad (1.4.11)$$

Since we have

$$\vec{\nabla}_i\Phi_{ij}^{int} = \vec{\nabla}_{ij}\Phi_{ij}^{int} = -\vec{\nabla}_j\Phi_{ij}^{int},$$

the equation (1.4.11) above for the work done due to the pair of i-th and j-th particle will be of the form

$$W_{12}^{int}(i,j) = -\int \vec{\nabla}_{ij}\Phi_{ij}^{int}\cdot d\vec{r}_{ij}.$$

Since every such expression for the work done involves a pair of particles, the total work done W_{12}^{int} by the internal forces will be the sum of such pairs:

$$W_{12}^{int} = -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\int_1^2 \vec{\nabla}_{ij}\Phi_{ij}^{int} = -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\Phi_{ij}^{int}\Bigg|_1^2 = \Phi_1^{int} - \Phi_2^{int}$$

where,

$$\Phi_1^{int} = -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\Phi_{ij}^{int}\Bigg|_1$$ =potential energy at configuration 1 arising from all the pairs of the internal forces,

and

$$\Phi_2^{int} = -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\Phi_{ij}^{int}\Bigg|_2$$ =potential energy at configuration 2 due to all the pairs of the internal forces.

The factor $\frac{1}{2}$ appears here to avoid double counting during the summation to include all the possible pairs of interactions.

Conservation law for total energy

We have seen that the work done due to the external forces and internal forces both in carrying the system from configuration 1 to configuration 2, can be expressed as the gradient of corresponding scalar functions called the potential energies. The sum total of the work done W_{12} can be found by adding the potential energies arising from the external forces and from all possible pairs of internal forces, as

$$
\begin{aligned}
W_{12} &= W_{12}^{e} + W_{12}^{int} \\
&= \Phi_1^e - \Phi_2^e + \Phi_1^{int} - \Phi_2^{int} \\
+\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} \Phi_{ij}^{int} &= (\Phi_1^e + \Phi_1^{int}) - (\Phi_2^e + \Phi_2^{int})
\end{aligned}
$$

or,
$$W_{12} = \Phi_1 - \Phi_2 \tag{1.4.12}$$

where, $\Phi_1 = \Phi_1^e + \Phi_1^{int}$ is the sum of the potential energies due to the external and internal forces at configuration 1, and $\Phi_2 = \Phi_2^e + \Phi_2^{int}$ is the corresponding term at configuration 2.

Comparing the equation (1.4.12) with (1.4.9) for the kinetic energy and the potential energy considerations, we have

$$W_{12} = T_2 - T_1 = \Phi_1 - \Phi_2$$

$$\Longrightarrow \qquad T_1 + \Phi_1 = T_2 + \Phi_2 = E \quad \text{(say,)}$$

i.e., the sum of the total kinetic energy and the total potential energy of the system at the configuration 1 equals that at the configuration 2; meaning that *the total energy E of a system of particles is conserved during its motion attaining different positions or the configurations.* This is the conservation law of Energy.

Example 1.4.1 *A system consisting of four balls, with mass of* 0.5 *kg each, is moving on a plane surface with the speed of the first ball* 4 *m/s along north, second ball with* 10 *m/s along south, the third ball having* 3 *m/s along west and the fourth ball is with* 11 *m/s speed along east direction. Find out the linear momentum of this system.*

Solution: We take two-dimensional cartesian coordinates so that the positive of the x-axis lies along the east and positive y-axis along north. Designating the four balls as A, B, C and D respectively,

(a) For the ball A, moving north, the mass, velocity and the lineare momentum respectively are,
$M_A = 0.5$ Kg, $\quad \vec{V}_A = 4\hat{j}$ m/s $\quad \vec{P}_A = M_A\vec{V}_A = 0.5 \times 4\hat{j} = 2\hat{j}$ Kg-m/s

(b) For the ball B, moving south,
$M_B = 0.5$ Kg, $\quad \vec{V}_B = -10\hat{j}$ m/s $\quad \vec{P}_B = M_B\vec{V}_B = 0.5 \times (-10)\hat{j} = -5\hat{j}$ Kg-m/s

(c) For the ball C, moving along west,
$M_C = 0.5$ Kg, $\quad \vec{V}_C = -3\hat{i}$ m/s $\quad \vec{P}_C = M_C\vec{V}_C = 0.5 \times (-3)\hat{i} = -1.5\hat{i}$ Kg-m/s

(d) For the ball D, moving along east,
$M_D = 0.5$ Kg, $\quad \vec{V}_D = 11\hat{i}$ m/s $\quad \vec{P}_D = M_D\vec{V}_D = 0.5 \times 11\hat{j} = 5.5\hat{i}$ Kg-m/s

The linear momentum of the system is the sum of the linear momentum of the constituent parts:

$$\begin{aligned} \therefore \quad \vec{P} &= \vec{P}_A + \vec{P}_B + \vec{P}_C + \vec{P}_D \quad \text{Kg-m/s} \\ &= 2\hat{j} - 5\hat{j} - 1.5\hat{i} + 5.5\hat{i} \quad \text{Kg-m/s} \\ &= 4\hat{i} - 3\hat{j} \quad \text{Kg-m/s} \end{aligned}$$

The magnitude of the linear momentum of the system is

$$P = |\vec{P}| = \sqrt{4^4 + (-3)^2} = 5 \text{ Kg-m/s}$$

Example 1.4.2 *A disk is spinning at a rate of 10 rad/s about an axis passing perpendicularly through its centre. A second identical disk (same mass and shape) with no spin, is placed on the top of the first disk. Friction acts between the two disks until both are eventually traveling at the same speed. Find the final angular velocity of the system of the two disks?*

Solution: Considering the energy loss due to the friction between the plates to be much small compared to the overall energy of the sysstem, we use the principle of conservation of total angular momentum for a system. The angular momentum of a rotating object is given by the product of the moment of inertia about the axis of rotation and the angular velocity of the rotation. In this case, let the moment of inertia of the 1st disk be I. Initially the angular momentum of the system of one disk is entirely from this rotating disk. When the 2nd disk is added, as it has the same moment of inertia as the 1st disk about the same axis, the momoent of inertia of the final system is given by the sum of the individual moments of inertia, *i.e.*, $2I$

Thus the initial moment of inertia=I;
Initial angular velocity=$\omega_i = 10$ rad/s
and hence the initial angular momentum $L_0 = I\omega_i = 10I$,

where I is the moment of inertia of the rotating disk about the axis passing through its centre.

The final moment of inertia of the system=$2I$
Let the final angular velocity of the system be ω_f.

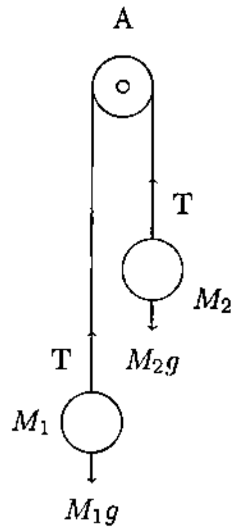
Hence the final angular mementum of the system=$L_f = 2I\omega_f$

By conservation of angular momentum,

$$\begin{aligned} L_0 &= L_f \\ \Rightarrow 10I &= 2I\omega_f \\ \Rightarrow \omega_f &= 5 \end{aligned}$$

Thus the *two-disk system* has a final angular velocity of 5 rad/s.

Example 1.4.3 *A system is considered consisting of two objects with masses m_1 and m_2 connected to the two ends of a light, inextensible string of length say l and the latter passing through a small pulley, so that the two masses are hanging under the action of the surface gravity of the earth. Such a system is called the* Atwood Machine. *Describe the dynamics of an Atwood machine.*



Solution: In the diagram, we see that the masses M_1 and M_2 are hanging through the location A with the help of a light inextensible string of length l. The force of gravity acts on the mass M_1 with magnitude $M_1 g$ and on M_2 with magnitude $M_2 g$, where g is the acceleration due to gravity at the given location. The string also generates within it a force, called the tension force in response, as the reaction to the externally applied force, *i.e.*, the downward pull of gravity. The tension forces are as shown in the figure. Here we take upward direction as positive coordinate x-axis, and consider that the mass M_1 is accelerating upward due to the acceleration in M_2.

From the figure, the Newton's second law applied to the mass M_1 yields

$$M_1 \ddot{x} = T - M_1 g \tag{1.4.13}$$

And applied to mass M_2 gives

$$-M_2 \ddot{x} = T - M_2 g \tag{1.4.14}$$

Subtracting (1.4.14) and (1.4.13),

$$\ddot{x} = \left(\frac{M_2 - M_1}{M_1 + M_2} \right) g \tag{1.4.15}$$

and therefore,

$$T = \left(\frac{2M_1 M_2}{M_1 + M_2} \right) g \tag{1.4.16}$$

1.5 Motion in a Central Force Field

A large variety of problems in the mechanics involve force fields, directed towards, or away from a point. Such a field is called the central force field. The motion of the planets around the sun, the satellites around the planets, or two charged particles around each other are some examples where central force fields are associated.

The study of particle motion under central force fields helps in the understanding of Lagrange's and Hamilton's equations which will be discussed in the subsequent units, and particularly the perturbation theory in the classical mechanics. Newton and Euler carried out extensive studies of the problems involving motion in the central force field, and these studies laid the foundation for further studies in this direction.

Definition 1 *Force field:*
The force field is a region in space at every point of which we can associate a unique force vector.

We can take the gravitational force field, or the gravitational field as an example of a force field. Here we can define a region around a given mass, say M and at every point outside the mass, we can associate a force which is felt by a test mass, say m. Since gravitational force is attractive by nature, the test mass will be attracted towards the mass M as a result of this force.

Definition 2 *Central Force:*
A central force is a force whose line of action is always directed towards or away from a fixed point, called centre or origin of the force and whose magnitude depends on the radial distance from the centre.

If the interaction between any two objects is represented by a central force, then the force is directed along the line joining the centres of two objects. Mathematically,

$$\vec{F}(r) = \hat{e_r} F(r) \tag{1.5.1}$$

where $\hat{e_r}$ is the unit vector along the direction of the position vector $\vec{r}$ and r is the magnitude of $\vec{r}$. If $\vec{F}(r)$ is positive, the force is repulsive and if negative, the force is attractive in nature.

As examples of central force, we can cite the attractive Gravitational force between two mass points, the attractive electrostatic force between two unlike charges, and repulsive in nature is the electrostatic force between the like charges as in the case of scattering of alpha particles by nuclei, etc.

As we can see, the motion of a particle in a central force field involves potential energy which depends only on the distance r of the moving particle from a fixed point in space. This fixed point is the source of the force and is called the center or the origin of the force. Generally this fixed point is chosen to be at the origin of the coordinate system so that it can be expressed as $\vec{F}(r) = \hat{e_r} F(r)$.

1.5.1 The equavalent one body problem

Let us consider a two-body system with two particles, particle 1 and particle 2, with masses m_1 and m_2, and position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively with respect to a given coordinate system. We will show here that the dynamics of this system of two bodies is equivalent to a system consisting of only one body with its mass given by a specific combination of the given two masses, called the *reduced mass* of the system and located at a specific position from the origin of the coordinate system.

Let the external forces $\vec{F}_1^{ext}$ and $\vec{F}_2^{ext}$ act on two particles of mass m_1 and m_2 respectively.
The total external force $\vec{F}^{ext}$ on the system is the sum of the forces acting individually on the particles, *i.e.*,

$$\vec{F}^{ext} = \vec{F}_1^{ext} + \vec{F}_2^{ext}. \tag{1.5.2}$$

Further, let the particle with mass m_1 be acted on by the internal force $\vec{F}_{21}^{int}$ due to the mass m_2 and the particle with mass m_2 by $\vec{F}_{12}^{int}$ due to the mass m_1. Then by Newton's third law of motion, we have

$$\vec{F}_{12}^{int} = -\vec{F}_{21}^{int} \tag{1.5.3}$$

These forces are the corresponding *Action* and the *Reaction* forces as laid down in Newton's third law of motion.

The equations of motion of the two particles of the system can be written as:

$$m_1\ddot{\vec{r}}_1 = \vec{F}_1^{ext} + \vec{F}_{21}^{int} \tag{1.5.4a}$$

$$\text{and} \qquad m_2\ddot{\vec{r}}_2 = \vec{F}_2^{ext} + \vec{F}_{12}^{int} \tag{1.5.4b}$$

Adding equations (1.5.4a) and (1.5.4b) with the consideration of (1.5.3) we find

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{F}_1^{ext} + \vec{F}_2^{ext} = \vec{F}^{ext} \tag{1.5.5}$$

$$\text{or,} \qquad (m_1 + m_2)\left(\frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2}\right) = \vec{F}^{ext} \tag{1.5.6}$$

From the definition of the centre of mass of the system,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{1.5.7a}$$

$$\text{and hence} \qquad \ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2} \tag{1.5.7b}$$

with the total mass of the system as $M = m_1 + m_2$, the equation of motion then reduces to

$$M\ddot{\vec{R}} = \vec{F}^{ext} \tag{1.5.8}$$

Thus equation (1.5.8) suggests that the dynamics of the given two-particle system can be described as equivalent to a new system with mass equal to the total mass of the two-body system and kept at its centre of mass; acted on by a force equal to the total forces acting on the two-particle system.

Let us further consider the position vector of particle 1 relative to particle 2. The separation vector is then

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{1.5.9}$$

Using the expression of the centre of mass (1.5.7a) we can write the position vectors $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{r}$, *i.e.*,

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \tag{1.5.10a}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r} \tag{1.5.10b}$$

Now multiply equation (1.5.4a) by m_2 and (1.5.4b) by m_1 and subtract the latter from the former. This yields

$$m_1 m_2 \ddot{\vec{r}}_1 - m_1 m_2 \ddot{\vec{r}}_2 = (m_2 \vec{F}_{21}^{int} - m_1 \vec{F}_{12}^{int}) + m_1 m_2 \left(\frac{\vec{F}_1^{ext}}{m_1} - \frac{\vec{F}_2^{ext}}{m_2} \right) \tag{1.5.11}$$

$$m_1 m_2 \ddot{\vec{r}} = (m_1 + m_2) \vec{F}_{21}^{int} + m_1 m_2 \left(\frac{\vec{F}_1^{ext}}{m_1} - \frac{\vec{F}_2^{ext}}{m_2} \right) \tag{1.5.12}$$

Dividing equation (1.5.12) throughout by $(m_1 + m_2)$ and defining the reduced mass of the system as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.5.13}$$

or

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{1.5.14}$$

where, μ is known as the *reduced mass*, the motion of the system of two bodies can always be reduced to an equivalent one-body problem. The equation (1.5.12) then reduces to

$$\mu \ddot{\vec{r}} = \vec{F}_{21}^{int} + \mu \left(\frac{\vec{F}_1^{ext}}{m_1} - \frac{\vec{F}_2^{ext}}{m_2} \right) \tag{1.5.15}$$

The following cases are in order.

Case 1 : If no external force acts on the system. Under such circumstances, we have

$$\vec{F}_1^{ext} = \vec{F}_2^{ext} = 0 \tag{1.5.16}$$

Case 2 : If external force $\vec{F}_1^{ext}$ and $\vec{F}_2^{ext}$ are proportional to the masses of the particles on which they act and produce equal accelerations in the two particles, *i.e.*,

$$\frac{\vec{F}_1^{ext}}{m_1} = \frac{\vec{F}_2^{ext}}{m_2} \tag{1.5.17}$$

Under both the cases referred above, the governing equations for the equivalent one body system reduce to

$$\mu\ddot{\vec{r}} = \vec{F}_{21}^{int} \tag{1.5.18}$$

Thus we note that the equation of motion of the original system of two particles reduces to an equivalent one body problem, with the mass equal to the reduced mass μ of the two originial particles and moving under the action of force $\vec{F}_{21}^{int}$. In other words, we can assume that there is a centre of force at the location of the first particle to govern the dynamics of a particle of mass μ located at the second particle. Such a reduction of a two body problem to an equivalent one-body problem proves very convenient in tackling complicated problem involving two masses.

In case $m_1 >> m_2$, we can ignore m_2 in the denominator of (1.5.13) and hence the reduced mass is approximated as

$$\frac{m_2}{\mu} = \left(\frac{m_2}{m_1} + 1\right) \approx 1 \qquad \Longrightarrow \qquad \mu \approx m_2$$

In such a case the problem reduces to just one-particle problem and can be solved by the Newton's laws for a single particle.

Although any classical system of two particles is by definition, a two body problem, in majority of the realistic cases, one may have to deal with a situation of two bodies with one body significantly heavier than the other, *e.g.*, the Earth and the Sun. In such cases, the heavier particle is approximately the centre of mass and the reduced mass is approximately the lighter mass. Hence, the heavier mass may be treated roughly as a fixed centre of force and the motion of the lighter mass may be solved for directly by one-body methods.

1.5.2 Particle Motion under Central Force Field

Let us consider a particle of mass m moving under the attractive gravitational field generated by a point mass M. For convenience, let the mass M be located at the origin and the position vector of the mass m be $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ of cartesian coordinate system so that the distance r between the masses is given by $r^2 = x^2 + y^2 + z^2$. The force experienced by the mass m is given by the universal law of gravitation, *i.e.*,

$$\vec{F}(r) = -\frac{GMm}{r^2}\hat{e}_r$$

where G is the universal gravitational constant. The negative sign here is to signify the attractive nature of the force. As the force acts along the line joining the two particles, *i.e.*, along the radial direction from the origin, a corresponding unit vector $\hat{e}_r$ is associated with the force vector.

The magnitude of the gravitational force is given by

$$F(r) = -\frac{GMm}{r^2}$$

We see that the force of gravitation depends only on the radial distance between the two masses and is directed towards the origin signified by the negative sign. The gravitational force between the two masses is therefore a central force. We shall now show that this force is conservative.

The given force field is,

$$\vec{F}(r) = \hat{e}_r F(r)$$

The unit vector $\hat{e}_r$ can be written in terms of the cartesian coordinates as

$$\hat{e}_r = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore,

$$\vec{F} = \frac{F(r)}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Resolving the force in three cartesian components

$$F_x = \frac{x}{r}F(r), \qquad F_y = \frac{y}{r}F(r), \qquad F_z = \frac{z}{r}F(r)$$

The curl of the force is

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

The components of the curl of the force is then given as

$$\therefore \quad \vec{\nabla} \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Now,

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{r}F(r)\right) = z\frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left(\frac{F(r)}{r}\right)$$

Similarly,

$$\frac{\partial F_y}{\partial z} = y\frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left(\frac{F(r)}{r}\right)$$

and so on.

The terms like $\frac{\partial r}{\partial y}$ or $\frac{\partial r}{\partial z}$ can be evaluated by writing r in terms of x, y and z.

$$r = \sqrt{(x^2 + y^2 + z^2)}$$

This gives,

$$\frac{\partial r}{\partial x} = \frac{2x}{2(x^2 + y^2 + z^2)} = \frac{x}{r},$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \text{and} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Consider the x-component of the curl, *i.e.*,

$$\begin{aligned} (\vec{\nabla} \times \vec{F})_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ &= z\frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left[\frac{F(r)}{r}\right] - y\frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left[\frac{F(r)}{r}\right] \\ &= z\frac{y}{r}\frac{\partial}{\partial r}\left[\frac{F(r)}{r}\right] - y\frac{z}{r}\frac{\partial}{\partial r}\left[\frac{F(r)}{r}\right] \\ \therefore (\vec{\nabla} \times \vec{F})_x &= 0 \end{aligned}$$

Similarly we can show that

$$(\vec{\nabla} \times \vec{F})_y = 0$$

and

$$(\vec{\nabla} \times \vec{F})_z = 0$$

Combining these results, we get

$$(\vec{\nabla} \times \vec{F}) = 0,$$

which is the condition for a conservative force field *i.e.*, the force field $\vec{F}$ is conservative.

Potential energy and Central Force field

As we have seen, the gravitational force is a central force depends only on the distance between two particles and is a conservative force field. Consequently we can define a potential energy function Φ such that

$$\vec{F} = -\vec{\nabla}\Phi$$

Since the force $\vec{F}$ depends only on r, so also does Φ, *i.e.*, Φ is dependent on r and not upon orientation. Hence the system governed by central force field has a spherical symmetry.

An interesting result can be proved in regards of the angular momentum of a moving particle in a central force field. The angular momentum of a particle moving in the central force field about the force centre remains constant for each orbit of the particle. This can be shown as the following:

The angular momentum in a central force field

The torque (the agent which causes a particle to execute rotational motion) is given by,

$$\vec{\tau} = \vec{r} \times \vec{F}(r)$$

where $\vec{r}$ is the position vector of the moving particle measured from the force centre.

The angular momentum $\vec{L}$ of the particle is given by,

$$\vec{L} = \vec{r} \times \vec{p}$$

Now,
$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}(\vec{r} \times \vec{p}) && [\ \vec{p} \text{ is the linear momentum of the particle}] \\ &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} && \because \vec{v} = \frac{d\vec{r}}{dt} \text{ and } \vec{p} = m\vec{v} \\ &= \vec{r} \times \vec{F}(r) = \vec{\tau}\end{aligned}$$

Here $\vec{\tau}$ is the torque vector acting on the system.

But since a cenral force acts along the direction of the separation vector, *i.e.*, along $\vec{r}$, the cross product, and hence the torque vector vanishes.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}(r) = 0$$

because of the cross product of two unidirectional vectors.

Integrating the expression we find

$$\vec{L} = \text{ a constant vector} \quad \Rightarrow \quad \boxed{\vec{r} \times \vec{p} = \text{ constant vector}}$$

In other words, the plane containing $\vec{r}$ and $\vec{p}$ remain perpendicular to $\vec{L}$ throughout the motion. Hence, the component of $\vec{L}$ along any axis through the centre of the force is a constant quantity.

This fact further implies that as there is no component of $\vec{F}(r)$ perpendicular to $\vec{r}$ and $\vec{p}$, the motion will always be confined in the plane containing the vectors $\vec{r}$ and $\vec{p}$, *i.e.*, the plane containing the initial position and momentum.

Thus, *the motion of a particle under central force field always remains confined in a plane.*

The motion of a particle in a central force field can be classified as:

1. **Bounded motion:**
   In this type of motion the distance between two bodies never exceeds a finite limit. For example, the motion of planets around the sun.

2. **Unbounded motion:**
   In this type of motion the distance between two bodies is infinite initially and finally. For example, scattering of alpha particles by the nuclei of a gold foil in Rutherford experiment.

