MANGALAYATAN
UNIVERSITY
*Learn Today to Lead Tomorrow*

# CALCULUS

**MAO-1111**

*Self Learning Material*

॥ विश्वं ज्ञाने प्रतिष्ठितम् ॥

Directorate of Distance & Online Education

**MANGALAYATAN UNIVERSITY**
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# PREFACE

*In this course, we shall deal with various aspects of* Calculus

- Limit and Continuity
- Successive Differentiations
- Partial Differentiation
- Tangent and Normal
- Curvature
- Asymptotes and Singular Points
- Differentiability
- Taylor's theorem

# SYLLABUS

**Unit 1:**

Limits of functions, Sequential criterion for limits, Divergence criteria. Limit theorems, One-sided limits, Infinite limits and limits at infinity, Continuous functions, Sequential criterion for continuity and discontinuity, Algebra of continuous functions, Properties of continuous functions on closed and bounded intervals; Uniform continuity, Non-uniform continuity criteria, Uniform continuity theorem.

**Unit 2:**

Differentiability of functions, Successive differentiation, Leibnitz's theorem, Partial differentiation, Euler's theorem on homogeneous functions. Tangents and normals, Curvature, Asymptotes, Singular points.

**Unit 3:**

Differentiability of functions, Algebra of differentiable functions, Carathdodory's theorem and chain rule; Relative extrema, Interior extremum theorem, Rolle's theorem, Mean-value theorem and its applications, Intermediate value property of derivatives - Darboux's theorem.

**Unit 4:**

Taylor polynomial, Taylor's theorem with Lagrange form of remainder, Application of Taylor's theorem in error estimation; Relative extrema, and to establish a criterion for convexity; Taylor's series expansions of simple trigonometric and exponential functions.

# CONTENTS

Chapter 1

# Limit and Continuity

## STRUCTURE

- Limits of functions and theorems on limits
- Infinite limit and limits at infinity
- Continuous Function
- Algebra of Continuous functions and properties of continuous function
- Uniform Continuity
  - Summary
  - Objective Evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:

- Limits of function and related theorem
- Continuous functions and its properties
- Standard results of continuity
- Concept of uniform continuity
- How to classify the continuity, uniform continuity and non-uniform continuity

## 1.1 INTRODUCTION

The most important idea in calculus is that of *limit*. The concept of the limit is the foundation of atmost all of mathematical analysis. In this chapter we shall introduce the notion of limits and continuity of a special class of functions whose domain is an interval and range is contained in R. These functions are known as *real valued functions of a single variable.* Since, we shall throughout be concerned with real valued functions only, the word *function* will stand for a real valued function.

## 1.2 GRAPH OF A FUNCTION

The graph of a function, always play an important role in discussing the nature of a function $f(x)$. It is defined as follows "*If $f : X \to Y$, be a function, then the set of all ordered pair $(x, y)$ in which $x \in X$, appears as a first element and its image appears as its second element is called the graph of f.*

*i.e.*, Graph of a function $f : X \to Y$ is $[\{(x, f(x)) : x \in X, f(x) \in Y]$.

**For example.** Consider the function

$$f(x) = \sin \frac{1}{x}, x \neq 0$$

Then, the graph of $f(x)$ is given.

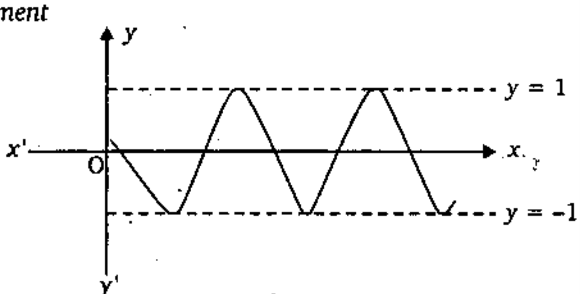


**Fig.1**

### REMARK

- By Dedekind Cantor axiom, we know that to every real number, there correspond a unique point on a directed line and vice versa. Let us consider two mutually prependicular directed straight lines in a plane intersecting at a point $O$ such that the point $O$ represents the real number 0 (zero). We observe that to every ordered pair of real numbers there correspond a point in the plane and vice -versa. Thus a graph of the function can be regarded as a collection of points in the plane.