There are two approaches to analyse the motion of a particle in central force field - the *Integrals of energy approach* and, the *Differential equation of orbit approach.*

In the first approach Newton's laws of motion is directly used and the related conserved quantities in the system are found out. In particular, Newton's second law is first written for particle of mass m moving with velocity $\vec{v}$:

$$m\dot{\vec{v}} = \vec{F} = F\hat{e}_r \tag{1.5.19}$$

The scalar product of (1.5.19) with $\vec{v}$, we get

$$m\dot{\vec{v}}.\vec{v} = F\vec{v}.\hat{e}_r \tag{1.5.20}$$

The left hand side of (1.5.20) can be written as

$$\vec{v}.\dot{\vec{v}} = \frac{d}{dt}\left(\frac{1}{2}\vec{v}.\vec{v}\right) = \frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

To evaluate the right hand side of (1.5.20), we observe that the velocity vector can be expressed in terms of polar 2-components as $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$. where $\hat{e}_r$ and $\hat{e}_\theta$ are the unit vectors along the radial and the tangential directions respectively. Hence we have

$$\vec{v}.\hat{e}_r = (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta).\hat{e}_r = \dot{r} = \frac{dr}{dt}$$

Therefore, (1.5.20) simplifies to,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = F\frac{dr}{dt} \tag{1.5.21}$$

Let F depends only on the magnitude of the radial vector, *i.e.*, $F = F(r)$, then integration of (1.5.21) with respect to time t yields

$$\frac{1}{2}mv^2 = \int F(r)dr + E, \tag{1.5.22}$$

where E is the constant of integration, depending on the initial conditions of the motion. The integral (1.5.22) nothing but the work done by the force F in displacing the particle along the orbit.

Let us consider the conservative force field *i.e.*, $\vec{F} = -\vec{\nabla}\Phi$, which implies that there exists a scalar function $\Phi(r)$, the potential energy of the particle, such that

$$\vec{F}(r) = -\frac{\partial\Phi}{\partial r} \tag{1.5.23}$$

Then (1.5.22) becomes

$$\frac{1}{2}mv^2 + \Phi(r) = E, \tag{1.5.24}$$

which states that the sum of the kinetic energy and the potential energy, *i.e.*, the total energy E of a moving particle under the central force field is constant. This is the law of conservation of energy. Now, solving (1.5.24) for v, we get,

$$v = \frac{dr}{dt} = \pm\left[\frac{2}{m}(E - \Phi)\right]^{\frac{1}{2}} \tag{1.5.25}$$

which on integration, yields

$$t \equiv t(r) = \int_{r_1}^{r_2} \frac{dr}{\left[\pm\left(\frac{2}{m}(E - \Phi)\right)^{\frac{1}{2}}\right]} \tag{1.5.26}$$

The motion of the particle, *i.e.*, the location of the particle as a function of time, $r(t)$, can then be found by inverting the equation (1.5.26).

In polar coordinates,(1.5.25) can also be rewritten as

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \Phi(r) = E \tag{1.5.27}$$

In the second approach, *i.e.*, the *Differential equation of the orbit approach*, one attempts to develop differential equations of the orbit (dependance of one coordinate on the other, say $y(x)$ in two dimensional cartesian and $r(\theta)$ in the 2-D polar coordinates) that a moving particle follows under the governing central force field.

We write down Newton's second law in two dimensional polar coordinates,

$$m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + \frac{\partial}{\partial t}(mr^2\dot{\theta})\hat{e}_\theta = F(r)\hat{e}_r \tag{1.5.28}$$

The radial and tangential components of the equation of motion are then

$$m\ddot{r} - mr\dot{\theta}^2 = F(r) \tag{1.5.29}$$

$$\frac{\partial}{\partial t}(mr^2\dot{\theta}) = 0 \tag{1.5.30}$$

Now from (1.5.30) we see that

$$mr^2\dot{\theta} = L = \text{constant.} \tag{1.5.31}$$

So we find that the quantity $L = mr^2\dot{\theta} = mv_\theta r$ is the angular momentum of the particle moving under central force, and is a conserved quantity.

For convenience in deriving the equation of the orbit, let us re-express the equations by substituting r as

$$u = \frac{1}{r}$$

Then

$$\dot{\theta} = \frac{L}{mr^2} = \frac{L}{m}u^2 \tag{1.5.32}$$

Now,

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{L}{m}\frac{du}{d\theta} \tag{1.5.33}$$

And

$$\frac{d^2r}{dt^2} = -\frac{L}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta}$$

So that

$$\frac{d^2r}{dt^2} = -\left(\frac{Lu}{m}\right)^2\frac{d^2u}{d\theta^2} \tag{1.5.34}$$

Using (1.5.32) through (1.5.34) in (1.5.29) we have

$$-m\left(\frac{Lu}{m}\right)^2\frac{d^2u}{d\theta^2} - \frac{m}{u}\left(-\frac{L}{m}u^2\right)^2 = F\left(\frac{1}{u}\right)$$

Simplifying,

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F\left(\frac{1}{u}\right) \tag{1.5.35}$$

which is the differential equation in u as a function of θ and is known as the polar equation of the orbit. This equation describes the motion of a particle under central force field.

Numerical Problems

Example 1.5.1 *Show that for a particle travelling in a cycloidal path $r = a(1-\cos\theta)$ in a central force field, the force law goes as r^{-4}.*

Solution: The orbit equation is $r = a(1-\cos\theta)$,

Putting $u = \frac{1}{r}$,

$$u = \frac{1}{r} = \frac{1}{a(1-\cos\theta)} = \frac{1}{2a}\ \text{cosec}^2\frac{\theta}{2}$$

$$\begin{aligned}\frac{du}{d\theta} &= \frac{1}{2a}\left(-2\ \text{cosec}^2\frac{\theta}{2}\times\cot\frac{\theta}{2}\times\frac{1}{2}\right)\\ &= -\frac{1}{2a}\ \text{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}\end{aligned}$$

And,

$$\frac{d^2u}{d\theta^2} = \frac{1}{2a}\,\mathrm{cosec}^2\frac{\theta}{2}\left[\cot^2\frac{\theta}{2} + \frac{1}{2}\,\mathrm{cosec}^2\frac{\theta}{2}\right]$$

$$\begin{aligned}
\therefore\quad \frac{d^2u}{d\theta^2} + u &= \frac{1}{2a}\,\mathrm{cosec}^2\frac{\theta}{2}\left[\cot^2\frac{\theta}{2} + \frac{1}{2}\,\mathrm{cosec}^2\frac{\theta}{2}\right] + \frac{1}{2a}\,\mathrm{cosec}^2\frac{\theta}{2} \\
&= \frac{1}{2a}\,\mathrm{cosec}^2\frac{\theta}{2}\left[\cot^2\frac{\theta}{2} + \frac{1}{2}\,\mathrm{cosec}^2\frac{\theta}{2} + 1\right] \\
&= \frac{1}{2a}\,\mathrm{cosec}^2\frac{\theta}{2}\left[\mathrm{cosec}^2\frac{\theta}{2} + \frac{1}{2}\,\mathrm{cosec}^2\frac{\theta}{2}\right] \\
&= \frac{3}{4a}\,\mathrm{cosec}^4\frac{\theta}{2}
\end{aligned}$$

So, using the differential equation of the orbit,

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= -\frac{m}{L^2u^2}F\left(\frac{1}{u}\right) \\
\Longrightarrow\quad \frac{3}{4a}\,\mathrm{cosec}^4\frac{\theta}{2} &= -\frac{m}{L^2u^2}F\left(\frac{1}{u}\right) \\
\Longrightarrow\quad F\left(\frac{1}{u}\right) &= -\frac{L^2u^2}{m}\frac{3}{4a}\,\mathrm{cosec}^4\frac{\theta}{2} \\
&= -\frac{L^2u^2}{m}\frac{3}{4a}(2au)^2 = -\frac{3aL^2}{m}u^4 \\
\Longrightarrow\quad F(r) &= -\frac{3aL^2}{mr^4} \propto \frac{1}{r^4}
\end{aligned}$$

So, the force goes as r^{-4}

Example 1.5.2 *An object of unit mass orbits in a central potential $V(r)$, whose orbit is described by $r = a\exp(-\alpha\theta)$, θ is azimuthal angle measured in the orbital plane. Find the form of $V(r)$.*

Solution:

$$\text{Let}\quad u = \frac{1}{r}.$$

The given orbit can then be rewritten as

$$\begin{aligned}
u &= \frac{1}{r} = \frac{\exp(\alpha\theta)}{a} \\
\frac{du}{d\theta} &= \frac{\alpha}{a}\exp(\alpha\theta) = \alpha u \\
\frac{d^2u}{d\theta^2} &= \frac{\alpha^2}{a}\exp(\alpha\theta) = \alpha^2 u
\end{aligned}$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = (1+\alpha^2)u = -\frac{1}{L^2u^2}F\left(\frac{1}{u}\right)$$

$$\Longrightarrow \quad F\left(\frac{1}{u}\right) = -L^2(1+\alpha^2)u^3$$

$$\Longrightarrow \quad F(r) = -L^2(1+\alpha^2)\frac{1}{r^3} = -\frac{dV}{dr}$$

Hence
$$V = \int F(r)dr = L^2(1+\alpha^2)\int \frac{1}{r^3}dr$$

or,
$$V = -\frac{L^2(1+\alpha^2)}{2r^2} + V_0$$
V_0 is the constant of integration.

1.6 Motion in inverse square law force field

1.6.1 Introduction

The inverse square law, quite often the most interesting amongst the laws in physics, is characterised by a force inversely proportional to the square of the distance from the centre of the force. Newton's law of universal gravitation and the electrostatic coulombs law are two examples of the laws of physics of this kind. The inverse-square law generally applies when some force, energy, or other conserved quantity is radiated outward from a point source in three-dimensional space. In electrostatics, the electrostatic coulomb force between two charged particles is inversely proportional to the distance of separation between the charges. Increasing the separation between objects decreases the force of attraction or repulsion between them, and on decreasing, it does increase. Below is a discussion on the inverse square law with some necessary consequences in the motion of particles. In particular, we make an attempt to deduce the equation of orbit under the inverse square law of forces.

Newton's universal law of Gravitation

Let m_1 and m_2 be the masses of the two particles with a distance r between them. Then the magnitude of the force of attraction F on m_2 due to m_1 is given by

$$F \propto \frac{m_1 m_2}{r^2}$$

or,

$$F = -G\frac{m_1 m_2}{r^2}$$

Here G is universal constant of gravitation and the negative sign indicates that the force is attractive.

1.6.2 Equation of motion in inverse square law force field

Let us consider a two-particle system with masses m_1 and m_2 in which the force of interaction between the particles varies inversely as the square of the distance between them. Then

$$F(r) = \frac{k}{r^2}, \qquad k = \text{constant of proportionality.}$$

We have already seen that two-particle system can be equivalently described as a one-body problem under central force, where a reduced mass given by $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ moves under a center of force given by the inverse square law. As the inverse square law of force concerns only the radial distance, it is a central force and is also conservative. So we can express it in terms of the gradient of a scalar, say $\Phi(r)$, *i.e.*,

$$\begin{aligned} F(r) &= -\frac{d\Phi}{dr}, \\ \text{or,} \quad -\frac{d\Phi}{dr} &= \frac{k}{r^2}; \end{aligned}$$

Integrating we have

$$\Phi(r) = \frac{k}{r}$$

The equation of motion of the equivalent mass under inverse square force field in two dimensional polar coordinates is then given by (1.5.27), *i.e.*,

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \Phi(r) = E$$

where first term corresponds to the kinetic energy of the mass μ along the radial direction, while the second term in the bracket refers to an *equivalent potential energy* term corresponding to the centripetal force on the mass. The term on the right hand side is the total energy of the system which is the sum of the kinetic energy, the potential energy due to the centripetal force and the potential energy corresponding to the inverse square force. The centripetal term arises when the initial velocity of the mass contains a tangential component. It is interesting to note that the centripetal force which arises due to the tangential motion of particles is actually radially directed. The potential energy (Φ_c) due to centripetal force term can be written in terms of the total angular momentum on the equivalent mass μ as,

$$\Phi_c = \frac{L^2}{2\mu r^2}; \qquad \because L = \mu r^2\dot{\theta}$$

We consider the total energy (E) as the sum of the kinetic energy and a single effective potential energy term (Φ_e), where

$$\Phi_e = \frac{k}{r} + \Phi_c = \frac{k}{r} + \frac{L^2}{2\mu r^2} \tag{1.6.1}$$

Thus when a particle free particle (a particle having only kinetic energy) enters the region of influence of an effective potetial, its motion is governed by the effective potential leading to definite path, or an orbit appropriate to the values of the associated potential and the kinetic energy. In general, the nature of the orbit will depend upon the sign of k. If we plot effective potential

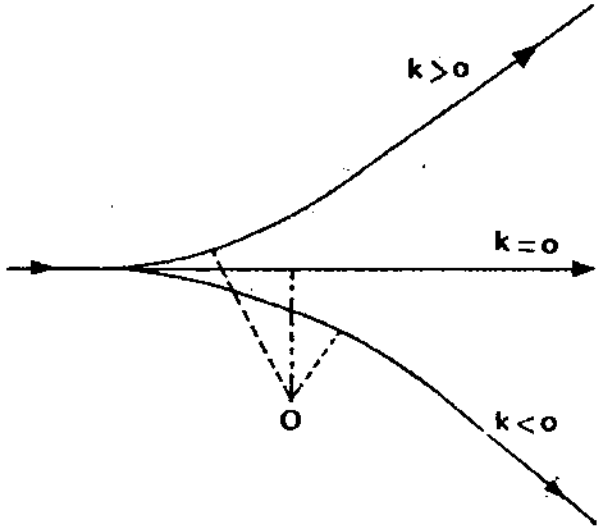


Figure 1.2: Types of orbits for unbounded motion, for different values of k.

energy Φ_e against the distance r between the particles for various values of k then we see that for a repulsive force $k \geq 0$ the potential energy curve does not have a minimum. Similar is the case for a force of attraction (when $k < 0$).

Under this circumstance, the particle, initially having an arbitrary value of energy approaches the center of force from infinity, reaches the closest distance of approach and turns around to move back to infinity. The nature of motion in such cases is called unbounded motion.

We can further see from the plots that the orbit formed with $k = 0$ is just a straight line maintaining the same initial direction, since this case implies that there is no force on the particle and therefore according to the newton's first law of motion, the particle continues to move along the same direction without any change of velocity.

If the force is of attractive nature then the form of the potential energy is

$$\Phi_e = -\frac{|k|}{r} + \frac{L^2}{2\mu r^2}$$

which creates a hump in the potential energy curve and the particle approaching the centre of force will be trapped inside the hump. This *trapping* renders a closed orbit of the particle around the centre of force and the motion will be bounded.

For a repulsive force on the other hand, the particle will be deflected away from the centre of the force and it eventually moves towards infinity.

Few special cases of motion of a particle under inverse square law of force are in order:

1. For total energy E_1, the intersection of the straight line equation E_1=costant with the potential energy curve is at $r = r_1$, at which the radial component of kinetic energy of the particle

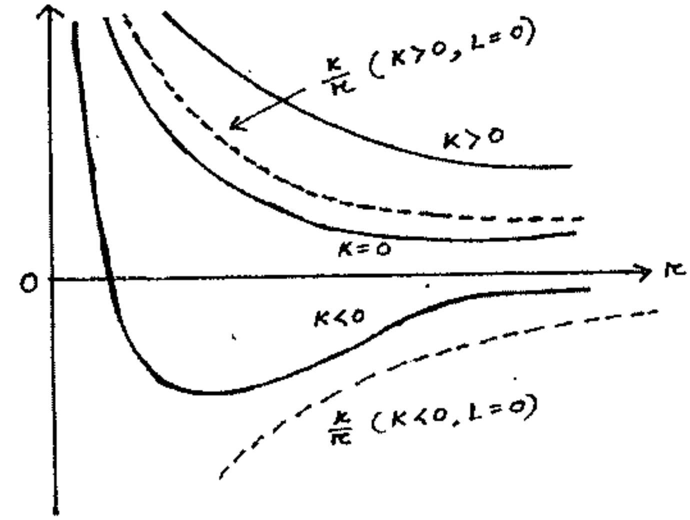


Figure 1.3: Variation of the effective potential Φ_e with distance r for the positive, zero and negative values of k in the potential energy term $\frac{k}{r}$.

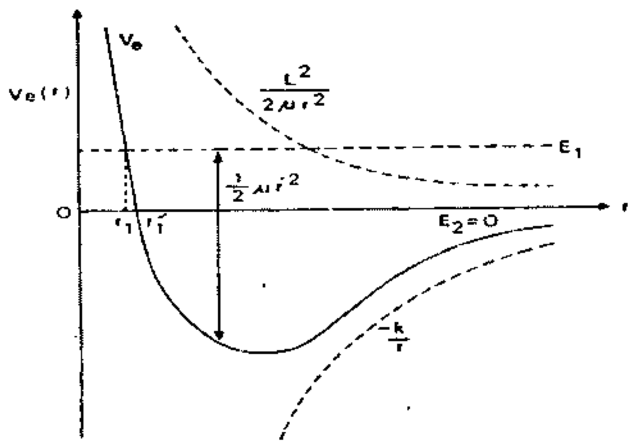


Figure 1.4: Variation of the effective potential Φ_e with radial distance r for inverse square law of force. The dotted curves are with $L = 0$.

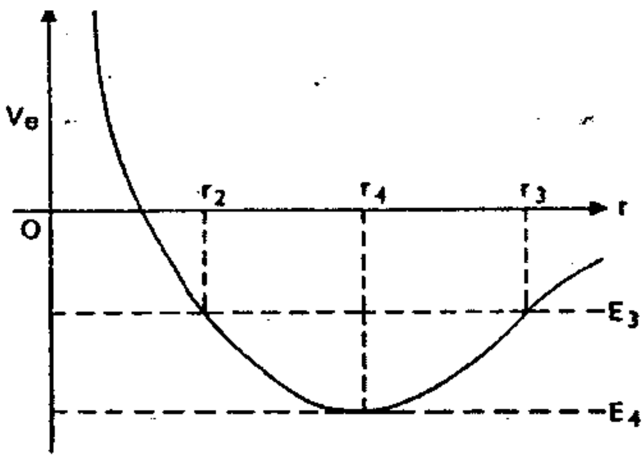


Figure 1.5: Variation of Φ_e with the radial distance r at minimum (r_2) and maximum (r_3) radii of orbit for the total energy $E_3 < 0$ and intersecting Φ_e at two points. For the lowest value of energy (E_4), the orbit is a circle with radius r_4.

is zero. In this case r_1 is a real root of equation

$$E_1 = \Phi_e = -\frac{|k|}{r} + \frac{L^2}{2\mu r^2}.$$

The particle moves in such away that it has one turning point at $r = r_1$. The motion corresponds to an unbounded motion.

2. If the energy of the particle is $E_2 = 0$, the roots of the equation in this case are $r = r_1$ and ∞. The particle goes to infinity but its radial velocity falls off continuously and becomes zero at infinity.

3. For energy $E_3 < 0$, the roots of equation are r_2 and r_3, both real and distinct. The motion of particle is bounded between these two distances, *i.e.*, the orbits are closed and in general, elliptic.

4. For total energy E_4, the two real roots of the equation coincide and the corresponding motion will be circular around the centre of force of the first particle.

1.6.3 Deduction of equation of orbit

We now deduce the equation of motion of a particle subjected to a central force. Newton's second law adapted to central force gives (refer to (1.5.29))

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = F(r)$$

$$\implies \quad \mu\ddot{r} = F(r) + \frac{L^2}{\mu r^3} \qquad \because \quad L = \mu r^2\dot{\theta} \tag{1.6.2}$$

Let us substitute $u = \frac{1}{r}$. Then

$$\frac{du}{d\theta} = -\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \qquad \left[\because \quad \frac{dr}{dt} = \dot{r}, \qquad \frac{d\theta}{dt} = \dot{\theta}\right]$$

$$\frac{du}{d\theta} = -\frac{\dot{r}}{r^2\dot{\theta}} = -\frac{\mu}{L}\dot{r} \qquad [\text{ since, } \quad L = \mu r^2\dot{\theta}\ ]$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-\frac{\mu}{L}\dot{r}\right) = \frac{d}{dt}\left(-\frac{\mu}{L}\dot{r}\right)\frac{dt}{d\theta} = -\frac{\mu}{L}\left(\frac{\ddot{r}}{\dot{\theta}}\right) = -\frac{\mu^2 r^2}{L^2}\ddot{r}$$

Substituting these in (1.6.2),

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2}F\left(\frac{1}{u}\right)$$

If the force field obeys the inverse square law then

$$F(r) = \frac{|k|}{r^2} \qquad \text{and,} \qquad F\left(\frac{1}{u}\right) = |k|u^2$$

Hence,
$$\frac{d^2u}{d\theta^2} + u = \frac{|k|\mu}{L^2}$$

Let
$$y = u - \frac{|k|\mu}{L^2}; \qquad \text{so that} \qquad \frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2}$$

Hence, the equation is of the form,

$$\frac{d^2y}{d\theta^2} + y = 0 \tag{1.6.3}$$

The equation (1.6.3) is a second order ordinary differential equation in y as a function of θ and has a solution

$$y = A\cos(\theta - \theta_0)$$

where A and θ_0 are constants.

Hence we have,
$$y = A\cos(\theta - \theta_0) = \frac{1}{r} - \frac{|k|}{L^2}\mu$$

$$\text{or,} \quad \frac{1}{r} = A\cos(\theta - \theta_0) + \frac{|k|}{L^2}\mu = \frac{|k|}{L^2}\mu\left[1 + \frac{L^2A}{|k|\mu}\cos(\theta - \theta_0)\right]$$

$$\text{or,} \quad \frac{L^2}{|k|\mu r} = 1 + \frac{L^2A}{|k|\mu}\cos(\theta - \theta_0) \tag{1.6.4}$$

The equation (1.6.4) is similar to the equation

$$\frac{l}{r} = 1 + e\cos\theta \tag{1.6.5}$$

which is an equation of a conic section with the origin at the focus. Now, comparing equation (1.6.4) with this equation we have

$$\text{Semi latus rectum} \qquad l = \frac{L^2}{|k|\mu} \tag{1.6.6a}$$

$$\text{Eccentricity} \qquad e = \frac{L^2 A}{|k|\mu} \tag{1.6.6b}$$

There is also an alternative way by which the expression for latus rectum and the eccentricity can be found. From the energy equation (1.5.27) for mass μ and the inverse square attractive potential $V = -\frac{|k|}{r}$ and further using the fact that the angular momentum is given by $L = \mu r^2\dot{\theta}$ we see that,

$$\begin{aligned}
\frac{1}{2}m(\dot{r}^2 &+ r^2\dot{\theta}^2) + V = E \\
\Longrightarrow \qquad \dot{r}^2 &= \frac{2}{\mu}(E - V) - \frac{L^2}{\mu^2 r^2} \\
\Longrightarrow \qquad \left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)^2 &= \left(\frac{dr}{d\theta}\frac{L}{\mu r^2}\right)^2 = \frac{2}{\mu}(E - V) - \frac{L^2}{\mu^2 r^2} \\
\Longrightarrow \qquad \frac{dr}{d\theta} &= \frac{r^2}{L}\sqrt{2\mu\left(E - V - \frac{L^2}{2\mu r^2}\right)}
\end{aligned}$$

Integrating θ with respect to r yields

$$\theta(r) - \theta_0 = \int \frac{\left(\frac{L}{r^2}\right) dr}{\sqrt{2\mu\left(E - V - \frac{L^2}{2\mu r^2}\right)}}$$

Now, putting $\qquad V = -\frac{|k|}{r}, \quad u = \frac{1}{r}, \qquad$ we have

$$\theta - \theta_0 = -\int \frac{du}{\sqrt{\frac{2\mu E}{L^2} + \frac{2\mu k u}{L^2} - u^2}} \tag{1.6.7}$$

To integrate the of equation (1.6.7) we can use the standard result

$$\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{-c}}\cos^{-1}\left(-\frac{b + 2cx}{\sqrt{b^2 - 4ac}}\right).$$

This gives

$$\cos(\theta - \theta_0) = \frac{uL^2 - \mu k}{\sqrt{\mu^2 k^2 + 2\mu E L^2}}$$

or,

$$\left(\frac{L^2}{\mu k}\right) u = \frac{\left(\frac{L^2}{\mu k}\right)}{r} = 1 + \left(\sqrt{1 + \frac{2EL^2}{\mu k^2}}\right)\cos(\theta - \theta_0) \tag{1.6.8}$$

Comparison of (1.6.5) and (1.6.6) with (1.6.8) yields

$$A = \frac{k\mu}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

Turning points

When a particle moves under a central force, there might be a situation where the particle encounters a potential barrier and the total energy of the particle is not sufficient to overcome it. In that case the particle will either be trapped or get bounced back, depending on the nature of the central force, *i.e., attractive* or *repulsive*. The locations at which this occurs are called the *Turning points*. At the turning points the radial velocity of the particle vanishes, *i.e.*, $\dot{r} = 0$. Essentially, the turning points are dependent on the initial kinetic energy and the potential energy arising from the central force. Mathematically the turning points can be found by searching the locations at which the energy of the particle just equals the potential energy. *i.e.*, $E - V_{eff} = 0$

For a particle moving under inverse square potential we can calculate the turning points from the equation of orbit (1.6.4). For this, we need first to find the values of θ for which $\dot{u} = 0$ in $0 \leq \theta \leq 2\pi$. This gives $\theta = 0, \pi$ for the turning points. Hence the radial distance at the turning points are given by

$$\frac{1}{r_1} = \frac{\mu|k|}{L^2} + A$$

$$\frac{1}{r_2} = \frac{\mu|k|}{L^2} - A$$

The value of constant A cannot exceed $\frac{\mu|k|}{L^2}$, otherwise r_2 will be negative, which is not real.

The the turning points are also the root of the equation

$$E - \Phi_e(r) = E + \frac{|k|}{r} - \frac{L^2}{2\mu r^2} = 0$$

This equation is a quadratic in $\frac{1}{r}$. Hence its root can be written as

$$\frac{1}{r_{1,2}} = \frac{\mu|k|}{L^2} + \frac{\mu|k|}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

Comparing this with (1.6.5) and (1.6.6) we have,

$$A = \frac{\mu|k|}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

$$e = \frac{L^2 A}{\mu|k|} = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

Nature of orbit

The nature of the orbit is determined by the value of eccentricity e of the orbit. The value of orbit depends upon total energy E. we have found the following cases:

1. For energy $E > 0$, the eccentricity $e > 1$ and the orbit will be a hyperbola.

2. For energy $E = 0$, the eccentricity $e = 1$ and the orbit will be a parabola.

3. For energy $E < 0$, the eccentricity $e < 1$ and the orbit will be an ellipse.

4. Finally for energy, $E = V_{e_{min}}$, the value of eccentricity $e = 0$ and the orbit will be a circle.

1.7 Kepler's laws of motion

We know that the kinetic energy of a particle of mass m moving under central force is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\ &= \frac{1}{2}m\left[\frac{L^2}{m^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \frac{L^2}{m^2}u^2\right] \\ &= \frac{1}{2}\frac{L^2}{m}\left[\left(\frac{\partial u}{\partial \theta}\right)^2 + u^2\right] \\ &= \frac{L^2}{2m}\left[A^2\sin^2(\theta-\theta_0) + \left(\frac{mk}{L^2} + A\cos(\theta-\theta_0)\right)^2\right], \qquad \text{(refer to equation (1.6.4))} \\ &= \frac{L^2}{2m}\left[A^2 + \frac{m^2k^2}{L^4} + \frac{2mkA}{L^2}\cos(\theta-\theta_0)\right] \end{aligned}$$

The potential energy of such a particle is of the form

$$V = -\frac{k}{r} = -ku = -\frac{mk^2}{L^2} - Ak\cos(\theta-\theta_0) \tag{1.7.1}$$

$$\begin{aligned} \therefore \quad \text{Total energy } E &= T + V \\ &= \frac{1}{2m}L^2A^2 + \frac{1}{2}\frac{mk^2}{L^2} + kA\cos(\theta-\theta_0) - \frac{mk^2}{L^2} - Ak\cos(\theta-\theta_0) \\ &= \frac{1}{2m}L^2A^2 - \frac{1}{2}\frac{mk^2}{L^2} \\ &= \frac{1}{2m}L^2\left[A^2 - \frac{m^2k^2}{L^4}\right] \end{aligned}$$

$$\frac{2mE}{L^2} = A^2 - \frac{m^2k^2}{L^4}$$

$$\text{or,} \quad A^2 = \frac{m^2k^2}{L^4} + \frac{2mE}{L^2}$$

$$A = \frac{mk}{L^2}\left[1 + \frac{2EL^2}{mk^2}\right]^{\frac{1}{2}}$$

$$\frac{AL^2}{mk} = e = \left(1 + \frac{2EL^2}{mk^2}\right)^{\frac{1}{2}}$$

So when

$e > 1;$	$E > 0$	$\rightarrow$	*hyperbola*
$e = 1;$	$E = 0$	$\rightarrow$	*parabola*
$0 < e < 1;$	$E < 0$	$\rightarrow$	*ellipse*
$e = 0;$	$E = -\dfrac{mk^2}{2L^2}$	$\rightarrow$	*circle*

Kepler's laws of planetary motion

Acceleration to Newton's law gravitation the force of attraction between sun of mass M and planet m is given by

$$F = -\frac{GMm}{r^2} = -\frac{k}{r^2} \tag{1.7.2}$$

Here r is the distance between the Sun and the planet. This is clearly a central force of attraction with the centre of force at the sun's geometric centre. The equation of motion along r and θ direction of polar coordinates for the planet under the centre of force of the sun is given as

$$m\ddot{r} - mr\dot{\theta} = -\frac{k}{r^2} \tag{1.7.3a}$$

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0 \tag{1.7.3b}$$

Again from equation (1.7.3b) we have, $r^2\dot{\theta} = \dfrac{L}{m} = h$, where L is a constant and the angular momentum of the planet. The constant h is therefore the angular momentum per unit mass of the planet.

Substitution of $r = \dfrac{1}{u}$ and following the steps to deduce the equation of orbits as discussed in the beginning of this section, finally yields

$$\frac{\partial^2 u}{\partial\theta^2} + u = \frac{k}{mh^2} \tag{1.7.4}$$

Substitute $y = u - \dfrac{k}{mh^2}$, to reduce the equation to

$$\frac{\partial^2 y}{\partial\theta^2} + y = 0 \tag{1.7.5}$$

The standard solution of this equation is,

$$y = A\cos(\theta - \theta_0)$$

or, in this case,

$$u = \frac{1}{r} = \frac{k}{mh^2} + A\cos(\theta - \theta_0) \tag{1.7.6}$$

$$\text{or,} \qquad \frac{\frac{mh^2}{k}}{r} = 1 + \frac{mh^2 A}{k}\cos(\theta - \theta^0)$$

$$\text{or,} \qquad \frac{l}{r} = 1 + G\cos(\theta - \theta_0)$$

which is the equation of a conic with semi-centre $l = \dfrac{mh^2}{k}$ and eccentricity $G = \dfrac{mh^2 A}{k}$.

Now we use the equation (1.7.6) with the maximum and minimum values of the cosine term to find the minimum and maximum distances.

$$r = \frac{1}{\frac{k}{mh^2} + A\cos(\theta - \theta_0)} \tag{1.7.7}$$

$$r_{max} = \frac{1}{\frac{k}{mh^2} - A} \tag{1.7.8}$$

$$r_{min} = \frac{1}{\frac{k}{mh^2} + A} \tag{1.7.9}$$

As r is always a positive quantity, it is clear that A must be less than $\dfrac{k}{mh^2}$,

$$i.e., \qquad A < \frac{k}{mh^2} \tag{1.7.10}$$

$$\Longrightarrow \qquad \frac{Amh^2}{k} < 1 \tag{1.7.11}$$

$$\therefore \qquad G < 1 \tag{1.7.12}$$

So the orbit of the planet is elliptical. This means that the planet always moves round the sun in elliptical orbit, which is the *Kepler first law.*

Again the equation (1.7.3b) is cast to

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{1.7.13}$$

$$mr^2\dot{\theta} = L = \text{constant}, \tag{1.7.14}$$

The area swept out by the radius under the planet of in moving from say P to Q,

$$dA = \text{Area of the } \triangle OPQ = \frac{1}{2}r.rd\theta = \frac{1}{2}r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \frac{h}{2} = \text{constant}$$

i.e., the area swept by radius vector is constant, This is *Kepler second law.*

For deducing the Képlar's third law, we calculate the time period of revolution of planet. This is given by

$$\begin{aligned} T &= \frac{\text{Area of the ellipse}}{\text{Areal velocity}} \\ &= \frac{\pi ab}{\frac{h}{2}} \qquad \left(\because r^2\dot{\theta} = h = \frac{L}{m}\right) \\ &= \frac{2\pi ab}{h} \end{aligned}$$

Here a is semi major axis and b is semi minor axis of the ellipse. The relation between a, b and the semi latus rectum l of the ellipse is given by

$$l = \frac{b^2}{a}$$

$$\text{Therefore,} \quad T = \frac{2\pi a(al)^{\frac{1}{2}}}{h}$$

$$\begin{aligned} \text{or,} \quad T^2 &= \frac{4\pi^2 a^3 l}{h^2} \\ &= \frac{4\pi^2 a^3}{h^2}\left(\frac{mh^2}{k}\right) \\ &= \frac{4\pi^2 m a^3}{k} \\ &= \frac{4\pi^2}{GM} a^3 \end{aligned}$$

$$Hence \quad T^2 \propto a^3$$

i.e., the square of the period of revolution is proportional to the cube of the semi major axis of the ellipse. This is the *Keplar's third law.*

Summary

In this chapter we have discussed the Newton's laws of motion for a single particle as well as for a system of particles and deduced the conservation laws. We have further discussed the central force field-the definition and properties, and a method to reduce a two body problems to equivalent one body problem for describing the motion under central force field. Also discussed the equation involving the energy of particles in motion and deduced orbit equations for analysing the motion bounded as well as the unbounded motion. It is apparent that the choice of the coordinate system plays an important role in the convenience of finding the solution of a problem. We see that for analysing the problems involving the central force and particularly the inverse square law of force, polar coordinate system has an advantage over the conventionally taken cartesian coordinate system.

Taking the inverse square potential as a specific case of the central force field, we derived the associated orbit equation and analysed the nature of orbits for different energy values. Finally we have used them to establish the Keplar's laws of motion.

Self study questions:

1. Write a note illustrating the interrelations amongst the Newton's three laws of motion.

2. Compare the dynamics and the methods of analysis for a single particle and a system of particles.

3. A particle of mass m freely falls vertically downwards from a height h above the surface of the earth. Now assume that the earth's gravity suddenly switches off when the particle reaches its midpoint of the verical fall. Describe the dynamics of the particle for the entire course of its motion. Will the particle touch the ground eventually? Explain.

4. Check if the following force vector is conservative or not.
$$\vec{F} = x\hat{i} - y^2\hat{j} + z^3\hat{k}.$$

5. Show that the angular momentum and kinetic energy of a system of particles can be expressed in terms of, and motion about the centre of mass of the system.

6. What is a central force? Is this force a conservative one? Write down the characteristics of a central force and give examples.

7. Imagine a particle moves under a inverse cube of forces. Deduce the relevant equations to check if the angular momentum of the particle is conserved or not.

8. What are bounded and unbounded motion? Explain the term turning points in the case of a particle motion under inverse square law of force.

9. Starting with the inverse square force field of attractive nature, deduce the form of the differential equation of the orbit of a particle moving under the influence of the field.

10. Deduce the Keplar's laws of motion for the attractive gravitational force between the sun and planets.

UNIT 2

Lagrangian Formulation of Dynamics-I

Preparatory inputs to this unit

1. Problem solving skills in simple dynamical problems using Newton's laws (Preceding materials useful).
2. Concepts in the Central force problem, Inverse square potentials and Keplar's laws on planetary motion.
3. Basics of vector algebra, trigonometry, coordinate geometry.

2.1 Introduction

As is well known, the motion of material bodies involves two aspects *viz.*, kinematics and dynamics. Kinematics is the geometrical description of motion of material bodies, *i.e.*, the motion of the material bodies are described in terms of quantities involivng geometric measurements such as displacement, velocity etc. The dynamics on the other hand, investigates the source or the cause of the motion along with the motional properties of a system through the equation of motion involving the force. For centuries the problem of motion and its causes was a central theme of natural philosophy. The issues related to the motion and its causes was greatly resolved, in a wide spectrum of situations by Sir Isaac Newton (1642-1727) through the enunciation of the laws of motion. He also formulated the law of universal gravitation. But today, it is obvious that Newton's theory is an approximation to be valid under certain given circumstances only. In this chapter, we proceed to discuss the technical inconveniences of using the Newtonian laws of motion and discuss alternative formalisms to overcome the difficulties faced, through the introduction of the concept of constraint and the related principles.

2.1.1 Newtonian Dynamics

Newtonian dynamics is a mathematical model which aims to predict the motions of particles and objects under a variety situations surrounding us. The general principles towards developing this model were first put forward by Sir Isaac Newton in his seminal work entitled *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), first published in 1687 and is more commonly referred to as the *Principia*. Until the beginning of the 20th century, Newton's theory of motion was thought to be capable of providing a complete description of all types of motions taking place in the Universe.

Newtonian dynamics are based on three axioms which are known as **Newton's laws of motion**. They are

1. Every body continues to be in its state of rest or of uniform motion in a straight line unless impressed by external forces to change the state.

2. The time rate of change of momentum of a system in motion, is proportional to the magnitude of the external force and the change takes place in a direction same as that of the force.

3. To every action there is an equal and opposite reaction. Alternatively, the mutual actions and reactions of any two particles are always equal and oppositely directed along the straight line joining the particles.

Newton's first law of motion

Newton's first law of motion essentially states that a point object subject to zero net external force moves in a straight line with a uniform speed (*i.e.*, it does not accelerate). However, this is only true in special frames of reference called inertial frames. We can consider the Newton's first law for the definition of an inertial frame. An inertial frame of reference is the one in which a particle subject to zero net external force moves in a straight line with constant speed.

In particular, Newton's first law of motion tells us about the motion of a body when no force acts on it, but unable to provide the details of the motion arising from the force. Nor can it illustrate how does the force cause the motion in a quantitative manner. It does simply tell us what happens to the particle when the force is absent.

Newton's second law of motion

Newton's second law of motion tells us that the motion in a point particle experiencing an external force, $\vec{F}$, is governed by an equation given by

$$\vec{F} \propto \frac{d\vec{p}}{dt}$$
$$\text{or,} \qquad \vec{F} = k\frac{d\vec{p}}{dt}$$

where k is the constant of proportionality, the momentum, p, is the product of the object's inertial mass, m, and its velocity, $\vec{v}$. Here the units of the force is so chosen that the constant of proportionality takes in a value of unity, *i.e.*, $k = 1$. Considering that the mass m is not a function of time, the above expression reduces to the familiar equation

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

This equation is only valid in a inertial frame. Clearly, the inertial mass of an object measures its *reluctance* to change from its existing state of motion. The above equation of motion can only be solved if the form of the force $\vec{F}$ is known *apriori* via some prescription for the expression of the force.

N.B. : Newton's second law is applicable only if the force is the net external force. Further, it does not apply directly to situations where the mass is changing, either through the loss or gain of materials, or because the object is traveling close to the speed of light where relativistic effects must be included. It does not apply directly on the very small scale of the atom where quantum mechanics must be used.

Newton's third law of motion

This law says that all forces in the universe occur in oppositely directed pairs of equal magnitude, and the existence of isolated force is an impossibility. For every external force that acts on an object there is a force equal in magnitude but with opposite direction acting back on the object which exerts the external force. In the case of internal forces (forces acting from and to within a system), the force on one part of a system will be countered by an equal reaction so that in an isolated system there is no net internal force on the system as a whole. A system cannot *bootstrap* itself into motion with the internal forces exclusively. To achieve a net force and an acceleration, it must interact with an object external to itself.

Newton's third law is one of the fundamental symmetry principles of the universe. Since we have no examples of it being violated in nature, it is a useful tool for analyzing situations.

2.1.2 Inconveniences with Newtonian dynamics

Newton's laws of motion directly serves as the governing laws for investigating dynamical systems. But as will be clear soon, the applications of Newton's laws for solving dynamical problems has some practical limitations. This is not because of the laws are wrongly formulated or wrong interpretations of its underlying meanings. The limitations are from the point of view of the inconveniences and difficulties in using Newtonian dynamics. Some of such difficulties are mentioned below.

1. Newton's laws are valid only in inertial frames. Inertial frames are frames which are mutually unaccelerated frames, *i.e.*, if the motion of bodies are described through Newton's laws in a given frame, say frame A, the same law will be valid in some other frame, say B, which is either at rest or of uniform motion with respect to the frame A. The frame B must be unaccelerated with respect to frame A.
   For non-inertial frames, such as a rotating frame, which involves acceleration, a new, transformed version of Newton's equations of motions are required for correct description of dynamics, but this exercise involves the appearance of pseudoforces like coriolis forces or centrifugal forces. As for example we may consider a dynamical problem in a frame attached on the surface of earth. In this case, the frame of referece attached to the earth rotates with it around the earth's axis of rotation. The motion of any object on the surface of the earth is thus a motion in non-inertial frame of reference.

2. Newton's laws require an *apriori*, complete specification of all the external forces acting on the body to correctly describe the dynamics of a moving body. So, one is not supposed to miss out any force acting on the body, lest a correct description of the dynamics will allude. This fact causes a real difficulty in achieving the desired solution to a dynamical problem. This mostly happens in the case of constrained motion, *i.e.*, motion cannot take place in all the available coordinate dimensions.

2.1.3 Constraints and Degrees of freedom

Constraints

Consider the motion of a free particle. To describe this motion, we need a coordinate system with three independent co-ordinates, such as the cartesian co-ordinates (x, y, z) or the spherical co-ordinates (r, θ, ϕ) and so on. The particle is free for motion along any one of the coordinate axis independent of the change in other co-ordinates. So, the motion of the free particle can be expressed in terms of three independent coordinates. Now when the motion of the particle is restricted along one coordinate direction, we say that one constraint is imposed on the motion of the particle and under such circumstances we need only two independent coordinates to locate the position of the moving particle at any instant of time. In general, when the motion of a system is restricted in some way, constraints are said to have been introduced. Constraints are generally expressed in the form of relationships between the co-ordinates of the constituent particles of a system, or their derivatives and the time. Such a relationship is also known as *constraint relations or constraint equations*. Thus, if r_i are the co-ordinates of the particles of a system consisting of N particles, then

$$f(r_1, r_2, ..., r_N;\ \dot{r}_1, \dot{r}_2, \ldots, \dot{r}_N;\ t) = 0 \tag{2.1.1}$$

is a constraint relation or constraint equation. Representing the coordinates as $r \equiv (r_1, r_2, \ldots, r_N)$ or their time derives as $\dot{r} \equiv (\dot{r}_1, \dot{r}_2, \ldots, \dot{r}_N)$, we rewrite the equation of constraints as

$$f(r, \dot{r}, t) = 0 \tag{2.1.2}$$

Another example of constraint relation is

$$f(r, \dot{r}, t) \gtreqless 0 \tag{2.1.3}$$

The constraints represented by (2.1.2) are called *bilateral* constraints, while those represented by (2.1.3) are known as *unilateral* constraints. If a constraint equation involves co-ordinates as well as their derivatives, the corresponding constraint is termed as *differential* or *kinematic.* In contrast, constraints whose equations contain only co-ordinates are called *geometric.* Thus, while equations (2.1.2) and (2.1.3) represent differential constraints, the following equations exemplify geometric constraints:

$$f(r, t) = 0 \tag{2.1.4}$$

$$f(r) = 0 \tag{2.1.5}$$

The equation (2.1.4) is explicitly dependent on time; and equations of this type are called *rheonomous* (or moving) constraints. On the other hand, the equation (2.1.5) is independent of time, and constraints of this types are called *scleronomous* (or stationary) constraints.

A more useful classification of constraints is based on the fact whether some of the co-ordinates can be expressed in terms of the remaining ones. If a constraint equation expresses a relationship between the co-ordinates and time explicitly through the equality sign, the constarint is said to be *holonomic* and all the cases otherwise, the constraints are *nonholonomic.* It is obvious that unilateral geometric constraints are nonholonomic.

If a particle is confined to move inside a sphere, the constraint relation is given by

$$(x^2 + y^2 + z^2) - a^2 \leq 0 \tag{2.1.6}$$

where a is the radius of the sphere. The constraint is unilateral and therefore nonholonomic.

Degrees of freedom

We have seen that if a system requires M coordinates to specify its unconstrained state, and if k number of constraints are imposed on it, the constrained system requires only $(M - k)$ independent coordinates to represent it, as k number of coordinates will be related through the equations of constraint. We then say that the system possesses $n = (M - k)$ number of *degrees of freedom.* If g of the k constraints are nonholonomic, then the degrees of freedom are still $(M - k)$, but the number of independent coordinates required to specify the position of the system is $[M - (k - g)]$, since only $(k - g)$ coordinates, corresponding to holonomic constraints, can be eliminated.

Examples

1. A marble rolling on the surface of a table has only two degrees of freedom. Had the table been not there, the marble would have occupied any of the locations in the 3-dimensional space above the surface of the earth, and we would require three independent coordinates say, $(x,\ y,\ z)$ to specify the location of the marble at any instant of time in cartesian coordinate system. Hence the system has $M = 3$. But as the marble is constrained to move on the plane of the table, which is say $z = constant$ plane for our convenience, we require only two coordinates $(x,\ y)$, as the z-coodinate is already fixed, or known for the marble. So, the restriction imposed on the movement of the marble is through $z = constant$, and this is the constraint equation for the motion of the marble, *i.e.*, $k = 1$ in this case. The degrees of freedom of the marble rolling on the plane of the table is then $n = M - k = 3 - 1 = 2$, corresponding to two free directions of movement.