Notes

## 1.3 LIMIT OF A FUNCTION

Let $f(x)$ be a function defined in some interval I containing a point $a$, but may or may not be defined at $a$ itself. We consider the behaviour of $f(x)$ as $x \to a$. It may happen that the values of $f$ become closer and closer to a number $l$ as $x \to a$ *i.e.,* the absolute value of the difference $(f(x)-l)$ can be made smaller than any pre-assigned positive number $\varepsilon$, however small, by taking sufficiently close to $a$. In such a case, we can say that $f(x)$ approaches or converges or tends to the limit $l$ as $x \to a$. We can write

$$\lim_{x \to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a.$$

Formally, we define.

**Definition.** *Let $f$ be a function defined in a neighbourhood of a except possible at a. Then a real number $l$ is said to be the limit of $f$ as $x$ tends to $a$ if given $\varepsilon > 0$, however small, there exists $\delta > 0$ (depending upon $\varepsilon$) such that*

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

*i.e.,* $$l - \varepsilon < f(x) < l + \varepsilon, \text{ whenever } x \in \,]a-\delta, a[\,\cup\,]a, a+\delta[.$$

## 1.4 ONE SIDED LIMITS

**(i) Right hand limit.** A function $f$ is said to approach $l$ as $x$ approaches $a$ from right if corresponding to an arbitrary positive number $\varepsilon$, there exists a positive number $\delta > 0$ such that

$$|f(x)-l| < \varepsilon \text{ whenever } a < x < a + \delta$$

It is written as $f(a+0)$ or $\lim_{x \to a+0} f(x) = l$

and $f(a+0) = \lim_{h \to 0} f(a+h)$

**(ii) Left hand limit.** A function $f$ is said to approach to $l$ as $x$ apporaches $a$ from the left, if corresponding to an arbitrary positive number $\varepsilon$, there exists a positive number $\delta > 0$ such that

$$|f(x)-l| < \varepsilon \text{ whenever } a - \delta < x < a$$

It is written as $f(a-0)$ or $\lim_{x \to a-0} f(x) = l$

If both, right hand limit (RHL) and left hand limit (LHL) of $f$ as $x \to a$ exist and are equal in value, then their common value will be the limit of $f$ as $x \to a$.

**REMARK**

- If either or both of these limits do not exists, the limit of $f$ as $x \to a$ does not exist. Even if both these limits exist but are not equal in value, then also the limit of $f$ as $x \to a$ does not exist.

WORKING PROCEDURE

**(i)** To find the limit on right, put $a+h$ for $x$ in $f(x)$ and then take limit as $h \to 0$.

$$\Rightarrow \quad \lim_{x \to a+0} f(x) = \lim_{h \to 0} f(a+h).$$

**(ii)** To find the limit on left, put $a-h$ for $x$ in $f(x)$

and then take limit as $h \to 0$.

$$\Rightarrow \quad \lim_{x \to a-0} f(x) = \lim_{h \to 0} f(a-h).$$

### 1.4.1 GRAPHICAL REPRESENTATION OF RHL AND LHL

Notes

Let $y = f(x)$ be a function.

If $x \to a$, then for those values of $x$ which greater than $a$, let $l_1$ be the limit of $f(x)$

*i.e.,*

$$\lim_{x \to a+0} f(x) = l_1 \text{ or } \lim_{x \to a-0} y = l_1.$$

This has been shown in the figure (2) by an arround from the right because for RHL $x \to a$ from the right similarly, the LHL = $l_2$, is shown in the same figure adjoining by an arrow from left.