2. A simple pendulum is a case of restricted motion of its bob, which is normally made to oscillate on a vertical plane, maintaining a fixed distance from the point of suspension through the constant length, say l, of the string of the pendulum. The otherwise unconstrained motion of the bob requires three independent coordinates, say $(x,\ y,\ z)$ in cartesian coordinates chosen with origin at the point of suspension and the XZ-plane as the plane of oscillation. In this case we have $M = 3$. The restrictions imposed are

   (a) the plane of oscillation (the XZ-plane) yielding $y = 0$, and

   (b) maintaining fixed distance (l) during the oscillation, *i.e.*, $x^2 + z^2 = l^2$.

   So we have $k = 2$. Therefore the degrees of freedom in the simple pendulum is $n = M - k = 3 - 2 = 1$ and we need only one independent coordinate to completely describe the dynamics of a simple pendulum.

3. We take example of double pendulum (a system of two pendulums, the point of suspension of the second pendulum is at the bob of the first pendulum) moving in a vertical plane. We would require four independent coordinates *viz.*, $(r_1,\ \theta_1)$ for the first and $(r_2,\ \theta_2)$ for the second pendulum, to describe the system completely. But as we know, the length of a pendulum is a constant. This means, we need to impose two restrictions $(r_1 = constant = l_1)$ and $(r_2 = constant = l_2)$ on the system to correctly represent its motion. These equations, representing the restrictions are the equations of constraints. So, in this case we say that two of the coordinates are eliminated by the equations of constraints and the system is completely described by the remaining two independent coordinates; with $M = 4$, $k = 2$ and $n = M - k = 4 - 2 = 2$.

4. We consider a rigid body. A rigid body is defined as a system of particles in which the relative distances of the constituent particles are fixed and does not vary with time. In this case, the constraints are expressed by the equations of the form

   $$r_{ij} = c_{ij},$$

   in which c_{ij} and r_{ij} denote the distances between i-th and j-th particles. With the positions of the particles in cartesian coordinates as $r_i(x_i, y_i, z_i)$ and $r_j(x_j, y_j, z_j)$ respectively, the conditions are expressed as:

   $$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = c_{ij}^2 \qquad (2.1.7)$$

5. Consider a circular disc rolling on a horizontal plane without sliding. The constraint of no sliding implies

$$dx = rd\phi \sin\theta$$

$$dy = rd\phi \cos\theta$$

where θ is the angle between the plane of the disc and the YZ-plane, while ϕ is the angle associated with the rolling of the disc. Both of the above constraints are nonholonomic. The disc has only two degrees of freedom but requires four independent coordinates to specify its position. Thus, the nonholonomic constraints do not lead to the reduction in the number of independent coordinates required to specify the position of the system, though they restrict the degrees of freedom of the system.

A dynamical system with N particles, when in unconstrained state, the maximum number of allowed independent coordinates is given by $M = 3N$ corresponding to 3 coordinates for each of the particles. Such a system when is restricted by k number of constraints will have $n = 3N - K$ number of degrees of freedom.

2.1.4 Difficulties with constraints

Constraints introduce two kinds of difficulties in studying dynamics.

1. They bind the coordinates r_i by the constraint equations, so that all the coordinates are no longer independent. They are related through the equations of constraint. Consequently, the equations of motion are also not independent.

2. The forces of constraints are not known a priori. They are to be determined from the solution of the problem. In the absence of the knowledge of all the forces, the Newtonian equations of motion will not truely reflect the realistic dynamical situation.

If the constraints involved are holonomic, the first difficulty can be overcome by introducing what is called the *generalised coordinates*. The second difficulty is overcome by formulating the problem of motion of a system in such a way that an explicit knowledge of all the forces acting on the system is not necessary.

2.1.5 Generalized coordinates and Generalised Velocity

Generalized coordinates are in general some parameters which are used to describe the configuration of the system relative to some reference configuration. The basic requirement for a set of numbers to act as generalised coordinates is its uniqueness, in the sense that these parameters must uniquely define the configuration of the system in relation to a given reference configuration. Further the parameters are also required to be indendent of each other so that any change in any of these parameters must not affect other members of this parameter set. Another term, which also finds its place during the description of a dynamical system is the generalized velocity. Generalised velocities are defined as the time derivatives of the generalized coordinates of the system.

An example of a generalized coordinate is the angle that locates a point moving on a circle. The term *generalised* distinguishes parameters from the use of the term coordinate to refer to Cartesian coordinates. For example, we normally describe the location of the point on the circle by (x, y) coordinates in the cartesian coordinate system. But this is to be remembered that these coordinates are not independent of each other; they are connected by a relation $x^2 + y^2 = r^2$, where r is the radius of the circle. We can specify the location of the point on the circle by a single parameter, the angle θ measured from a pre-defined straight line, called the *prime line*. The parameter θ here serves as the generalised coordinate of the point on the circumference of the circle.

There may be multiple choices for generalized coordinates that are used to describe a physical system; the best choice of the set is motivated by the prospective convenience and ease with which one obtains the solution of the equations describing the system. As these parameters need to be independent of one another, the number of independent generalized coordinates is decided by the number of degrees of freedom of the system.

Thus the term generalised coordinates refer to a set of parameter, independent of each other and used to uniquely and completely specify a system. They are not necessarily the physical coordinates describing some linear distance. For a system of N particles with position vectors $\{\vec{r}_i\} \equiv (\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ with k number of say holonomic constraint relations will have $n = 3N - k$ degrees of freedom and we need to choose n number of independent parameters as generalised coordinates for specifying the system. Let the generalised coordinates be $\{q_i\} \equiv (q_1, q_2, \ldots, q_n)$. The components of the position vectors for each of the N particles will be functions of the generalised coordinates expressible as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n)$$

The differential of $\vec{r}_i$ can be found by using the chain rule as,

$$d\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1} dq_1 + \frac{\partial \vec{r}_i}{\partial q_2} dq_2 + \ldots + \frac{\partial \vec{r}_i}{\partial q_n} dq_n = \sum_j^n \frac{\partial \vec{r}_i}{\partial q_j} dq_j$$

When $\vec{r}_i$ has dependence on time, *i.e.*, when $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n; t)$ the differential change is given by

$$d\vec{r}_i = \sum_j^n \left(\frac{\partial \vec{r}_i}{\partial q_j} dq_j \right) + \frac{\partial \vec{r}_i}{\partial t} \tag{2.1.8}$$

2.2 Virtual Work and Virtual Displacement

2.2.1 Virtual displacement

A mechanical system is said to be in equilibrium if the the resultant *i.e.*, the vector sum of all the forces acting on it is zero. Under this condition, the position occupied by, or the configuration of the system at any instant of time is called an equilibrium position or configuration.

We thus see that, consistent with the constraints imposed on it, the system can have several positions of equilibrium at any given time.

We can consider the differences of any two such closely separated equilibrium positions at the same instant of time as the *distance or displacement* between them. Such *displacement*, which are considered have occured without involving a progress in time, is termed as the *virtual displacement.*

A virtual displacement may not coincide with any of the actual displacements that a system undergoes during it's motion. Virtual displacement is an important concept since it allows us to investigate the conditions under which a mechanical system will be in equilibrium.

So the virtual displacement in a dynamical system is defined as a change in the configuration of the system as the result of any arbitrary infinitesimal change in the coordinates δr_i, consistent with the forces and constraints imposed on the system at a given instant t. The displacement is called virtual in order to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be operating.

Virtual displacement in generalised coordinates

Let us consider a system described by n generalised coordinates q_j $(j = 1, 2, ..., n)$. Let us suppose the system undergoes certain displacement in the configuration space in such a way that it does not take any time and that it is consistent with the constraints on the system. Then the virtual displacement is defined in terms of the generalised coordinates as

$$\delta r_i = \left(\frac{\partial r_i}{\partial q_j}\right) \delta q_j$$

where δq_j are the virtual displacements associated with the generalised coordinates.

2.2.2 Virtual Work

The Virtual work is the work associated with the virtual displacement and is defined as the work done by a force $\vec{F}$ during a virtual displacement δr in a particle. Thus

$$\delta W = \vec{F}.\delta\vec{r}$$

For a system consisiting of N particles, the virtual work on the i-th particle due to a force $\vec{F_i}$ on it is defined as

$$\delta W_i = \vec{F_i}.\delta\vec{r_i}$$

And the total virtual work done on all the particles is expressed as

$$\delta W = \sum_i \delta W_i = \sum_i \vec{F_i}.\delta\vec{r_i}$$

A comparison of virtual displacement and real displacement

	Virtual Displacement		Real Displacement
1.	Not a change in the coordinates in the physical space, merely a shift in the allowable configurations.	1.	A change in the coordinates in the physical space.
2.	Does not involve any time in the change of the configurations.	2.	Not instaneous, involves a progression in time.
3.	Does not represent the true motion in the particles.	3.	Represents true motion in the particles.
4.	Associated work done always zero	4.	Associated work done may not be zero

It should be noted that as the virtual displacement does not represent any physical displacement and does not involve any time, the virtual work done by the forces of constraints is always zero. Thus if $\vec{f}_i$ is the force of constraint on the i-th particle and $\delta\vec{r}_i$ is the corresponding virtual displacement, then

$$\sum_i \vec{f}_i.\delta\vec{r}_i = 0. \tag{2.2.1}$$

2.2.3 Principle of Virtual work

The principle of virtual work states that the necessary and sufficient condition for a system to be in equilibrium is that the virtual work done by all the forces acting on the system is zero. Thus

$$\delta W = \sum_i \vec{F}_i.\delta\vec{r}_i = 0 \qquad \text{for equilibrium.}$$

But as the forces acting on any system can be the externally applied ($\vec{F}^e$) or the forces of constraints ($\vec{f}$) we can write the equation as

$$\sum_i \vec{F}_i^e.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0 \qquad \text{for equilibrium.}$$

which, on consideration of equation (2.2.1), reduces to

$$\boxed{\sum_i \vec{F}_i^e.\delta\vec{r}_i = 0} \tag{2.2.2}$$

Thus the principle of virtual work can be rephrased as :

The necessary and sufficient condition for a system to be in equilibrium is that the virtual work done by all the exterally applied forces acting on the system is zero.

Generalized coordinates and virtual work

The principle of virtual work states that if a system is in static equilibrium, the virtual work of the applied forces is zero for all virtual movements of the system from this state, that is, $dW = 0$ for any variation $\delta\vec{r}$. When formulated in terms of generalized coordinates, this is equivalent to the requirement that the generalized forces for any virtual displacement are zero.

In order to establish this, let us consider that the force $\vec{F}_j$ acts on j-th particle (position vector r_j) of a system consisting of N particles. Suppose that the system has n number of degrees of freedom. The virtual work generated by a virtual displacement from the equilibrium position is then given given by

$$\delta W = \sum_{j=1}^{N} \vec{F}_j.\delta\vec{r}_j \tag{2.2.3}$$

Now assume that each $\delta\vec{r}_j$ depends on the generalized coordinates q_i, where $i = 1, ..., n$, then

$$\delta\vec{r}_j = \frac{\partial\vec{r}_j}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_j}{\partial q_2}\delta q_2 + ... + \frac{\partial\vec{r}_j}{\partial q_n}\delta q_n$$

and hence

$$\begin{aligned}\delta W &= \left(\sum_{j=1}^{N} \vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_1}\right)\delta q_1 + \left(\sum_{j=1}^{N} \vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_2}\right)\delta q_2 + ... + \left(\sum_{j=1}^{N} \vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_n}\right)\delta q_n \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{N} \vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_i}\right)\delta q_i \\ &= \sum_{i,j}^{n,N}\left(\vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_i}\right)\delta q_i = \sum_{i,j}^{n,N} Q_j\delta q_i \end{aligned} \tag{2.2.4}$$

where

$$Q_j = \vec{F}_j.\frac{\partial\vec{r}_j}{\partial q_i} \tag{2.2.5}$$

The left hand side of the equation has the dimension of work and so also has to be on the right hand side, *i.e.*, $Q_j\delta q_i$ has the dimension of work. Since the right hand side is a product of Q_j and the generalised virtual displacement δq_i, the quantity Q_j must be identified with *generalised forces* acting on the system. It is to be noted that Q_j need not have a dimension of a force just as the generalised coordinates does not need to have a dimension of length. *The only requirement on Q_j and δq_i is that their product must have a dimension of work.*

2.2.4 D'Alembert's principle

The principle of virtual work can be extended to cover mechanical systems in motion. Named after the discoverer Jean le Rond D'Alembert, a French physicist and mathematician, the D'Alembert's principle, is a statement of the fundamental classical laws of motion and is the dynamic analogue to the principle of virtual work for applied forces in a system.

Let us suppose an applied force $\vec{F}_i$ acts on a particle i of a system producing in it a momentum $\vec{p}_i$. We may then write the equation of motion for the particle as $\vec{F}_i - \dot{\vec{p}}_i = 0$, and assume that the particle is in equilibrium under the joint influence of the applied force $\vec{F}_i$ and a *reverse effective force* $\dot{\vec{p}}_i$ acting along the direction opposite to that of $\vec{F}_i$. The reverse effective force is also known as the *kinetic reaction*. Thus, the consideration of the principle of virtual work under the action of these joint forces can then be written as

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \tag{2.2.6a}$$

$$\sum_i \vec{\mathcal{F}}_i.\delta\vec{r}_i = 0 \tag{2.2.6b}$$

Equation (2.2.6b) is the statement of the principle of virtual work for the case of the resultant force $\vec{\mathcal{F}}$. We see that the principle can be applicable for the case of a moving system. This extension of the principle of virtual work to the case of a moving system is known as the *D'Alembert's Principle.*

The principle states that the total virtual work done by sum of the forces acting on a system of particles and the kinetic reaction time derivatives of the momenta of the system itself along any virtual displacement consistent with the constraints of the system, is zero.

We can ignore internal forces, as these occur in pairs, and decompose the force into an applied force $\vec{F}_i^e$, and a (holonomic) constraint force $\vec{f}_i$. Considering the fact that constraint forces does not do any work, the D'Alembert's principle can be rewritten as

$$\sum_i (\vec{F}_i^e - \dot{\vec{p}}_i) = \sum_i \left[\vec{F}_i^e - \frac{d}{dt}(m_i\dot{\vec{r}}_i)\right] = 0. \tag{2.2.7}$$

2.3 Lagrange's equations for holonomic constraints

Consider a dynamical system with N particles, the mass and position vector of the i-th particle being m_i and $\vec{r}_i$, respectively. Let the system moves under a set of applied forces, the force $\vec{F}_i$, being applied to the i-th particle and so on. Further the forces are assumed conservative such that they can be expressed as the gradient of a scalar potential energy function $V(\{\vec{r}_i\})$:

$$\vec{F}_i = -\vec{\nabla}_i V$$

where $\quad \vec{\nabla}_i = \hat{i}\frac{\partial}{\partial x_i} + \hat{j}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i}$ is the gradient in reference to the i-th particle.

To describe this system we would apply the priciple for dynamical equilibrium, *i.e.*, the D'Alembert's principle, which avoids the forces of constraints and requires only the number of degrees of freedom. We assume here that the N-system is under k number of holonomic constraints. The degree of freedom is therefore $n = 3N - k$. This means we need n generalised coordinates. Let the generalised coordinates be $\{q_j\} \equiv (q_1, q_2, q_3, \ldots, q_n)$. Thus the expression of virtual displacement in terms of these generalised coordinates is written was

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \tag{2.3.1}$$

where the term like $\frac{\partial \vec{r}_i}{\partial t}\delta t$ will not appear in the expression because virtual displacement does not involve any time. The actual displacement, of course, is dependent on time and gives rise to the concept of velocitiy which we know is the rate of change of the position vector over a time, *i.e.*,

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \tag{2.3.2}$$

We now transform the D'Alembert's principle in generalised coordinates. The first term is the amount of virtual work in the system. (Recall the steps leading to equation (2.2.5)). We write the first term as

$$\sum_i \vec{F}_i.\delta\vec{r}_i = \sum_j Q_j \delta q_j \tag{2.3.3}$$

Next we wish to rewrite the second term of the principle in terms of the virtual displacements of the gneeralised coordinates. So, the term is

$$\begin{aligned}
\sum_i \dot{\vec{p}}_i.\delta\vec{r}_i &= \sum_i \frac{d}{dt}(m_i\dot{\vec{r}}_i).\delta\vec{r}_i \\
&= \sum_i \frac{d}{dt}(m_i\dot{\vec{r}}_i).\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j \\
&= \sum_{ij}\left(m_i\ddot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j}\right)\delta q_j
\end{aligned} \tag{2.3.4}$$

We rewrite the term within the bracket of equation (2.3.4) above, which is the coefficient of δq_j as the following:

$$\sum_i \left(m_i\ddot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_i \left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i\dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \tag{2.3.5}$$

Using Equation (2.3.2), we can interchange the order of differentiation in the last term of the equation (2.3.5) as the following:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) &= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} \\
&= \frac{\partial \dot{\vec{r}}_i}{\partial q_j}
\end{aligned} \tag{2.3.6}$$

This expression shows that while differentiating the position coordinates, one can interchange the order of differentiation with respect to the generalised coordinates and time.

Further, differentiation of the (2.3.2) with respect to $\dot{q}_j$ yields

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{2.3.7}$$

We now substitute the results of equations (2.3.6) and (2.3.7) into equation (2.3.5) to get

$$\sum_i \left(m_i \ddot{\vec{r}}_i . \frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i . \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - m_i \dot{\vec{r}}_i . \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right]$$
$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2\right)\right] - \frac{\partial}{\partial q_j}\left[\left(\sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2\right)\right]$$
$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \tag{2.3.8}$$

where $T = \sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2$ is the total kinetic energy of the system of N particles.

We are now in a position to write down the D'Alembert's principle after full conversion in terms of generalised coordinates and generalised velocities. Combining equations (2.3.8), (2.3.4) and recalling the expression for the generalised forces (2.2.3) and (2.2.4), we write the D'Alembert's principle as

$$\sum_i \left[\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right) - Q_j\right] \delta q_j = 0 \tag{2.3.9}$$

Now since the generalised coordinates chosen for a system are always independent of each other, the corresponding virtual displacements δq_j in the generalised coordinates are also arbitrary. The only way the right hand side becomes zero is to consider each of the bracketted quantity in the sum will be separately zero, *i.e.*, we must have, for any value of j,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \tag{2.3.10}$$

This set of n equations is called Lagrange's equations. Note that while deducing (2.3.10) we have not used any specific property of the applied force, *viz*, whether the force is conservative or non-conservative. So (2.3.10) will be equally valid for both conservative and non-conservative forces. However for the case of the conservative forces, the force vectors can be derived from potential energy function V. We can write the cartesian components of the force on the i-th particle as,

$$F_{ix} = -\frac{\partial V}{\partial x_i}, \quad F_{iy} = -\frac{\partial V}{\partial y_i}, \quad F_{iz} = -\frac{\partial V}{\partial z_i}$$

Further, we consider that the potential energy function V is a function of position, ($\vec{r}_i$ or q_j) and is not dependent on velocity ($\dot{\vec{r}}_i$ or $\dot{q}_j$). Hence the expressions for the generalised forces will be as

$$Q_j = \sum_i \vec{F} . \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}\right]$$
$$= -\sum_i \left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right]$$
$$= -\frac{\partial V}{\partial q_j} \tag{2.3.11}$$

As the potetial energy function is considered not dependent on velocity, we have $\frac{\partial V}{\partial \dot{q}_j} = 0$.

Therefore the form of the Lagrange's equation under conservative force field is given by

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$$

or,
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{2.3.12}$$

where $L = L(q_j, \dot{q}_j, t) = T - V$ is called the Lagrangian function for the system. The equation (2.3.12) is known as the Lagrange's equations of motion for a conservative holonomic system.

Note:

1. Lagrange's equation of motion is an equation on the scalar L whereas Newton's equations of motion is dependent on vector quantities in 3-dimensional space, which can be broken down to three equivalent scalar equations in its components.

2. Deducing Lagrange's equations from D'Alembert's principle actually uses the Newton's laws of motion. Therefore, the resultant equation, *i.e.*, the Lagrange's equations for a system will be equivalent to Newton's laws of motion. Thus we see,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

which is same in the form with the Newton's second law of motion in the say, j-th component,

$$\frac{d}{dt}(p_j) = F_j$$

where p_j and F_j are the j-th component of the momentum and force respectively. Comparison of these two equations serves to generalise the the concept of momentum and the force. *i.e.*, the term $\frac{\partial L}{\partial \dot{q}_j}$ is the equivalent momentum term known as the j-th component of *generalised momentum* or *canonically conjugate momentum* or simply *conjugate momentum*. Similarly the equivalent force term $\frac{\partial L}{\partial q_j}$ is known as the *conservative generalised force*.

2.3.1 Cyclic coordinates and conservation laws

An important property of the Lagrangian is that it is easy to read off the conserved quantities or the conservation laws present in the system. Suppose a particular generalised coordinate, say q_i does not appear in the expression of a Lagrangian L, then the latter does not depend on the particular coordinate q_i. Then the corresponding generalised momentum is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Writing the i-th component of the Lagrange's equations in terms of p_i we find that

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = 0, \qquad \Longrightarrow\ p_i = \text{constant in time.}$$

This means, the i-th component of the generalised momentum is conserved. Thus, it is easy to see which generalised coordinates do not appear in the Lagrangian and figure out the corresponding generalised momenta to be conserved quantities. Such coordinates are called *cyclic* or *ignorable*.

For example, the conservation of the generalised momentum, say,

$$p_2 = \frac{\partial L}{\partial \dot{q}_2}$$

can be directly seen if the Lagrangian of the system is of the form

$$L(q_1, q_3, q_4, \ldots; \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \ldots; t)$$

Numerical examples

Example 2.3.1 *Find the Lagrangian the the Lagrange's equation of motion for a simple pendulum, for a small amplitude of oscillation.*

Solution:

Let the pendulum oscillate in the surface gravity of the earth in the $z = 0$ vertical plane of the cartesian coordinates, with the x-axis placed horizontal, y-axis vertically up and the origin being at the point of suspension. Let g be the acceleration due to gravity, m the mass of the bob and l be the length of the pendulum. The system can be described by a single variable θ, the angle made by the thread in any given instant, with the vertical y-axis. Therefore, the degree of the system is 1 with θ chosen as the generalised coordinate.

We write the coordinates of the mass at any given instant in terms of the generalised coordinates as $\vec{r} = (l\sin\theta, -l\cos\theta)$ *and the components of the velocity as* $\dot{\vec{r}} = (l\cos\theta\dot{\theta}, l\sin\theta\dot{\theta})$.

The kinetic energy of the system is then given by

$$T = \frac{1}{2}m|\dot{\vec{r}}|^2 = \frac{1}{2}m(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta) = \frac{1}{2}ml^2\dot{\theta}^2$$

Considering the horizontal line passing through the point of suspension of the pendulum as the reference of zero potential, the potential energy of the system is

$$V = -mgl\cos\theta$$

Thus the Lagrangian of the system is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

so that

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

The Lagrange's equation of motion is then given through

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$or, \quad \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

$$or, \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This is the equation of motion of the pendulum for an arbitrary amplitude of oscillation. If we consider the pendulum for small amplitude, the angle θ must be small enough so that $\sin\theta \approx \theta$. So the corresponding equation of motion will be

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Example 2.3.2 *Deduce the equation of motion for an Atwood maching by using Lagrage's equations.*

Solution:

An Atwood maching a system of two masses, say m_1 and m_2 connected at the ends by a light inextensible string of length l and the length passes over a frictionless pulley of negligible weight. The masses hang from the pulley under the surface gravity of the earth.

It is observed that the motion of the masses are not independent, they are related through the string. Hence the degree of freedom of the system must be 1. Consider 1-dimensional coordinate frame with its axis (say x-axis) vertically downwards for positive coordinates and the origin be at the pulley. So, this coordinate be taken as the generalised coordinates for the system.

Let the coordinate of the mass m_1 be x. Then the coordinate of the mass m_2 must be $l - x$.

The corresponding velocity of the masses will then be $\dot{x}$ and $-\dot{x}$ respectively.

The kinetic energy of the system is then the sum of the individual kinetic energies of the two masses, i.e.,

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Considering the horizontal line through the pulley to be the reference for zero potential, with +ve values upward and -ve values downward, the gravitational potential energy of the system is

$$V = -m_1gx - m_2g(l - x) = -(m_1 - m_2)gx + m_2gl$$

Hence the Lagrangian of the system is given by

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx - m_2gl$$

so that

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g \quad ; \qquad \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)g\dot{x}$$

The Lagrange's equation of motion is then given as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or, $$(m_1 + m_2)g\ddot{x} - (m_1 - m_2)g = 0$$

or, $$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)}g$$

which is the equation of motion for the masses in the Atwood machine.

Example 2.3.3 *Use Lagrange's equation to find the equations of motion of a compound pendulum in a vertical plane about a fixed horizontal axis.*

Solution:

Let the compound pendulum be suspended from a point S with C as a center of mass and execute oscillating motion in a vertical plane.

It's moment of inertia about the axis of rotation through the point S is given by

$$I = I_c + Ml^2 = M\left(K^2 + l^2\right)$$

Here the mass of the pendulum is taken as M.. The term $I_c = MK^2$ is the moment of intertia of the pendulum about an axis passing through it's centre of mass C, K is the radius of gyration about a parallel axis through the centre of mass and l is the distance between center of suspension and the center of mass.

Here the degrees of freedom is 1 and so we take θ the instantaneous angle, which the line joining S and C makes with the vertical axis through S, as the generalised coordinate. The kinetic energy of the compound pendulum oscillating in the given vertical plane is

$$T = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\left(K^2 + l^2\right)\dot{\theta}^2$$

We take the horizontal plane through S as the reference for zero potential and find the potential energy of the pendulum as $V = -Mgl\cos\theta$.

The Lagrangian is then

$$L = T - V = \frac{1}{2}M\left(K^2 + l^2\right)\dot{\theta}^2 + Mgl\cos\theta$$

Therefore,

$$\frac{\partial L}{\partial \theta} = -Mgl\sin\theta \quad ; \qquad \frac{\partial L}{\partial \dot{\theta}} = M\left(K^2 + l^2\right)\dot{\theta}$$

Hence the Lagrange's equation in θ coordinate is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \text{Therefore,} \qquad \frac{1}{2}M\left(K^2+l^2\right)\ddot{\theta} + Mgl\sin\theta &= 0 \\ \ddot{\theta} + \left(\frac{gl}{K^2+l^2}\right)\sin\theta &= 0 \end{aligned}$$

This is the equation of motion of the compound pendulum. If θ is small, then $\sin\theta \approx \theta$, so the equation of motion will reduce to,

$$\ddot{\theta} + \left(\frac{gl}{K^2+l^2}\right)\theta = 0$$

Example 2.3.4 *Find the equation of motion of a spherical pendulum using Lagrange's equations.*
Solution:
Here,

$$x = l\sin\theta\cos\phi$$

$$y = l\sin\theta\sin\phi$$

$$z = l\cos\theta$$

Then

$$\dot{x} = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi$$

$$\dot{y} = l\dot{\theta}\sin\phi + l\dot{\phi}\sin\theta\cos\phi$$

$$\dot{z} = -l\sin\theta\dot{\theta}$$

Therefore, the kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$$

and the potential energy is given by

$$V = -mgl\cos(\pi-\theta) = mgl\cos\theta$$

Therefore the Lagrangian is formed as

$$L = T - V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - mgl\cos\theta$$

Now, the equation of motion for θ-co-ordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Longrightarrow \qquad ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0 \qquad (A)$$

and the equation of motion for ϕ-coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\implies \quad ml^2\ddot{\phi}\sin^2\theta + 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0 \qquad (B)$$

Equations (A) and (B) are the equations of motion for a spherical pendulum.

Example 2.3.5 *A bead slides on a wire in the shape of an upright cycloid described by the equations*

$$x = a\,(\theta - \sin\theta)$$

$$y = a\,(1 + \cos\theta)\,,$$

where $0 \leq \theta \leq 2\pi$. Find the Lagrangian function and the equation of motion for the bead.

Solution:
Given that the path of the bead is represented by the equations

$$x = a\,(\theta - \sin\theta)$$

$$y = a\,(1 + \cos\theta)$$

The kinetic energy T of the bead is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}ma^2\left[\left(\dot{\theta} - \cos\theta\dot{\theta}\right)^2 + a^2\left(-\sin\theta\dot{\theta}\right)^2\right] = \frac{1}{2}ma^2\dot{\theta}^2\,(2 - 2\cos\theta) = ma^2\dot{\theta}^2(1-\cos\theta)$$

The potential energy of the bead is given by

$$V = mgy = mga\,(1 + \cos\theta)$$

The Lagrangian is therefore,

$$L = T - V = ma^2\dot{\theta}^2\,(1 - \cos\theta) - mga\,(1 + \cos\theta)$$

from which we have

$$\frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta}\,(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}^2\sin\theta + mga\sin\theta$$

The Lagrangian equation or equation of motion for the bead is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\implies \quad \frac{d}{dt}\left\{2ma^2\dot{\theta}(1 - \cos\theta)\right\} - \left(ma^2\dot{\theta}^2\sin\theta + mga\sin\theta\right) = 0$$

$$\implies \quad 2ma^2\ddot{\theta}\,(1-\cos\theta) + 2ma^2\dot{\theta}\sin\theta\dot{\theta} - ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$$

$$or, \qquad \ddot{\theta}\,(1-\cos\theta) + \frac{1}{2}\dot{\theta}\sin\theta - \frac{g}{2a}\sin\theta = 0$$

which is the required equation of motion.

Example 2.3.6 *Derive the equation of motion for a particle moving under central force. Find the form of the equation when the particle moves under an attractive inverse square law of force whose magnitude is given by*

$$\left(F = -\frac{k}{r^2}\right)$$

Solution:
Whenever the particle moves under a central force, the force has to be conservative and the motion must be confined to a single plane.

Let $(r,\ \theta)$ be the plane polar coordinates of the particle of mass m.

The kinetic energy for the particle is,

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Hence the Lagrangian is,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

where $V(r)$ is the potential energy in the central force field.
We find

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}, \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial L}{\partial \theta} = 0, \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Hence Lagrange's equation of motion for the r-coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\implies \qquad m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad (A)$$

and for θ-coordinates, it is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\implies \qquad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$or, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \qquad (B)$$

Now, from the given form of the attractive inverse square law force, we have $F = -\frac{\partial V}{\partial r} = -\frac{k}{r^2}$, *therefore* (A) *reduces to,*

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0 \qquad (C)$$

Equations (B) *and* (C) *are the required equations of motion.*

2.3.2 Summary

This unit of study has been associated with the discussions on the technical limitations in the use of the Newton's laws of motion. The presence of the forces of constraints in the dynamical systems constitute the prime cause of the said incoveniences. We have studied the characteristics of these constraints, its types and tried to see how constraints cause the difficulties in the use of Newtonian dynamics. For holonomic systems the limitations can be addressed through the use of the generalised coordinates and the introduction of virtual displacement and virtual work, along with the principle of virtual work to finally write down the equations of motion which are free from the forces of constraints. We donot need to know the forces of constraints *apriori*. This results in the development of an alternative formalism to tackle the dynamical problems, called the Lagrangian formalism, where the key parameter of motion is a scalar quantity called the Lagrangian. The equation of motion is a set of differential equations in the Lagrangian unlike the Newtonian formalism which requires the involved forces to be known, including the forces of constraints. Moreover in the Lagrangian formalism, the coordinates do not limit themselves only to the dimension of physical length - any free parameters of the system can serve as the coordinates, thus befitting the name of the generalised coordinates. The cyclic coordinates in the Lagrangian is directly helpful to count the number of conserved quantities associated with the motion.

Self study questions:

1. What are constraints? How do they affect the motion of a system? Explain the nature of the constraint forces with examples.

2. Distinguish between the holonomic and nonholonomic constraints, scleronomous and rheonomous constraints. Cite examples for each category of the constraints.

3. What are the degrees of freedom and the generalised coordinates? Explain the advantage of their use. Can the charge flowing through an electrical circuit be considered as generalised coordinates? Explain.

4. What are virtual displacement and virtual work? State and explain the principle of virtual work. Extend this to deduce the D'Alemberts principle.

5. Use D'Alemberts principle to deduce the Lagrange's equations of motion in the case of a conservative force and analyse the motion when the Lagrangian is an explicit function of time.

6. What is a cyclic coordinate? Explain the use of cyclic coordinates. Write down the consequences when all the generalised coordinates of a given Lagrangian are cyclic.

7. Write down the Lagrangian for a mass m in projectile motion with the initial velocity u_0 projected with an angle θ made with the horizontal and use the Lagrange's equations to write down the equations of motion for the projectile.

8. Deduce the equation of motion for a simple pendulum of length l and the bob mass m whose point of suspension is moving horizontally with an uniform velocity of magnitude u.

UNIT 3

Lagrangian Formulation of Dynamics-II

Preparatory inputs to this unit

1. Problem solving skills in dynamical problems using Lagrange's equations (Preceding materials useful).
2. Maxwell's equations: The basic equations of electromagnetic theory.

3.1 Lagrange's equations for Velocity Dependent Potential

We have already seen in the preceeding chapters that while deriving the Lagrange's equation of motion, the associated potential energy is a function of the distance between two locations, and not on the actual path traversed by the particle. Nor the corresponding forces and the potential energy is a function of time or velocity. Such a force is called a conservative force. For a non-conservative system the equation of motion in general cannot be written in terms of Lagrange's equation of motion. But in some special cases, it is still possible to write down the equations in terms of Lagrange's equations of motion and hence obtain an analytical solution.

Let us assume that the generalised forces Q_j can be written in terms of a scalar function $U(q_j, \dot{q}_j)$, a function dependent on velocity term $\dot{q}$ apart from the distance q, as the following:

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Then the Lagrange's equation of motion for the total kinetic energy T can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

where $L = T - U$ is the corresponding *Lagrangian* for such a system. The scalar function $U(q_j, \dot{q}_j)$ is called a *velocity dependent potential.*

One example of such a potential is found in electrodynamics, where Maxwell equations are the governing equations of motion for electric charges and related electromagnetic fields:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad \vec{\nabla}.\vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \qquad \vec{\nabla}.\vec{B} = 0$$

Here $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields generated through a free charge density ρ in motion, $\vec{J}$ being the corresponding current density, and $\vec{D}$ and $\vec{H}$ are the electric displacement vector and the Magentic induction respectively. A free charge q moving in such electromagnetic fields with a velocity, say $\vec{v}$ will experience a Lorentz force $\vec{F}$ given by

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}).$$

This system is a non-conservative one and the charge q experiences a force which is dependent on velocity.

As $\vec{\nabla}.\vec{B} = 0$, we can write the magnetic field as a curl of a vector function $\vec{A}$, *i.e.*, $\vec{B} = \vec{\nabla} \times \vec{A}$, where $\vec{A}$ is called the magnetic vector potential. We can then write

$$\begin{aligned} & \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \\ \Longrightarrow \quad & \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi, \\ \text{or,} \quad & \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \end{aligned}$$

where ϕ is the electrostatic scalar potential function. The expression for the Lorentz force then reduces to

$$\vec{F} = q\left[-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right].$$

The x-component of the Lorentz force $\vec{F}$ is calculated as

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + (\vec{v} \times (\vec{\nabla} \times \vec{A}))_x\right]$$

Now, we know that

$$\begin{aligned} \frac{dA_x}{dt} \equiv \frac{d}{dt}A_x(x, y, z, t) &= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z \end{aligned}$$

and therefore,

$$\begin{aligned}
(\vec{v} \times (\vec{\nabla} \times \vec{A}))_x &= v_y(\vec{\nabla} \times \vec{A})_z - v_z(\vec{\nabla} \times \vec{A})_y \\
&= v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - v_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\
&= v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z} \\
&= \frac{\partial}{\partial x}(\vec{v}.\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}.
\end{aligned}$$

The expression for the x-component of the Lorentz force then reduces to

$$\begin{aligned}
F_x &= q \left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}(\vec{v}.\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \right] \\
&= q \left[-\frac{\partial}{\partial x} \left(\phi - \vec{v}.\vec{A} \right) - \frac{d}{dt} \left(\frac{\partial}{\partial v_x}(\vec{v}.\vec{A}) \right) \right] \\
&= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial v_x}
\end{aligned}$$

where $U = q\phi - q\vec{v}.\vec{A}$. The electrostatic potential ϕ is in general independent of velocities.

From the above discussions we see that U is a form of generalised potential independent of velocities and hence we can write the corresponding Lagrangian for the charged particle as

$$L = T - q\phi + q\vec{v}.\vec{A}.$$

3.1.1 Conservation theorems and symmetry properties

We have so far seen that in case of a dynamical system one can frame the equations of motions in different formalisms- Newtonian formalism, Lagrangian formalism etc. Newtonian formalism requires the external applied forces to be completely specified, whereas the Lagrangian formulation requires the information about the kinetic energies and the potential energies for development of the equations of motion. These equations of motion are in general 2nd order differential equations with n number of degrees of freedom. So integration of such equations will involve $2n$ number of constants of integration, which are determined from the initial conditions. Many a times, the equations are not integrable and hence are not amenable to analytic solutions in the closed form or in terms of known functions. In case of the equations being nonlinear differential equations, no general methods are available to solve them. Under such circumstances, studying some properties of the differential equation itself and extraction of some information about the dynamical system will help in the understanding of the system to a considerable extent. Therefore, it becomes imperative to look into the properties of the differential equations and relate them with some information of the system such that maximum can be stated about the system without actually requiring a complete integration of the problem.

In many problems it is possible to reduce the dynamical equations into the so called first integrals, *i.e.*, of the form

$$f(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) = \text{constant}, \tag{3.1.1}$$

which are nothing but first-order diffential equations. The advantage of this type of reduction is that many conservation laws in connection of the system are actually in this form, as we have already seen in the case of Newtonian and Lagrangian formalisms.

To elaborate the point, we consider a system of mass points under the influence of forces which are derived from some scalars called potentials, depending only on position. We can then write

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} &\equiv \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_i \frac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \\ &= m_i \dot{x}_i = p_{ix} \end{aligned}$$

Here p_{ix} is the x component of the linear momentum associated with the i-th particle of the system. This expression actually serves to define the concept of momentum in more general sense. Thus the *generalised momentum*, also know as *canonical momentum* or *conjugate momentum* associated with the generalised coordinate q_j is the defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

It is relevant to note here that just as the generalised coordinates need not have the dimension of length, the canonical momentum also does not necessarily have the dimension of momentum.

Now, if the Lagrangian of a system does not contain a particular generalised coordinate, say q_j, then the coordinate is said to be *cyclic* or *ignorable.* The Lagrange's equation of motion corresponding generalised coordinate is then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

or,

$$\frac{dp_j}{dt} = 0 \tag{3.1.2}$$

which implies p_j=constant.

Hence we can state as a general conservation theorem that *the conjugate momentum corresponding to a given cyclic coordinate is conserved, or a constant of motion.*

Equation (3.1.2) is in fact a first integral of the equations of motion of the form as given by (3.1.1). The first integrals can be utilized for elimination of the cyclic coordinate from the problem, which can then be solved entirely in terms of the remaining generalised coordinates.

If a coordinate corresponding to a displacemnet is cyclic, it means that a translation of the system along the cyclic corrdinate does not have any effect on the dynamics of the system. That is, if the system is invariant under translation along a given directin, the corresponding linear momentum is conserved. Similary, if a coordinate corresponding to the rotation of the system is cyclic implies that the system must be invariant under rotation about the given axis. Thus we can see that the momentum conservation theorems are intimately connected with the symmentry properties of the system. If the system is spherically symmetric we can say without much reflection that all the components of the angular momentum are conserve. Or if the system is symmetric only about the x axis, then only the x component of the angular momentum L_x will be conserved and so on for the other axes. These symmetry considerations can be extended beyond to include complicated problems to determine by inspection whether certain constants of the motion exist.

As an example we consider a general Lagrangian corresponding to a system, $L(q_j, \dot{q}_j, t)$ which is a function of the coordinates q_j and the velocities $\dot{q}_j$ and may also depend explicity on the time. The total time derivative of such a Lagrangian is then

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} \right) + \frac{\partial L}{\partial t} \tag{3.1.3}$$

We can use the Lagrange's equations

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right)$$

to wrtie the equation (3.1.3) as

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

or,

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) + \frac{\partial L}{\partial t}$$

We can write this as

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0 \tag{3.1.4}$$

We denote the quantity inside the parenthesis as H, called the Hamiltonian function,

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L.$$

With this, (3.1.4) can be written as

$$\frac{dH}{dt} = \frac{\partial L}{\partial t} \tag{3.1.5}$$

This expression suggests that if the Lagrangian is not an explict function of time, *t does not appear in the expression for L* explicitly but only through the time variation of q and $\dot{q}$, then equation (3.1.5) is a conserved quantity of the motion. In fact, in the dynamical problems, where the generalised coordinates does not involve the time explicitly and the potential energy does not depend on generalised velocity and time, the function H can be identified as the sum of the total kinetic energy T and total potential energy V, or the total energy E,

$$H = T + V = E,$$

i.e., the total energy is a conserved quatity.

3.2 Lagrange's multiplier for holonomic and nonholonomic systems

3.2.1 Lagrange's Undetermined Multipliers

If a physical system is constrained in its motion then its degrees of freedom are reduced. We use the equations of constraint to eliminate dependent variables and we work with a new set of independent variables. Sometimes it is difficult or inconvenient to remove these dependent variables. Under these circumstances use of Lagrange's multipliers gives an alternative technique to solve the problems. Consider a simple case of a function $f = f(x, y, z)$ of three independent variables. The function f has an extremum value when

$$df = 0 \tag{3.2.1}$$

Since

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{3.2.2}$$

so for $df = 0$ the necessary and sufficient conditions are

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \tag{3.2.3}$$

Let the equation of constraint be

$$g(x, y, z) = 0 \tag{3.2.4}$$

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz \tag{3.2.5}$$

Because of equation (3.2.4) of constraint, condition (3.2.3) is no longer valid since there are now only two independent variables. If there are x and y, then dz is no longer arbitrary but will be related to changes in x and y. Multiplying (3.2.5) by λ and then adding with (3.2.2) we get

$$\begin{aligned} df + \lambda dg &= \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz \\ &= 0 \end{aligned} \tag{3.2.6}$$

The multiplier λ can be chosen by setting

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0 \tag{3.2.7}$$

where we assume that $\frac{\partial g}{\partial z}$ is non-zero. Now using equation (3.2.7) in equation (3.2.6) we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy = 0 \tag{3.2.8}$$

Since x and y are independent, their coefficients must vanish. Hence

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \tag{3.2.9}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 \tag{3.2.10}$$

Thus when equations (3.2.7) and ((3.2.9),(3.2.10)) are satisfied we get $df = 0$ or f has an extremum value. Now we have four variables x, y, z and λ and three equations (3.2.7), (3.2.9), (3.2.10). The fourth equation is actually the equation of constraints. Since in the solution we want to know only x, y, and z, the multiplier need not be determined. For this reason, it is called Lagrange's undetermined multiplier.

3.2.2 Application of Lagrange's Undetermined Multipliers

Particle on Sphere

Let us consider a particle of mass m moving under the action of gravity on the sueface of a smooth sphere of radius l.

We shall find its equation of motion and the angle θ_c at which the particle flies off from the surface. Let the origin of the co-ordinates be at the center of the sphere and let the z axis be vertically upwards.

In this case, the equation of the constraint is given by

$$r - l = 0$$

where r is the radial distance of the particle. From this we obtain, $dr = 0$. Hence, the co-efficients in the equation of the constraints are $a_r = 1$, $a_\theta = 0 = a_\Phi$. Let us suppose that the particle is initially at rest and let it slide down along the surface. The particle will obviously move in a vertical plane which we shall take for convenience $\Phi = 0$. The kinetic and potential energies and hence the Lagrangian for the particle are respectively

$$T = \frac{1}{2}m\left(\dot{r}^2 - r^2\dot{\theta}^2\right)$$
$$V = mgr\cos\theta$$
$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 - r^2\dot{\theta}^2\right) - mgr\cos\theta$$

The Lagrangian equations in x and θ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda$$

$$\Rightarrow - ml\theta^2 + mg\cos\theta = \lambda \tag{3.2.11}$$

and $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow ml^2\ddot{\theta} - mgl\sin\theta = 0 \tag{3.2.12}$$

where we used $r = l$ and $\dot{r} = \ddot{r} = 0$. The undetermined multiplier λ is dependent on θ in general. Differentiating equation (3.2.11) containing λ with respect to time, we get

$$-2ml\dot{\theta}\ddot{\theta} - mg\sin\theta\dot{\theta} = \frac{d\lambda(\theta)}{dt} = \frac{d\lambda}{d\theta}\dot{\theta}$$

Using the value of $\ddot{\theta}$ from equation (3.2.12) we get

$$-3mg\sin\theta = \frac{d\lambda}{d\theta}$$

Integrating we get

$$\lambda(\theta) = 3mg\cos\theta + c$$

At $\theta = 0$, $\lambda = mg$, this being the force of constraint at the top of the sphere. This gives

$$c = -2mg$$

Hence,

$$\lambda(\theta) = 3mg\cos\theta - 2mg$$

The particle will move on the surface as long as the force of constraint is positive, *i.e.*, as long as the surface pushes it outward. Corresponding condition is

$$\lambda(\theta) = 3mg\cos\theta - 2mg \geq 0$$

This equality holds for

$$\cos\theta_c = \frac{2}{3}$$

i.e., at the angle $\theta_c \cos^{-1}\frac{2}{3}$ the particle flies off the surface. Here we have neglected the force of friction.

3.2.3 Dynamical problems with Nonholonomic constraints

We know that if a system is under nonholonomic constraints, *i.e.*, the dynamic variables are either not related through equality sign or depends explicitly on the velocity terms, then the problem needs to be takled differently and we cannot straightway use the Lagrange's equations. Lagrange's equations were derive under the condition that the constraint relations are holonomic.