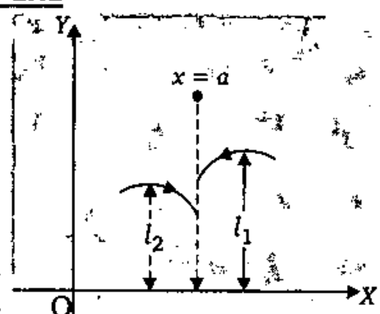


Fig. 2

## 1.5 LIMIT AT INFINITY AND INFINITE LIMITS

### 1.5.1 LIMITS AT INFINITY

(i) A function $f(x)$ is said to tends to a limit $l$ as $x \to \infty$ if for given $\varepsilon > 0$, however small, there exists a positive number $\delta$, such that

$$|f(x) - l| < \varepsilon \ \forall x \geq \delta$$

$$\Rightarrow \quad l - \varepsilon < f(x) < l + \varepsilon \ \forall x \geq \delta$$

and we write

$$\lim_{x \to \infty} f(x) = l$$

(ii) A function $f(x)$ is said to tends to a limit $l$ as $x \to -\infty$ if for given $\varepsilon > 0$, however small, there exists a positive number $\delta > 0$, such that

$$|f(x) - l| < \varepsilon \ \forall x \leq -\delta$$

$$\Rightarrow \quad l - \varepsilon < f(x) < l + \varepsilon \ \forall x \leq -\delta$$

and we write

$$\lim_{x \to -\infty} f(x) = l$$

### 1.5.2 INFINITE LIMITS

(i) A function $f: A \to R$, where $A \subseteq R$ is said to tend to the limit $+\infty$ as $x \to a$, if for any given positive number $\delta_1 > 0$, there exists a positive number $\delta_2$ such that

$$x \in A, \ 0 < |x - a| < \delta_2 \Rightarrow f(x) > \delta_1$$

and we write $\lim_{x \to a} f(x) = \infty.$

(ii) A function $f: A \to R$, where $A \subseteq R$ is said to tend to the limit $-\infty$ as $x \to a$, if for any given positive number $\delta_1, \exists$ a positive number $\delta_2$ such that

$$x \in A, \ 0 < |x - a| < \delta_2 \Rightarrow f(x) < -\delta_1$$

and we write $\lim_{x \to a} f(x) = -\infty$

(iii) If neither of the above two conditions are satisfied, then the function $f(x)$ is said to oscillate as $x \to a$, if a number $\delta_1$ can possibly be assigned such that

$$|f(x)| < \delta_1 \text{ whenever } 0 < |x - a| < \delta_2$$

then the function $f$ is said to oscillate finitely otherwise infinitely.

(iv) A function $f(x)$ is said to tend to $\infty$ as $x \to \infty$, if for any given positive number $N$, however large, $\exists$ a positive number $\delta$ such that $f(x) > N \ \forall x \geq \delta$

and we write $\lim_{x \to \infty} f(x) = \infty$

Notes

(v) A function $f(x)$ is said to tend to $-\infty$ as $x\to\infty$, if for any given positive number $N$, however large, $\exists$ a positive number $\delta$ such that $f(x) < -N\ \forall x \geq \delta$

and we write $\lim\limits_{x\to\infty} f(x) = -\infty$

(vi) A function $f(x)$ is said to tend to $\infty$ as $x\to -\infty$, if for any given positive number $N$, however large, $\exists$ a positive number $\delta$ such that $f(x) > N\ \forall x \leq -\delta$

(vii) A function $f(x)$ is said to tend to $-\infty$ as $x\to-\infty$, if for any given positive number $N$, however large, $\exists$ a positive number $\delta$ such that $f(x) < -N\ \forall x \leq -\delta$

**REMARK**

- If a function $f$ does not tend to a finite limit or to $\infty$ or $-\infty$ then
  (i) if it is bounded in a nbd of $a$, it is said to oscillate finitely.
  (ii) if it is unbounded in a nbd of $a$, it is said to oscillate infinitely.