When nonholonomic constraints are related by inequality sign, the corresponding co-ordinates are only restricted, but does not eliminate. In case of nonholonomic constraints depending on velocities, the constraint relations reduce to differential form. In either case, the introduction of non-holonomic constraints does not reduce the number of independent coordinates required for description of the system, whereas the degrees of freedom is reduced. This is unlike the case for a holonomic system where degrees of freedom decides the required number of independent coordinates for description of the system. Thus for a system of N-particles under k number of holonomic and k' number of non-holonomic constraints, the number of degrees of freedom is $n = 3N - k - k'$ and the number of independent coordinates required is $n + k'$.

If all the nonholonomic constraints are connected by

$$g_i = g_i(q_1, q_2, ..., q_{n+k'}, \dot{q}_1, \dot{q}_2, ...\dot{q}_{n+k'}, t) = 0$$

where $i = 1, 2, ...k'$. Given the form of nonholonomic constraint, it is possible to write down the generalised forces of costraints by the use of what is called *Lagrange's undetermined multiplier* as

$$(Q_j)_c = \sum_{i=1}^{k'} \lambda_i \frac{\partial g_i}{\partial \dot{q}_j}$$

where λ_i are the Lagrange's undetermined multipliers. Thus if the lagrangian for such a system involves the velocity independent potential, *i.e.*, $L = T - V$, then the Lagrange's equations for both holonomic and nonholonomic constraints can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = (Q_j)_c$$

or,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - \sum_{i=1}^{k'} \lambda_i \frac{\partial g_i}{\partial \dot{q}_j} = 0$$

On the other hand, sometimes constraints are expressible as a linear relations connecting differentials of generalised coordinates or time. Suppose there are l such relations expressible in the following form

$$\sum_k a_{ik} dq_k + a_{it} dt = 0. \tag{3.2.13}$$

Under such circumstances, the use of the method of *Lagrange's undetermined multipliers* yields the complete set of Lagrange's equations for the given nonholonomic constraints as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - \sum_{l=1} \lambda_l a_{lj}, \qquad j = 1, 2, ..., n \tag{3.2.14}$$

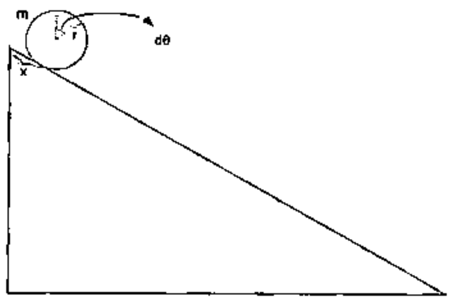
Here n is the degree of freedom of the system.

Numerical Examples

Example 3.2.1 *Consider a hoop of mass m and radius r, rolling without slipping, down an inclined plane. Find the equation of motion.*

Solution:

For describing the motion of the hoop down the inclined plane without slipping, we need two generalised coordinates, x, θ (see figure bleow)



The no slip condition on the hoop implies

$$r d\theta = dx \tag{3.2.15}$$

As the hoop advances down the inclined plane, the total kinetic energy of the hoop involves two terms: one due to kinetic energy of motion of its centre of mass down the inclined plane and the other, the rotational kinetic energy due to its rolling:

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} mr^2\dot{\theta}^2$$

The potential energy is

$$V = mg(l - x)\sin\alpha,$$

where l is the length of the inclined plane with α as the angle of inclination. The Lagrangian of the system is then given by

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m \left(\dot{x}^2 + r^2\dot{\theta}^2\right) - mg(l - x)\sin\alpha \end{aligned}$$

As there is only one constraint relation (3.2.13), we need only one undetermined multiplier, say λ. The coefficients of the differentials of the generalised coordinates are then

$$\begin{aligned} a_\theta &= r \\ a_x &= -1. \end{aligned}$$

Thus the Lagrange's equations corresponding to the two generalised coordinates x, θ are

$$m\ddot{x} - mg\sin\alpha + \lambda = 0, \tag{3.2.16}$$
$$mr^2\ddot{\theta} - \lambda r = 0 \tag{3.2.17}$$

The constraint relation (3.2.15) *can be written as*

$$r\dot{\theta} = \dot{x}$$

Differentiating this with respect to time,

$$r\ddot{x} = \ddot{x}.$$

Hence the Lagrange's equation (3.2.17) *reduces to*

$$m\ddot{x} = \lambda$$

and (3.2.16) *becomes*

$$\ddot{x} = \frac{1}{2}g\sin\alpha$$

from which

$$\lambda = \frac{1}{2}mg\sin\alpha$$

and

$$\ddot{\theta} = \frac{g\sin\alpha}{2r}$$

Thus we see that the acceleration with which the hoop rolls down the inclined plane is only one half of that we would expect if the hoop had slipped down a frictionless plane, and the friction force of constraints would be $\lambda = \frac{1}{2}mg\sin\alpha$.

Example 3.2.2 *Deduce the equation of motion for a particle under the action of gravity, moving on a frictionless surface on a sphere.*

Solution

Let us consider a particle of mass m*, which is kept on a frictionless spherical surface of radius* r*, and allowed to move under the action of gravity. We choose spherical polar coordinates* (r, θ, ϕ) *to describe the dynamics, where,* θ *is the angle in the vertical plane and* ϕ *is the azimuthal angle, the angle in the horizontal plane. We consider that the particle moves on the surface in the vertical plane under gravity so that the motion along* ϕ*-direction need not be considered,*

The Lagrangian L *of the system is then given given by*

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

As the particle moves on the surface, we have $dr = 0$. *So, we have the coefficients of the constraint equation as* $a_r = 1, \quad a_\theta = 0$. *Moreover, we have* $\dot{r} = 0$. *Therefore, the Lagrangian can be rewritten as,*

$$L = \frac{1}{2}mr^2\dot{\theta}^2) - mgr\cos\theta$$

The radial part of the Lagrange's equation will be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = a_r\lambda$$

$$\implies \qquad -mr\dot{\theta}^2 + mg\cos\theta = \lambda$$

Here λ is the Lagrange's undetermined multiplier, supposed to be dependent on the coordinates, which is θ here. Now, differentiating w. r. t. t, we have

$$-2mr\dot{\theta}\ddot{\theta} - mg\sin\theta\dot{\theta} = \frac{d\lambda}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{d\lambda}{d\theta}\dot{\theta}$$

$$or, \qquad -2mr\ddot{\theta} - mg\sin\theta = \frac{d\lambda}{d\theta} \tag{3.2.18}$$

The θ-part of the Lagrange's equation will be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = a_\theta\lambda$$

Using the value of a_θ we rewrite the equation as

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\sin\theta = 0$$

$$or, \qquad mr^2\ddot{\theta} - mgr\sin\theta = 0 \tag{3.2.19}$$

Eliminating $\ddot{\theta}$ from (3.2.18) *and* (3.2.19) *we have,*

$$-2mr\left(\frac{g\sin\theta}{r}\right) - mg\sin\theta = \frac{d\lambda}{d\theta}$$

$$or, \qquad \frac{d\lambda}{d\theta} = -3mg\sin\theta$$

Integrating this w. r. t. θ, we get

$$\lambda \equiv \lambda(\theta) = 3mg\cos\theta + A \tag{3.2.20}$$

where A is the constant of integration. At the maximum height of the particle on the surface of the sphere, we have $\theta = 0$ and the force of constraint at this point, $\lambda = mg$. Therefore,

$$mg = 3mg\cos 0 + A, \qquad \implies \qquad A = -2mg.$$

Putting this into (3.2.20)*, we get*

$$\lambda(\theta) = 3mg\cos\theta - 2mg$$

For the particle to continue its motion on the surface, the force of constraint should maintain a positive value so that the surface always pushes the particle outward, i.e.,

$$\lambda(\theta) \leq 0$$

$$\implies \qquad 3mg\cos\theta - 2mg \leq 0$$

$$\implies \qquad \cos\theta \leq \frac{2}{3}$$

That is, the critical angle θ_0, beyond which the particle will fly off, is given by

$$\cos\theta_0 = \frac{2}{3}.$$

3.3 Rayleigh Dissipation Function

3.3.1 Introduction

One of the classics of physics literature, this treatise contains a wealth of theorems and physical illustrations on all of the aspects of vibration theory. Rayleigh himself was responsible for developing much of the theory, especially the introduction of the dissipation function. His treatment is smooth-flowing and clear and contains rarely discussed topics, as on the effects of constraints and the stationary properties of the eigen frequencies. Rayleigh leans heavily on the work of Routh, who in his Adams Prize Essay of 1877 and in his text Rigid Dynamics was one of the first to give a systemic discussion of small vibration.
It can be shown that if a system involves frictional forces or in general dissipative forces,then in suitable circumstances, such a system can also be described in terms of extended Lagrangian formulation. We shall now see that how the Lagrange's equations get modified if the forces are non-conservative and dissipative.

Frictional forces are found to be proportional to the velocity of the particle so that in cartesian co-ordinates,components are

$$F_j^d = -k_j \dot{x}_j,$$

where k_j are constants. Such frictional forces are defined in terms of a new quantity called Rayleigh dissipation function given as

$$A = \frac{1}{2}\sum k_j x\dot{j}^2,$$

which yields

$$F^{d_j} = -\frac{\delta \Im}{\delta x\dot{j}}.$$

Writing the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_j}}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

in cartesian co-ordinates, assuming that this holds for such a system, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

where L contains the potential of conservative forces as described in previous cases; Q_j represents the forces which do not arise from potential *i.e.*,

$$\begin{aligned} Q_j^d &= F_j^d \\ &= -\frac{\partial \Im}{\partial \dot{x}_j} \end{aligned}$$

Thus equation can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} + \frac{\partial \Im}{\partial \dot{x}_j} = 0$$

Expressing Q_j in terms of generalised coordinates, we find

$$\begin{aligned} Q_j^d &= \sum_j F_i^d \frac{\partial x_i}{\partial q_j} = -\sum_j k_i \dot{x}_i \frac{\partial x_i}{\partial q_j} \\ &= -\sum_i k_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \qquad \left(\text{because} \qquad \frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) \\ &= \frac{\partial}{\partial \dot{q}_j}\left\{-\frac{1}{2}\Sigma k_i \dot{x}_i{}^2\right\} \\ &= -\frac{\partial \Im}{\partial q_j} \end{aligned}$$

Lagrange's equations now become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \Im}{\partial q_j} = 0$$

Thus for such a non-conservative system,to obtain equations of motion, two scalars L and $\Im$ are to be specified.

3.3.2 Summary

We have already seen that the Lagrange's equations of motion were derived under the assumption that the constraints present in the system are holonomic and the associated forces are conservative so that a suitable potential energy function is associated with the force. But the events in the real life is not confined to above assumptions and nonholonomic constraints and and nonconservative forces do associate with majority of the phenomena that we encounter. This unit of study discusses this point in some detail. Starting with some specific examples of the cases where the dissipative forces under some conditions can also be expressed in terms of potentials, the discussion progresses to include what is called the Rayleigh dissipative function. The method of Lagrange's undetermined multiplier is then discussed along with examples demonstrating how they can be used in real life problems. Finally using the method of undetermined multiplier the Lagrange's equation of motion has been extended to include the nonholonomic constraints along with the necessary mathematical deductions.

Self study questions:

1. What are velocity dependent potentials? Explain how the Lagrange's equations of motion is modified when dissipative forces are present.

2. Express all the components of the Lorentz force in terms of velocity dependent potentials.

3. Explain the term symmetry in the laws of motion. Establish the relation between the conservation laws with the existing symmetry of a given system.

4. Explain the role of Rayleigh Dissipation function.

5. What are Lagrange's undetermined multipliers? Under what circumstances one needs to use them? Illustrate the points with examples.

6. Deduce the form of the Lagrange's equations of motion when nonholonomic constraints are present in the given dynamical system.

UNIT 4

Hamiltonian Formulation of Dynamics

Preparatory inputs to this unit

1. Lagrange's equations: a critical look on the properties.
2. Basics of Ordinary differential equations.

4.1 Hamiltonian Formalism

4.1.1 Introduction

The Hamiltonian formulation which is alternative to the Lagrangian formulation, proves to be convenient and usefull particularly in dealing with problems of modern physics. No new physical concept is introduced in this formulation but we get another tool to work on the problems in physics. We can also obtain Hamilton's equation of motion for a system with n degree of freedom. Here is a brief description about the Hamiltonian formulation.

For a system with n degrees of freedom, there are n Lagrange's equations of motion. Each equation is a second order differential equation and the solutions of the n equations need $2n$ constants usually given by initial positions and initial velocities. The Lagrangian is a function of q_i and $\dot{q}_i$ and the motion of the system can be visualised in an n-dimensional configuration space.

4.1.2 Hamilton's Equation of Motion

We know that the Hamiltonian function is related to the Lagrangian function by the equation

$$H = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

Let the Hamiltonian be expressed as a function of generalised co-ordinates and generalised momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$, *i.e.*, $\dot{q}_i$ is replaced by p_i in the above expression and

$$H \equiv H(q_i, p_i, t)$$

We want to describe the motion of the system in terms of an equation of motion involving the Hamiltonian. This clearly becomes a problem of transformation from the set of variables $(q_i, \dot{q}_i)$ to a new set (q_i, p_i). In order to achieve this transformation we write the differential form

$$dH = \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \tag{4.1.1}$$

But from the definition of H stated above, we have

$$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - dL \tag{4.1.2}$$

Since $L \equiv L(q_i, \dot{q}_i, t)$, we get

$$\begin{aligned} dL &= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt \\ &= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt \end{aligned}$$

where we have used the definition of generalised momenta $p_k = \frac{\partial L}{\partial \dot{q}_k}$. Substituting this value of dL in equation (4.1.2) we get

$$\begin{aligned} dH &= \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt \\ &= \sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \end{aligned} \quad (4.1.3)$$

Comparing the co-efficients of dp_k and dq_k in equations (4.1.1) and (4.1.3), we get

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

and

$$-\dot{p}_k = \frac{\partial H}{\partial q_k}$$

These two equations are called Hamilton's equations or Hamilton's canonical equations of motion. They form a set of $2n$ first order differential equations of motion and replace the n-Lagrange equations of second order.

4.2 Legendre Dual Transformation: Alternative Deduction of Hamilton's Canonical Equations

4.2.1 Legendre Dual Transformation

The important object for determining the motion of a system using the Lagrangian approach is not the Lagrangian itself but its variation, under arbitrary changes in the variables q and $\dot{q}$ treated as independent variables. It is the vanishing of the variation of the action under such variations which determines the dynamical equations. In the phase space approach, we want to change variables treated as independent variables. It is the vanishing of the variation of the action under such variations which determines the dynamical equations. In the phase space approach, we want to change variables $\dot{q} \to p$ where the p_i are part of the gradient of the Lagrangian with respect to the velocities. This is an example of a general procedure called the Legendre transformation. We will discuss it in terms of the mathematical concept of a differential form. Because it is the variation of L which is important, we need to focus our attention on the differential dL rather than on L itself. want to give a formal dentition of the differential, which we will do first for a function f($x_1, ..., x_n$) of n variables, although for the Lagrangian we will later subdivide these into coordinates and velocities. We will take the space in which x takes values to be some general space we call M, which might be ordinary Euclidean space but might be something else, like the surface of a sphere. Here M is a manifold Given a function f of n independent variables x_i, the differential is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

The term dx_i which appear in the above form can be thought of as operators acting on this vector space argument to extract the i0th component, and the action of df on the argument $(x, \vec{v})$ is

$$df(x, \vec{v}) = \sum_i \frac{\partial f}{\partial x_i} v_i.$$

This differential is a special case of a 1-form, as in each of the operators dx_i. All n of these dx_i form a basis of 1-forms, which are given more generally as

$$\omega = \sum^i \omega_i x dx^i$$

and considering $L(q_i, v_j, t)$ (here $v_i = \dot{q_i}$) and the differential

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i p_i dv_i,$$

and

$$dg = \sum_i dv_i p_i + \sum_i v_i p_i - dL = \sum_i p_i dv_i, \qquad v_i = \frac{\partial g}{\partial p_i}$$

This particular form of changing variables is called a Legendre transformation and the duality of the variables in this transformation is given by the theorem

Theorem 4.2.1 *Let a function $F(u_1, u_2, \cdots, u_n)$ have an explicit dependence on the n independent variables $u_1, u_2, \cdots, u_n$. Let the function F be transformed to another function G=$G(v_1, v_2, \ldots.., v_n)$ expressed explicitly in terms of a new set n independent variables $v_1, v_2, \ldots.., v_n$ where this new variables are connected to the old variables by the relations*

$$v_i = \frac{\partial F}{\partial u_i} \qquad i = 1, 2, 3, \cdots, n \tag{4.2.1}$$

and the form of G is given by

$$G(v_1, v_2, \cdots, v_n) = u_i v_i - F(u_1, u_2, \cdots, u_n).$$

Then the variables $u_1, u_2, \cdots, u_n$ satisfy the dual transformation namely the relations

$$u_i = \frac{\partial G}{\partial v_i}, \qquad i = 1, \cdots, n \tag{4.2.2}$$

and

$$F(u_1, u_2, \cdots, u_n) = u_i v_i - G(v_1, v_2, \cdots, v_n).$$

This is the duality of the two functions F and G along with the variables u_i, v_i. This above relations (4.2.1) *and* (4.2.2) *is called the Legendre dual transformation.*

Examples of Legendre dual transformation

In Thermodynamics the four thermodynamic potentials the internal energy $U' = U'(S', V')$ the free energy $F' = F(V', T')$, the enthalpy of $H' = H'(S', P')$ and Gibb's potential $G' = G'(P', T')$ are all connected by Legendre's dual transformation. Here the independent variables are any two of entropy, volume, temperature and pressure. A change in the pair of independent variables defines a new potential function which is connected to the old one by a suitable Legendre's dual transformation. For example a dual transformation $U'(S',V')$ can be the free energy $F'(V',T')$ where V' remains as the passive variable and the variable S' is transformed into T' which is possible if the relation $T' = \frac{\partial U'}{\partial S'}$ exists over the change of variables S' in U' to T' in F'. Then by Legendre's dual transformation we have

$$F'(V',T') = U'(S',V') - T'S', \quad S' = -\frac{\partial U'}{\partial S'}$$

and

$$\frac{\partial F'}{\partial V'} = \frac{\partial U'}{\partial V'}.$$

Similarly one can find that

$$H' = U' + P'V' \qquad \text{with } P' = -\frac{\partial U'}{\partial V'} \quad \text{and } V' = \frac{\partial H'}{\partial P'}$$

and

$$G' = F' + P'V' \qquad \text{with } P' = -\frac{\partial F'}{\partial V'} \quad \text{and } V' = \frac{\partial G'}{\partial P'}$$

as dual transformations of U' and V' respectively.

4.2.2 Deduction of Hamilton's Canonical Transformation from Legendre's Dual Transformation

We can apply the Legendre's dual transform to the lagrangian of a system

$$L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$$

with $\dot{q}_i$ and t as a passive variable. The dual variables of $\dot{q}_i$, with $i = 1, 2, \ldots, n$ are given by the generalised momenta

$$p_i = \frac{\partial l}{\partial \dot{q}_i}, \qquad (i = 1, 2, \ldots, n). \tag{4.2.3}$$

Hence the dual function of the lagrangian L is $H = p_i\dot{q}_i - L(q, \dot{q}, t)$ where

$$H = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t) = H(q, p, t)$$

and is called the Hamiltonian of the system. The dual transformation of (4.2.3) is

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad (i = 1, 2, \ldots, n). \tag{4.2.4}$$

and the equations for the passive variables take the form

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

and

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}, \qquad (i = 1, 2, \ldots, n).$$

We know that provided there are no potential forces, the system is holonomic and bilateral, and Euler-Lagrange's equations of motion are valid, one can write

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{p}_i.$$

Substituting in previous equation, we get

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad (\ i = 1, 2, \ldots, n\ ). \tag{4.2.5}$$

The two equations (4.2.4) and (4.2.5) together are called Hamilton's canonical equations of motion.

4.3 Hamilton's Variational Principle

Consider the integral

$$I = \int_{m}^{n} L dt \tag{4.3.1}$$

where the function L is the Lagrangian of the mechanical system defined as the difference between the kinetic T and potential V energies of the system,

$$L = T - V. \tag{4.3.2}$$

Hamilton's variational principle states that the integral I taken along a possible path of motion of a physical system is an extremum when evaluated along an actual path of motion. In other words, out of the number of possible ways in which a system could change its configuration during a time interval between say t_1 and t_2, nature chooses the way to either maximize or minimize the integral I which is called the action for the system. Mathematically, this statement can be expressed as follows

$$\delta I = \delta \int_{t_1}^{t_2} L dt \tag{4.3.3}$$

where δ means a variation in the entire integral about its extremum value. We assume that such a variation is obtained by varying the coordinates and velocities of a system at their values away from the actuals during the time the system evolves from t_1 to t_2, under the constraint that the variations of all the parameters at the end points of the motion at t_1, t_2 will be zero, *i.e.*, all parameters are unchanged at the extreme points of time in consideration.

4.3.1 Hamilton's equations from variational principle

Hamilton's canonical equations can be obtained from a variational principle, similar to the way the Lagrange equations are derived. However, the variations will be over paths in the (q, p) phase-space, which has $2n$ dimensions, twice the n dimensions of the coordinates, called the configuration-space. Note that the function inside the integral, upon which the variational principle will be applied, is the Lagrangian L, but now considered as a function of $q, \dot{q}, p, \dot{p}$ and t, so that

$$L(q, \dot{q}, p, \dot{p}, t) \equiv \sum_i \dot{q}_i p_i - H(q, p, t) \tag{4.3.4}$$

We write the *Action Integral* $I(q, p)$ of the system in concern as follows

$$I(q, p) \equiv \int_{q_1, p_1, t_1}^{q_2, p_2, t_2} L(q, \dot{q}, p, \dot{p}, t) dt \tag{4.3.5}$$

Inside the integral, $(q, \dot{q}, p, \dot{p})$ are actually implicit functions of time, *i.e.*, $(q(t), \dot{q}(t), p(t), \dot{p}(t))$

The variational principle states that the motion of a system from time t_1 to t_2 is such that the system extremizes the Action Integral in the phase space. This ensures the variation in the action integral in the phase space is zero, *i.e.*,

$$\delta I = \delta \int_{q_1, p_1, t_1}^{q_2, p_2, t_2} L(q, \dot{q}, p, \dot{p}, t) dt = 0$$

The variational principle is actually a special case of a result from the Calculus of variation. The study of the calculus of variation establishes the result that there exists a condition for the integral

$$J = \int_{x_1}^{x_2} f\left(y(x), \dot{y}, x\right) dt$$

involving a given functional $f\left(y(x), \dot{y}(x), x\right)$, with $\dot{y} = \dfrac{dy}{dx}$ to have an extremum value. The necessary and sufficient condition for the above integral J to be extremum is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \left(\frac{\partial f}{\partial y}\right) = 0.$$

Here the reference of the extremum value of the functional J is with respect to the δ-variation, which is the variation in the paths described by the integral between two fixed points in the given $(x,\ y$ space. More about the calculus of variation will follow in forthcoming chapters.

It is now obvious that Hamilton's variational principle is a version of the integral J for a dynamical problem defined for a given Lagrangian L as the required functional. The corresponding necessary and sufficient condition to be satisfied by the Lagrangian is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \tag{4.3.6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) - \left(\frac{\partial L}{\partial p_i}\right) = 0, \qquad i = 1, 2, ..., n \tag{4.3.7}$$

We now rewrite (4.3.6) and (4.3.7) by using the Lagrangian given in (4.3.4). It is easy to find from

$$L(q, \dot{q}, p, \dot{p}, t) \equiv \sum_i \dot{q}_i p_i - H(q, p, t)$$

that

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i$$

$$\frac{\partial L}{\partial p_i} = \dot{q}_i - \frac{\partial H}{\partial p_i}$$

$$\frac{\partial L}{\partial \dot{p}_i} = 0$$

Using these, (4.3.6) and (4.3.7) can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \quad \Longrightarrow \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{4.3.8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) - \left(\frac{\partial L}{\partial p_i}\right) = 0 \quad \Longrightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{4.3.9}$$

(4.3.8) and (4.3.9) are the Hamilton's canonical equations.

4.3.2 Properties of the Hamilton's equations of motion

From the Hamiltonian $h(q_k, p_k, t)$ the Hamilton's equation of motions are obtained by

$$\dot{q}_k = \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \tag{4.3.10}$$

$$\dot{p}_k = \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \tag{4.3.11}$$

The Poisson Bracket of any two dynamical variables $f(q_k, p_k, t)$ and $g(q_k, p_k, t)$ is defined by

$$[f, g] \equiv \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right) \tag{4.3.12}$$

One see that

$$[f, f] \equiv 0$$

$$[g, f] = -[f, g]$$

The total time derivative of f,

$$\frac{df}{dt} = \sum_k \left[\frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k\right] + \frac{\partial f}{\partial t} \tag{4.3.13}$$

$$= [f, g] + \frac{\partial f}{\partial t} \qquad \text{(from (4.3.3), (4.3.4), (4.3.5))}$$

In particular, if $f = H$ we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{4.3.14}$$

If H does not depend explicitly on time it is a constant of motion. Any invariant of the motion not containing t explicitly, has a vanishing Poisson bracket with H.

4.3.3 Cyclic Coordinate:

If the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be cyclic or ignorable. The Lagrange equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} - \frac{\partial L}{\partial q_k} = 0 \tag{4.3.15}$$

reduces, for a cyclic coordinate, to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} \tag{4.3.16}$$

or

$$\frac{dp_k}{dt} = 0$$

which means that

$$p_k = consatnt \tag{4.3.17}$$

Hence, we can state as a general conservation theorem that the generalised momentum conjugate to a cyclic coordinate is conserved.

According to the definition of Cyclic Coordinate q_k is one that does not appear explicitly in the Lagrangian; by virtue of Lagrange's equations its conjugate momentum p_k is then a constant. We know

$$\dot{p}_j = \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k}$$

A coordinate that is cyclic will thus also be absent from the Hamiltonian. Conversely, if a generalised coordinate does not occur in H, the conjugate momentum is conserved. The momentum conservation theorem can thus be transferred to the Hamiltonian formulation with no more than a substitution of H for L.

Note !

When some coordinates, say $q_1; q_2, \ldots ; q_m$; $(m < n)$, are cyclic, the Lagrangian of the system is of the form

$$L = L(q_{m+1}, \ldots, q_n, p_{m+1}, \ldots, p_n, b_1, \ldots ; b_n; t)$$

Thus $(n-m)$ coordinate and momenta remain,and the problem is essentially reduced to $(n-m)$ degree of freedom. Hamilton's equation corresponding to each of the $(n-m)$ degree of freedom can be obtained while completely ignoring the cyclic coordinates. The cyclic coordinate themselves can be found by integrating the equation of motion

$$\dot{q}_k = \frac{\partial H}{\partial p}; \qquad k = 1, 2, ..., m.$$

Routh has devised procedure that combine the advantage of the Hamilton formalism in handling cyclic coordinates with the Lagrangian formulation.

4.4 The Routhian

The advantage of the hamiltonian formulation in handling cyclic coordinates may be combined with Lagrangian procedure by a method devised by Routh. Essentially one carries out mathematical transformation from the $(q, \dot{q})$ basis to the (q, p) basis only for the cyclic coordinates, obtaining their equations of motion in the Hamiltonian form, while the remaining coordinates are governed by Lagrange equations. If the cyclic coordinate are labeled $q_{s+1}, \ldots, q_n$, then a function R, known as the *Routhian*, may be introduced, which is defined as,

$$R(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, q_{s+1}, \ldots, p_n; t) = \sum_{k=s+1}^{n} p_k \dot{q}_k - L \tag{4.4.1}$$

A differential of R is therefore given by

$$dR = \sum_{k=s+1}^{n} \dot{q}_k dP_i - \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt$$

from which it follows that

$$\frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k}, \qquad \frac{\partial R}{\partial \dot{q}_k} = -\frac{\partial L}{\partial \dot{q}_k}, \qquad k = 1, \ldots, s. \tag{4.4.2}$$

$$\frac{\partial R}{\partial q_k} = -\dot{p}_k, \qquad \frac{\partial R}{\partial p_k} = \dot{q}_k, \qquad k = s+1, \ldots, n. \tag{4.4.3}$$

Equation (4.4.3) for the $n - s$ ignorable coordinates $q_{s+1}, \ldots\ldots, q_n$ is in the form of Hamilton's equation of motion with R as the Hamiltonian; while equation (4.4.2) obey the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q_k}}\right) = \frac{\partial R}{\partial q_k} = 0, \qquad k = 1, \ldots, s. \tag{4.4.4}$$

with R as the Lagrangian.

If we restrict ourselves with situations where the Lagrangian is not an explicit function of time, then it follows that in steady motion, the cyclic coordinates are linear functions of time. This can be seen from (4.4.3) with a Routhian of the form (4.4.4); these equations imply that a generalised velocity of a cyclic coordinate q_k is given in terms of noncyclic variables by some relations of the type

$$\dot{q}_k = \dot{q}_k(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, \alpha_1, \ldots, \alpha_r), \qquad k = s+1, \ldots, n. \tag{4.4.5}$$

For a steady motion, $(q_1, \ldots, q_s)$ are constant; $(\dot{q}_1, \ldots, \dot{q}_s)$ are zero; and therefore $\dot{q}_k$ for a cyclic variable is constant and q_k varies linearly with time.

The advantage of the Hamiltonian approach, rather than, though it does, simplifying the solutions of mechanical problem actually provides the base from which one makes extensions to other fields. This point can be understood from the fact that the products of generalised momentum and generalised coordinates *i.e.*, $p_k q_k$ has always the dimension of action, *i.e.*, of energy multiplied by time. Thus the Hamiltonian formulation is particularly useful in making a transition from classical mechanics to quantum mechanics in which the action is quantized.

4.5 Hamilton's canonical equations: numerical examples

We have seen that dynamical problems can be solved easily using generalised co-ordinates and Lagrange's equations. In fact there is hardly a mechanical problem where generalised co-ordinates are not applicable. Even though from the operational point of view, the use of Hamilton's equations are generally not as convenient as the Lagrangian equations but proves very useful to delve into critical properties like symmetry and conserved quantities in advanced mechanics particularly to three fields: celestial mechanics, statistical mechanics and transition from classical mechanics to quantum mechanics.

Steps to write Hamilton's equations for dynamical systems

1. Find the Lagrangian L.

2. Calculate the generalised momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = L', \qquad \dot{q}_i = L''$$

3. Express L through L'

4. Express Hamiltonian in terms of (q_i, L', t) as

$$H = \sum p_j \dot{q}_j - L'(q_j, \dot{q}_j, t) = \sum p_j L'' - L'$$

5. Use Hamiltonian canonical equation of the motion.

Example 4.5.1 *Find the equations of motion for a simple pendulum using Hamilton's canonical equations.*

Solution: *The pendulum is assumed to oscillate in the $z = 0$ vertical plane of the cartesian coordinates, with the x-axis placed horizontal, y-axis vertically up and the origin being at the point of suspension. Let g be the acceleration due to gravity, m the mass of the bob and l be the length of the pendulum. The degree of the system is 1 and the angle of oscillation θ is chosen as the single generalised coordinate.*

We write the coordinates of the mass at any given instant in terms of the generalised coordinates as $\vec{r} = (l\sin\theta, -l\cos\theta)$ *and the components of the velocity as* $\dot{\vec{r}} = (l\cos\theta\dot{\theta}, l\sin\theta\dot{\theta})$.

The kinetic energy of the system is then given by

$$T = \frac{1}{2}m|\dot{\vec{r}}|^2 = \frac{1}{2}m(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta) = \frac{1}{2}ml^2\dot{\theta}^2$$

Considering the horizontal line passing through the point of suspension of the pendulum as the reference of zero potential, the potential energy of the system is

$$V = -mgl\cos\theta$$

Thus the Lagrangian of the system is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

and the generalised momentum p_θ is given as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \Longrightarrow \qquad \dot{\theta} = \frac{p_\theta}{ml^2}$$

Using this, the expression of the Hamiltonian H needs to be expressed in terms of the generalised coordinates and generalised momenta.

$$\begin{aligned} H = p_\theta\dot{\theta} - L &= p_\theta\dot{\theta} - \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \\ &= \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta \\ &= \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta \end{aligned}$$

from which we find

$$\frac{\partial H}{\partial \theta} = mgl \sin\theta$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

The equation of motion of the system is given by the Hamilton's canonical equations

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{4.5.1}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl \sin\theta \tag{4.5.2}$$

From (4.5.1) *we find on differentiation,* $\dot{p}_\theta = ml^2\ddot{\theta}$. *Substituting this to* (4.5.2), *we get*

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

Or, $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

which is the equation of motion for a simple pendulum.

Example 4.5.2 *Deduce the equations of motion for a particle under a central force by using Hamilton's canonical equations.*

Solution: *Central force is defined as the force which is directed outward from a fixed point or directed inward towards a fixed point. Such a force is always a function of the distance from the fixed point.*

We consider a particle of mass m moving in a central force field in a given plane (r, θ) in two dimensional polar coordinates. The particle is assumed to be attracted towards the origin of the coordinates. If we further assume that the central force field follows the inverse square law, i.e., the force is inversely proportional to the square of the radial distance of the particle from the origin. Therefore the force field is described by $F = -\frac{k}{r^2}$. *Here k is the constant of proportionality and the negative sign in the expression of the force is to signify that the nature of the force is attractive.*

If the instantaneous position of the particle is $(r,\ \theta)$, *then the radial and tangential velocity components will be* $(\dot{r},\ r\dot{\theta})$, *so that the square of the magnitude of the velocity vector will be* $(\dot{r}^2 + r^2\dot{\theta}^2)$. *Therefore, the kinetic energy of the particle will be given by*

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

The potential energy of the particle can be found by finding the work done by the force field in bringing the particle from ∞ *to the instantaneous radial distance* r, *i.e.,*

$$V = -\int_\infty^r F\,dr = -\int_\infty^r \left(-\frac{k}{r^2}\right)\,dr = -\frac{k}{r}$$

The Lagrangian for such a system can then be written as

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

so that the generalised momenta, i.e., the radial momentum p_r and the tangential momentum p_θ are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Using these, we now construct Hamiltonian H as a function of the generalised coordinates, generalised velocities and posibly time from the Lagrangian, i.e.,

$$\begin{aligned} H = \sum_i p_i\dot{q}_i - L &= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r} \\ &= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \end{aligned}$$

From the expression for H above, We see that

$$\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{k}{r^2}, \qquad \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\frac{\partial H}{\partial \theta} = 0, \qquad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

So, we can write Hamilton's canonical equations as

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}. \end{aligned}\right\} \tag{4.5.3}$$

and

$$\left.\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}, \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0. \end{aligned}\right\} \tag{4.5.4}$$

From (4.5.3) *and* (4.5.4) *we see that*

$$\begin{aligned} \ddot{r} = \frac{\dot{p}_r}{m} &= \frac{1}{m}\left(\frac{p_\theta^2}{mr^3} - \frac{k}{r^2}\right) \\ &= \frac{p_\theta^2}{m^2r^3} - \frac{k}{mr^2} \end{aligned}$$

or,

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}. \tag{4.5.5}$$

But

$$\frac{p_\theta^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3} = mr\dot{\theta}^2 = \frac{m(r\dot{\theta})^2}{r} = \frac{mv_\theta^2}{r} \tag{4.5.6}$$

With the tangential component of the velocity $v_\theta = r\dot{\theta}$ *the term* $\frac{mv_\theta^2}{r}$ *contained in* (4.5.6) *is identified with the centrifugal force and* $F \equiv F(r) = -\frac{k}{r^2}$ *as the central force field in which the particle executes motion. Therefore from the equation* (4.5.5) *we can write the radial part of the equation of motion of the particle can be written as*

$$m\ddot{r} = \frac{mv_\theta^2}{r} + F(r).$$

Example 4.5.3 *Write down the Hamiltonian function and Hamilton's equation of motion for a compound pendulum.*

Solution: *A compound pendulum consists of a comparatively large sized body which hangs from a point from within the body towards it's upper part so that its centre of gravity lies below the point of suspension. The effective length of the pendulum is the length between the point of suspension and the centre of gravity.*

A compound pendulum therefore executes a to-and-fro rotatory motion about the point of suspension, the kinetic energy of the motion being the corresponding kinetic energy of rotation, i.e.,

$$L = \frac{1}{2}I\dot{\theta}^2,$$

and the potential energy

$$V = mgh\cos\theta$$

Thus we have the Lagrangian L for compound pendulum is

$$L = \frac{1}{2}I\dot{\theta}^2 + mgh\cos\theta \tag{4.5.7}$$

We have

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \tag{4.5.8}$$

The Hamiltonian function H is given by

$$\begin{aligned} H &= \sum p_j\dot{q}_j - L(q_j, \dot{q}_j, t) \\ &= P_\theta\dot{\theta} - L \end{aligned}$$

Using equations (4.5.7) *and* (4.5.8), *H can be expressed as*

$$H = I\dot{\theta}\dot{\theta} - \frac{1}{2}I\dot{\theta}^2 - mgh\cos\theta \tag{4.5.9}$$

Now, from equation (4.5.8),

$$\dot{\theta} = \frac{P_\theta}{I}$$

With this substitution, the equation (4.5.9) *reduces to*

$$\begin{aligned} H &= \frac{1}{2} I \left(\frac{P_\theta}{I}\right)^2 - mgh\cos\theta \\ &= \frac{P_\theta^2}{2I} - mgh\cos\theta \end{aligned}$$

which gives

$$\frac{\partial H}{\partial P_\theta} = \frac{p_\theta}{I}, \qquad \frac{\partial H}{\partial \theta} = mgh\sin\theta.$$

The Hamilton's canonical equations are then

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial P_\theta} \\ \dot{P}_\theta &= \frac{\partial H}{\partial \theta} \end{aligned}$$

or,

$$\dot{\theta} = \frac{P_\theta}{I}, \tag{4.5.10a}$$

$$\dot{P}_\theta = -mgh\sin\theta \tag{4.5.10b}$$

Now, from equation (4.5.10a), *we have*

$$P_\theta = I\dot{\theta}$$

With this substitution, the equation (4.5.10b) *becomes*

$$I\ddot{\theta} + mgh\sin\theta = 0$$

or,

$$\ddot{\theta} + \frac{mgh}{I}\sin\theta = 0$$

But $I = mK^2$, K *is the radius of gyration. Thus the equation of the motion for the compound pendulum turns out to be*

$$\ddot{\theta} + \frac{gh}{K^2} = 0$$

4.6 Comparison of the Lagrangian and Hamiltonian equation

1. Both in the Lagrangian and Hamiltonian formulations, both L or H can be considered as the key functions for the system from which a complete set of equations of motion can be obtained.

2. The Lagrange's equation of motion can be derived from the D'Alembert's principle which is essentially a differential principle based on Newton's laws of motion. But a more general and deeper insight to the foundation of mechanics is actually provided through the Hamilton's principle, which is an integral principle.

3. In the Lagrangian formulation, L is a function of the mutually independent variables are the generalised co-ordinates q_j's, the generalised velocities $\dot{q}_j$'s and the time. For a system of n degrees of freedom, the Lagrangian, and hence the motion is conveniently described by an n-dimensional co-ordinate space called the *configuration space*. Thus, the Lagrange's equations are second order differential equations in the configuration space and hence $2n$ initial conditions are required to obtain the solutions completely.

   In the Hamiltonian formulation, H is a function of independent variables of the generalised coordinates q_j, the generalised momenta p_j and time. The space in which the motion of the system is described is the *phase space*. The phase space is described by the $2n$ independent variables- the n generalised coordinates and n generalised momenta, apart from the time. The Hamiltonian formulation consists of two sets of first order differential equations combining to $2n$ degree of freedom while in Lagrangian formulation there are n second order differential equations corresponding to n degree of freedom.

4. From definition of hamiltonian it follows that if L is explicitly independent of time, then H is also explicitly independent of time.

5. If H is explicitly independent of time t, then

$$H = H(q_j, p_j, t) \equiv H(q_j, p_j)$$

   Therefore,

$$\begin{aligned}\frac{dH}{dt} &= \sum \frac{\partial H}{\partial q_j}\dot{q}_j + \sum \frac{\partial H}{\partial p_j}\dot{p}_j \\ &= \sum \frac{\partial H}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j} = 0\end{aligned}$$

   i.e., H does not change with time. Thus if the Hamiltonian (or the Lagrangian) does not depend on time explicitly, it is a constant of motion.
   In this case,

$$H = \sum p_j \dot{q}_j - L(q_j, \dot{q}_j) = E$$

   turns out to be the energy integral for conservative system for which $H = T + V = E$. Generally, H and L are not functions of t. If they are, then in such cases, there exists a constant of motion called *Jacobi integral* given by

$$H = constant = J.$$

   which may not be identical with the actual energy E.

4.7 Summary

Lagrangian and Hamiltonian formulations provide the complete solutions of varieties of dynamical problems. It is seen that for systems where one can define a Lagrangian for holonomic systems with applied forces derivable from an ordinary or generalised potential and workless constraints, we have the Lagrangian formulation which eliminates the forces of constraints from the equations of motion. The usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extensions in many areas outside classical mechanics. The Hamiltonian formulation provides with broad framework for transitions to Statistical Mechanics and Quantum Mechanics.

Self study questions:

1. Find the Hamiltonian of the following form of the Lagrangian:

   (a) $L(x, \dot{x}, t) = \exp(-x^2)\left[\exp(\dot{x}^2) + 2\dot{x}\int_0^{\dot{x}} \exp(-\alpha^2)\, d\alpha\right].$

   (b) $L(x, \dot{x}, t) = \frac{1}{2}\exp(\alpha t)[\dot{x}^2 - \omega^2 x^2].$

2. What is Legendre dual transformation? How does this transformation help in deducing the Hamilton's canonical equations?

3. Compare the role of the cyclic coordinates in Lagrangian formulation with the Hamiltonian formalism.

4. Can Hamilton's canonical set of equation provide us with more information compared to that of either the Lagrangian or the Newtonian? Justify your answer.

5. Write down the Hamiltonian and the Hamilton's equation of motion for a simple pendulum whose string is a spring of unstretched length l, the mass of the bob being m and the spring constant of the spring is k.

6. Analyse the dynamics of the Atwood machine by using the Hamilton's canonical equations.

UNIT 5

Calculus of variation

Preparatory inputs to this unit

1. Hamilton's canonical equations: a critical look on the properties.
2. Basics of Integral Calculus and Ordinary differential equations.

5.1 Calculus of Variation

5.1.1 Overview of this unit

In this unit, a detail discussion of the Calculus of Variation will be presented, along with the deductions of relevant equations and some interesting applications. To begin with, a formal definition of functionals and their first variations will be presented. Getting along with the subject, the Euler-Lagrange Equation will be deduced through the use of some basic lemmas and proofs, which have already been established otherwise, but not incorporated in this discussion.

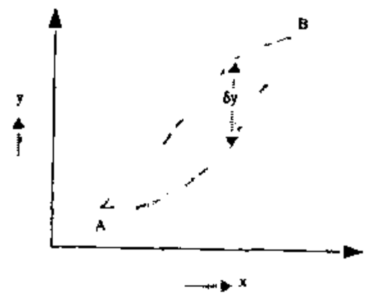
5.1.2 Introduction

Calculus of Variation is that branch of Mathematics which deals with extremum of functionals. A functional is a mapping which assigns a real number to each function or curve associated with some class of functions. The origin and the development of the Calculus of Variation is traced back to a challenge posed by Johann Bernoulli (1667 – 1748) to several of his contemporaries, including Newton. The challenge was to find the trajectory of an object moving in a uniform gravitational field so that, in the absence of friction, the time it takes for the object to travel from a point to another point, is the minimum.
As a generalisation to the problem as forwarded by Bernoulli, one would like to find the trajectory under certain given constraints or restrictions, which maximizes or minimizes a given mathematical quantity. Most of the problems it is found that the quantity to be minimised or maximised appears as an integral of certain functionals, *i.e.*, functions of several functions, over a given interval.

To go into the detail of the calculus of variations, we define an integral J such that

$$J = \int_a^b F(x, y, y')dx, \qquad \text{such that} \quad y(a) = A, \qquad y(b) = B$$

where $y = y(x)$ ranges over the set of all continuously differentiable functions defined on the interval $[a, b]$.



5.1.3 The technique of the Calculus of Variation

The basic problem of calculus of variation is to find a path, $y = y(x)$, in one dimension between x_1 and x_2, such that the line integral of some function $f(y, y', x)$, where $y' = \frac{dy}{dx}$, is an extremum, *i.e.*, maximum or minimum. The quantity f depends upon the functional form of the dependent variable $y(x)$ and is called a functional. Symbolically, the problem can be stated as: for a function $f(y, y', x)$, the integral

$$J = \int_{x_1}^{x_2} f(y, y', x)\, dx \tag{5.1.1}$$

along the path $y = y(x)$ between x_1 and x_2 is to be extremum.

Let (x_1, y_1) and (x_2, y_2) be two points in the space and consider two varied paths between two extreme points $y(x_1) = y_1$ and $y(x_2) = y_2$.

In order to find a path or paths which would give an extremum value of the integral, we state the problem in the language of differential calculus. For this, we associate a parameter α with all possible curves, *i.e.*, paths. The parameter α should be such that for some value of α, say $\alpha = 0$, the curve under examination would coincide with the path or paths that would give an extremum value for the integral. Then, y will be a function of both the independent variable x and the parameter α. We can always write $y(\alpha, x)$ as

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \tag{5.1.2}$$

where $\eta(x)$ is some function of x which has continuous first derivative and the function itself vanishes at both $x = x_1$ and $x = x_2$. The last condition, *viz.*, $\eta(x_1) = \eta(x_2) = 0$ ensures that the varied function $y(\alpha, x)$ will be identical to $y(x)$ at the extremities of the path.

With the dependence of y on α in addition to x, the integral is thus a function of the parameter α. So we have

$$J(\alpha) = \int_{x_1}^{x_2} f[y(\alpha, x), y'(\alpha, x), x]\, dx \tag{5.1.3}$$

Then, the condition that $J(\alpha)$ has an extremum value is

$$\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0 \tag{5.1.4}$$

This is only a necessary condition, but it is not sufficient. We shall use this condition to obtain an equation or equation that should be satisfied by f in order to have an extremum value.

Differentiating equation (5.1.3) under the integral sign, we get

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} f(y, y', x)dx\right] = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx$$

as we have $\dfrac{\partial x}{\partial \alpha} = 0$. Hence

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial \alpha \partial x}\right] dx \tag{5.1.5}$$

Integrating the second term in the integrand by parts,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right)dx = \frac{\partial f}{\partial y'}\left.\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}dx \tag{5.1.6}$$

But

$$\frac{\partial y}{\partial \alpha} = \eta(x)$$

and hence

$$\left.\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0 \tag{5.1.7}$$

Thus, the first term on the right hand side of equation (5.1.6) vanishes and equation (5.1.5) becomes

$$\begin{aligned}\frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\right]dx \\ &= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx\end{aligned} \tag{5.1.8}$$

It appears that equation (5.1.8) is independent of α. But, the functions y and y' with respect to which the derivatives of the functional f are taken, are functions of α. When $\alpha = 0$, we have $y(\alpha, x) = y(x)$, and the dependence on α disappears.

We want that $\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0$, and since $\eta(x)$ is an arbitrary function [such that $\eta(x_1) = \eta(x_2) = 0$], the integrand of equation (5.1.8) must vanish for $\alpha = 0$. Thus, we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{5.1.9}$$

Functions y and y' appearing in equation (5.1.9) are the original functions, independent of α. Equation (5.1.9) is called *Euler equation* and it represents the necessary condition that the integral J has the extremum value.

Alternate form of Euler equation

Euler equation (5.1.9) can also be put into another equivalent form. For this, consider

$$\begin{aligned}\frac{d}{dx}f(y, y', x) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}\end{aligned} \tag{5.1.10}$$

Now

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Substituting the value of $y''\frac{\partial f}{\partial y'}$ from equation (5.1.10), we get

$$\begin{aligned}\frac{d}{dx}\left(y'\frac{\partial f}{\partial y}\right) &= \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \\ &= \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \end{aligned} \tag{5.1.11}$$

But, the last term of the right hand side vanishes in view of equation (5.1.9) and equation (5.1.11) reduces to

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y}\right) = 0 \tag{5.1.12}$$

This is sometimes called the 'second form' of Euler equation. It is useful, particularly when $\frac{\partial f}{\partial x} = 0$, *i.e.*, when f does not depend explicitly upon x. Then, we have

$$f - y'\frac{\partial f}{\partial y} = \text{a constant} \tag{5.1.13}$$

Generalisation of Euler equation to several variables

Euler equation can be generalised for the case when f is a function of several dependent variables. Then, we can write

$$f = f\left[y_i(x), y_i'(x), x\right], \qquad i = 1, 2, \ldots, n. \tag{5.1.14}$$

In this case, we select the parametric equations similar to equation (5.1.2) as

$$y_i(\alpha, x) = y_i(0, x) + \alpha\eta_i(x) \tag{5.1.15}$$

The same procedure as described above can then be followed and we can get Euler-Lagrange equation in the form

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0, \qquad i = 1, 2, \ldots, n. \tag{5.1.16}$$

More generally

$$f = f\left[y_i(x_j), y_i'(x_j), x_j\right]$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Here, we have taken $x_1, x_2, \ldots, x_k$ as independent variables on which y depends. In this case, Euler-Lagrange equations take the form

$$\frac{\partial f}{\partial y_i} - \sum_{j=1}^{k}\frac{\partial}{\partial x_j}\frac{\partial f}{(\partial y_i/\partial x_j)} = 0 \tag{5.1.17}$$

The Euler-Lagrange equation in δ-notation

The results of the calculus of variation are very often expressed in terms of a compact δ-notation as follows:

We have, by equation (5.1.8)

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \frac{\partial y}{\partial \alpha} dx$$

Multiplying both sides of this equation by the differential $d\alpha$, we have

$$\frac{\partial J}{\partial \alpha}\, d\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \frac{\partial y}{\partial \alpha} d\alpha\, dx \tag{5.1.18}$$

Let

$$\frac{\partial J}{\partial \alpha} d\alpha = \delta J \; ; \qquad \text{and} \qquad \frac{\partial y}{\partial \alpha}\, d\alpha = \delta y \tag{5.1.19}$$

Hence, we get

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y\, dx \tag{5.1.20}$$

In this notation, the condition of extremum becomes

$$\delta J = \delta \int_{x_1}^{x_2} f(y, y', x) dx = 0 \tag{5.1.21}$$

Taking the symbol δ inside the integral sign, we get

$$\begin{aligned} \delta J &= \int_{x_1}^{x_2} \delta f\, dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\, \delta y + \frac{\partial f}{\partial y'}\, \delta y'\right]\, dx \end{aligned} \tag{5.1.22}$$

Now

$$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d(\delta y)}{dx}$$

$$\text{Hence} \qquad \delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\, \delta y + \frac{d}{dx}\frac{\partial f}{\partial y'}\, \delta y\right]\, dx \tag{5.1.23}$$

Integrating the second term on the right hand side by parts, we get

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \delta y\, dx \tag{5.1.24}$$

Here we used the condition that the variation in y at the end points is zero.
Since δy is arbitrary, we can get $\delta J = 0$ only if the integrand vanishes. This gives Euler-Lagrange equation as before.

5.1.4 Applications of Euler equation

1. *Geodesics in Euclidean plane:* The shortest distance between any two points in a space is called geodesics. An Euclidean or a flat surface is a surface, where the Pythagorus theorem, $ds^2 = dx^2 + dy^2$ is valid and constitutes a plane surface or a plane. Now, curves, joining any given two points, will have a distance. The question is, what is the shortest distance between the two points? Using the technique of the calculus of variations, the result can be obtained as follows:

   A small, elemental length, say ds in a plane can be expressed with the two components $(dx,\ dy)$ of the differentials in cartesian coordinates as

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}, \qquad \text{where} \quad y' = \frac{dy}{dx}$$

   The total length of the curve between any two given points, say $A \equiv (x_1,\ y_1)$ and $B \equiv (x_2,\ y_2)$ in the plane will be given by the integration of ds from A to B, *i.e.*,

$$I = \int_A^B ds = \int_A^B \left(\sqrt{1 + y'^2}\right) dx = \int_A^B f(x, y, y')\ dx$$

   Now in order for the distance between A and B to be the shortest, the variation of the above integral, the δ-variation of the integral I should be zero *i.e.*,

$$\delta I = \delta \int_A^B \left(\sqrt{1 + y'^2}\right) dx = 0$$

   The necessary condition for the vanishing of the delta-variation is the Euler equation in $f(x, y, y') = \sqrt{1 + y'^2}$, *i.e.*, Euler equation must be satisfied by $f(x, y, y')$:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \tag{5.1.25}$$

   Now

$$\frac{\partial f}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

   With this, the Euler equation, (5.1.25) reduces to

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0,$$

$$\text{or,} \qquad \frac{y'}{\sqrt{1 + y'^2}} = p, \qquad \text{where } p \text{ is a constant}$$

$$\Longrightarrow \qquad y' = \sqrt{\left(\frac{p^2}{1 - p^2}\right)} = m, \qquad m \text{ is a constant}$$

   Integratinig with respect to x, $\qquad y = mx + c, \qquad c$ is a constant.

which is an equation of a straight line. Therefore, we conclude that the shortest distance between two points in an euclidean plane is a straight line.

2. *Minimum surface of revolution:* The surface of revolution is formed by revolving a curve about a given axis. Set the curve passing through two end points $(x_1\ y_1)$ and $(x_2,\ y_2)$ is rotated about the y-axis, as shown in the figure. We now seek to find a curve which on revolving about the axis gives the geometry of minimum surface area. We consider an elemental *ribbon-shaped*

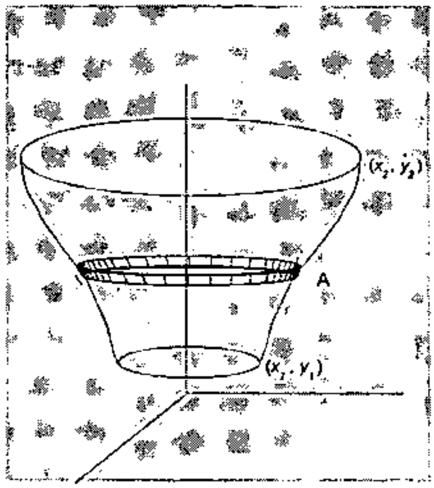


Figure 5.1: Minimum surface of revolution.

ring or strip at point A formed by the revolution of arc-length ds about y-axis. If x is the distance of thie arc from the ya-axis, the surface area of the strip is then

$$da = 2\pi x dx = 2\pi x \left(\sqrt{1+y'^2}\right) dx$$

The total surface area described by the curve about the axis of rotation is then given by

$$I = \int f(x,y,y')dx = \int_{x_1}^{x_2} 2\pi x\sqrt{(1+y'^2)}dx$$

where $f \equiv f(x,y,y') = 2\pi x\sqrt{(1+y'^2)}$.

This total surface area will be minimum when $\delta I = 0$, for which the equation to be satisfied is the Euler equation, *i.e.*,

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

We find

$$\frac{\partial f}{\partial y'} = \frac{2\pi x y'}{\sqrt{1+y'^2}}, \qquad \frac{\partial f}{\partial y} = 0$$

Substituting these values in the Euler equation, we get

$$\frac{d}{dx}\left(\frac{2\pi x y'}{\sqrt{1+y'^2}}\right) = 0,$$

or, $$\frac{xy'}{\sqrt{1+y'^2}} = a,$$ where a is a constant.

Simplifying,

$$y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

Integrating with respect to x,

$$y = a\int \frac{dx}{\sqrt{x^2-a^2}} + c,$$ c is the constant of integration.

or, $$x = a\cosh^{-1}\frac{x}{a} + c,$$

$\Longrightarrow$ $$x = a\cosh\frac{y-c}{a}.$$

which is the equation for a catenary.

3. *Propagation of light in an inhomogeneous medium:* Let us consider a 3-dimensional space filled with an optically inhomogeneous medium such that the velocity of propagation of light at a point is some function of its coordinates (x, y, z). The fundamental principle of light, the *Fermat's principle* suggests that light propagates from one point in the space to another along that curve for which the travel time is the minimum.

   Let A and B be two points in an inhomogeneous medium. Let us assume that the light moves from A to B along a curve given by

   $$y = y(x), \quad z = z(x)$$

   As light moves in inhomogeneous memdium, the velocity of propagation v will be a function of the coordinates, *e.g.*, $v \equiv v(x, y, z)$ in cartesian coordinates. So the time taken by the light to move from A to B is

   $$\int_{t_A}^{t_B} dt = \int_A^B \frac{ds}{v} = \int_a^b \frac{\sqrt{1+y'^2+z'^2}}{v}dx, \qquad \text{where } y' = \frac{dy}{dx}, \quad z' = \frac{dz}{dx}$$
   $$= T(y, z)$$

   Here we assume the coordinates of the end points are $A = A(a, a_y, a_z)$ and $B = B(b, b_y, b_z)$; v is the velocity of light at (x, y, z).

For minimum time, we must have

$$\delta \int_{t_A}^{t_B} dt = 0,$$

$$\Longrightarrow \qquad \delta \int_a^b \frac{\sqrt{1+y'^2+z'^2}}{v} dx = 0$$

from which follows the Euler equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \tag{5.1.26a}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0 \tag{5.1.26b}$$

with the function F given by

$$F(x, y, z, y', z') = \frac{\sqrt{1+y'^2+z'^2}}{v}.$$

From this we can calculate

$$\frac{\partial F}{\partial y} = -\frac{\sqrt{1+y'^2+z'^2}}{v^2}\frac{\partial v}{\partial y}$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{v\sqrt{1+y'^2+z'^2}}$$

$$\frac{\partial F}{\partial z} = -\frac{\sqrt{1+y'^2+z'^2}}{v^2}\frac{\partial v}{\partial z}$$

$$\frac{\partial F}{\partial z'} = \frac{z'}{v\sqrt{1+y'^2+z'^2}}$$

Substituting these into Euler equations (5.1.26a) and (5.1.26b), we have

$$\frac{\sqrt{1+y'^2+z'^2}}{v^2}\frac{\partial v}{\partial y} + \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}\,v}\right] = 0$$

$$\frac{\sqrt{1+y'^2+z'^2}}{v^2}\frac{\partial v}{\partial z} + \frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}\,v}\right] = 0$$

These are the governing differential equations for propagation of light in an inhomogeneous medium. Knowing the form of $v(x, y, z)$ one can solve these differential equations to yield the form of the trajectories of light propagation.

4. *The Physics of Soap films:* An interesting application of the calculus of variation, is the determination of the shape of a thin soap bubble suspended in space within a wire frame. Here, we will consider the case of a thin soap film suspended between two wire rings of radius ρ_A and ρ_B whose centres lie along a straight line, the axis. We set up a cylindrical coordinate system with the Z-axis oriented along the axis of symmetry. The origin is chosen on the axis such that the centres of the two rings lie equidistant and mutually opposite to the origin. In order to describe the surface of the soap film formed between the rings, it is sufficient to specify a function $\rho(z)$ that measures the distance of the surface from the z-axis. The surface tension

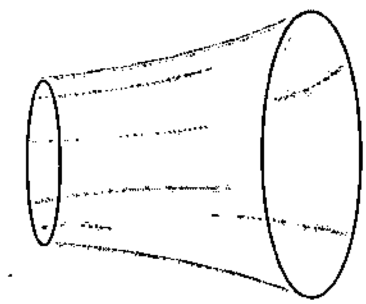


Figure 5.2: Soap film between two circular wire frames of different radii.

is the dominant force allows the soap film to retain its shape. The energy associated with the surface tension is contained in the surface area of the film. In general, given the situation, a system always tries to minimize its energy content and the corresponding configuration is such adjusted to minimize the energy content. In case of soap film too, same is the case; the film attempts to minimize its energy through minimization of its surface area. It therefore, adjusts its shape so that the surface area is minimal, subject to the requirement that it will be attached to the two wire rings. The surface area of the film is

$$S = \int_0^{2\pi} d\phi \int \rho(z) dl = 2\pi \int \rho(z) dl$$

where

$$dl = \left(dz^2 + d\rho^2\right)^{\frac{1}{2}} = \left[1 + \frac{d\rho}{dz}\right]^{\frac{1}{2}} dz$$

is the differential length along the curve $\rho(z)$. Our objective is to find the curve that minimizes the integral

$$S = 2\pi \int_{-L/2}^{L/2} \rho \left[1 + \frac{d\rho}{dz}\right]^{1/2} dz$$

where L is the distance between the two rings.

Setting

$$f \equiv \rho \left[1 + \frac{d\rho}{dz}\right]^{1/2} dz$$

we can write the Euler-Lagrange Equation as conditions to the required minimization, as

$$\begin{aligned}
& \frac{\partial f}{\partial \rho} - \frac{d}{dz}\frac{\partial f}{\partial \rho'} = 0 \\
\Longrightarrow \quad & \left(1 + \rho'^2\right)^{1/2} - \frac{d}{dz}\left(\frac{\rho\rho'}{(1+\rho'^2)^{1/2}}\right) \\
\Longrightarrow \quad & \frac{1}{\left(1+\rho'^2\right)^{1/2}} - \frac{\rho\rho''}{\left(1+\rho'^2\right)^{3/2}} = 0 \\
\Longrightarrow \quad & \frac{d}{dz}\left[\frac{\rho}{(1+\rho'^2)^{1/2}}\right] = 0 \\
\Longrightarrow \quad & \frac{\rho}{\left(1+\rho'^2\right)^{1/2}} = C_1 \\
\Longrightarrow \quad & \rho' = \pm\sqrt{\frac{\rho^2}{C_1^2} - 1} \\
\Longrightarrow \quad & \rho(z) = C_1 \cosh\left(\frac{z + C_2}{C_1}\right)
\end{aligned}$$

which is a catenary. Here C_1 and C_2 constants of integration to be determined by the requirement that the soap film is attached to the two wire rings, *i.e.*, $z(-L/2) = \rho_A$ and $z(L/2) = \rho_B$.

5.2 The Brachistochrone or shortest time problem

5.2.1 Historical perspective

The brachistochrone problem was one of the earliest problems raised during the development of the calculus of variations. It was said that Newton was challenged to solve the problem in 1696, which he could solve the very next day. In fact, the solution, which is a segment of a cycloid, was known to be found by Leibniz, L'Hospital, Newton and the two Bernoullis. Johann Bernoulli had solved the problem using the analogous technique used for finding the path of light refracted by transparent layers of varying density.

It was known that actually, Johann Bernoulli had initially found an incorrect proof; the the curve he found was a cycloid. His brother Jakob was then challenged to find the required proof for the curve. After Jakob correctly proved the solution, Johann substituted this proof for his own.

5.2.2 Brachistochrone: the details

The Brachistochrone, or the shortest time problem is the problem that involves finding the form of the curve joining the points A and B traversed by a particle falling under the influence of gravity from the point A of higher altitude to B, the lower altitude in the least possible time.

Let us choose A as the origin and y-axis vertically downwards, x-axis horizontally through A and y-axis vertically downwards.

Let a particle of mass m takes any position P on this curve at any given instant of time. If the particle moves an elemental distance ds at P in an infinitesimal interval of time dt, then the speed of the particle at P is given by,

$$v = \frac{ds}{dt}$$

$$\text{or,} \qquad dt = \frac{ds}{v}$$

Further the quantity ds can be expressed in terms of infinitesimals of the horizontal and vertical distances as

$$ds^2 = dx^2 + dy^2,$$

$$\text{or,} \qquad ds = \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] dx$$

$$= \left(\sqrt{1 + y'^2}\right) dx$$

The total time taken by the particle in falling from A to B can be found by integrating dt from t_A, the time when the particle is at A, to t_B when the particle is at B:

$$t_{AB} = \int_A^B \frac{ds}{v} \tag{5.2.1}$$

From the energetics of the Newtonian dynamics we know that if a particle at rest freely falls under gravity through a vertical distance y, picking up its velocity from 0 to v, the kinetic energy gain by the particle equals the loss of potential energy, *i.e.*,

$$\frac{1}{2}mv^2 = mgy,$$

from which we get

$$v = \sqrt{2gy}.$$

Further,

$$ds^2 = dx^2 + dy^2$$

$$\text{or,} \qquad ds = \left(\sqrt{1 + y'^2}\right) dx$$

where g is the acceleration due to gravity. Using these expressions, the integration (5.2.1) can be written as

$$t_{AB} = \int_A^B \left(\frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\right) dx = \int_A^B f(x, y, y')dx$$

with the integrand $f(x, y, y')$ as,

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}.$$

from which

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy(1 + y'^2)}}$$

For the time to be taken by the particle to be minimum, t_{AB} must have an extremal value. So $f(x, y, y')$ must satisfy the Euler Equation. We write down the second form of the Euler equation given by (5.1.13), *i.e.*,

$$f - y'\frac{\partial f}{\partial y'} = A, \qquad \text{where } A \text{ is a constant}$$

$$\implies \quad \frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y(1 + y'^2)}} = B, \qquad B = A\sqrt{2g} \text{ is another constant}$$

Simplifying,

$$\frac{1}{\sqrt{y(1 + y'^2)}} = B$$

$$\implies \quad y' = \sqrt{\frac{2a - y}{y}}, \qquad \text{with } 2a = \frac{1}{B^2},$$

Integration of this differential equation yields,

$$x = \int \left(\sqrt{\frac{y}{2a - y}}\right) dy + K, \qquad K \text{ is the constant of integrati}$$

To find the integration on the right hand side, we substitue

$$y = a(1 - \cos\theta), \tag{5.2.2}$$

$$\text{so that} \quad dy = a\ \sin\theta\ d\theta, \quad \text{and} \quad \sqrt{\frac{y}{2a - y}} = \tan\frac{\theta}{2}$$

Then the integration reduces to

$$\begin{aligned}
x &= \int \tan\frac{\theta}{2}\ a\ \sin\theta\ d\theta + K \\
&= \int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} .2a\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta + K \\
&= \int 2a\sin^2\frac{\theta}{2} d\theta + K \\
&= a\int (1 - \cos\theta) d\theta + K
\end{aligned}$$

$$\text{or,} \quad x = a(\theta - \sin\theta) + K \tag{5.2.3}$$

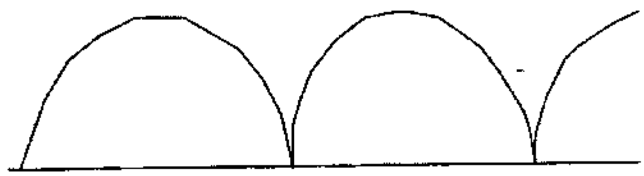


Figure 5.3: Cycloidal path for the Brachostochrone problem.

If the initial position of the particle is so chosen that when $x = 0$, $y = 0$ then $\theta = 0$. This reduces the constant K in (5.2.3) to zero. Therefore we have the solution to the Brachistochrone problem as

$$x = a(\theta - \sin\theta)$$

$$\text{and} \qquad y = a(1 - \cos\theta),$$

These are the equations of a cycloid which passes through the origin of the coordinate system chosen. Thus the path of the particle is a cycloid. The value of a in these equations can be so chosen to make the path passing through the second chosen point, say (x_1, y_1). Along the path, as depicted in the figure, the time of transit of the particle from the origin to (x_1, y_1) turns out to be the minimum.

5.3 Hamilton's variational principle

In order to solve a dynamical problem involving non-holonomic constraints, we use the Lagrangian and the Hamiltonian formulation which are obtained from the variational principle, called *Hamilton's variational principle.* This principle is therefore a general foundation to the Lagrangian formulation of mechanics. The variational principle is looked upon as an *integral principle.* Here, we consider the entire motion of the system between two instants t_1 and t_2 and consider small virtual variations of the entire motion of the system from the actual motion.

The principle is stated in a general form independent of any co-ordinate system and hence is useful in non-mechanical systems and fields.

Hamilton's principle may be stated as follows:
Of all the possible paths along which a dynamical system may move from one point to another within a given interval of time (consistent with constraints, if any) the actual path followed is that which minimizes the time integral of the Lagrangian.