## 1.6 UNIQUENESS OF LIMIT

**THEOREM 1.** *The limit of a function, if exists is unique.*

**Proof.** Let $f(x)$ be a function defined on an interval $I$. Let $a \in I$. Also, let us suppose

$$\lim_{x\to a} f(x) \text{ exist.}$$

Let if possible, $f(x)$ tends to two different limits $l_1$ and $l_2$ as $x\to a$. $(l_1 \neq l_2)$

Take $\varepsilon = \frac{1}{2}|l_1 - l_2| > 0$

Since $f(x)\to l_1$ as $x\to a$, $\exists\ \delta_1 > 0$ such that

$$|f(x) - l_1| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1 \quad \text{...(1)}$$

Now, since $f(x)\to l_2$ as $x\to a$, $\exists\ \delta_2 > 0$ such that

$$|f(x) - l_2| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_2 \quad \text{...(2)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \text{ whenever } 0 < |x-a| < \delta$$

$$\leq |f(x) - l_1| + |f(x) - l_2| \text{ whenever } 0 < |x-a| < \delta$$

$$< \varepsilon + \varepsilon = |l_1 - l_2|$$

$\Rightarrow \quad |l_1 - l_2| < |l_1 - l_2|$

which is a contradiction.

Hence, $\quad l_1 = l_2$

$\Rightarrow$ limit of a function, if exists is unique.

## 1.7 ALGEBRA OF LIMIT OF FUNCTIONS

**THEOREM 1.** *If* $\lim\limits_{x\to a} f(x) = l$ *and* $\lim\limits_{x\to a} g(x) = m$, *then*

*(i)* $\lim\limits_{x\to a} [f(x) \pm g(x)] = l \pm m$ $\qquad$ *(ii)* $\lim\limits_{x\to a} [f(x).g(x)] = l.m$

*(iii)* $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, *provided* $m \neq 0$.

**Proof.** (i) Given that

$\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$

By definition, for $\varepsilon > 0\ \exists\ \delta > 0$ such that $|f(x) - l| < \varepsilon/2$

and $|g(x)-m|<\varepsilon/2$ for $0<|x-a|<\delta$

Consider $|(f(x)\pm g(x))-(l\pm m)|=|(f(x)-l)\pm(g(x)-m)|\le|(f(x)-l)|\pm|(g(x)-m)|$

$<\varepsilon/2+\varepsilon/2=\varepsilon$ for $0<|x-a|<\delta$

$\Rightarrow$ $|(f(x)\pm g(x))-(l\pm m)|<\varepsilon$ whenever $0<|x-a|<\delta$ Hence, $\lim_{x\to a}[f(x)\pm g(x)]=l\pm m$

(ii) Since $\lim_{x\to a} f(x)=l$, then for $\varepsilon=1$ $\exists$ $\delta_1>0$ such that

$|f(x)-l|<1$ for $0<|x-a|<\delta_1$

or $|f(x)-l|+|l|<1+|l|$ for $0<|x-a|<\delta_1$

$\Rightarrow$ $|f(x)|\le|f(x)-l|+|l|<1+|l|$ for $0<|x-a|<\delta_1$ ...(1)

Also we have $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$

Then, for $\varepsilon>0$ $\exists$ $\delta_2>0$ such that

$|f(x)-l|<\varepsilon$ and $|g(x)-m|<\varepsilon$ for $0<|x-a|<\delta_2$ ...(2)

Now, consider

$|f(x).g(x)-lm|=|f(x)g(x)-f(x).m+f(x)m-ml|=|f(x)(g(x)-m)+m(f(x)-l)|$

$\le|f(x)|\,|g(x)-m|+|m|\,|f(x)-l|<(1+|l|+|m|)\varepsilon$ [Using (1) and (2)]

$=\varepsilon_1$ for $0<|x-a|<\delta$ where $\delta=\min\{\delta_1\delta_2\}$

$\Rightarrow$ $|f(x)g(x)-lm|<\varepsilon_1$ for $0<|x-a|<\delta$.

Hence, $\lim_{x\to a} f(x).g(x)=l.m$

(iii) Since, $\lim_{x\to a} g(x)=m\neq 0$, then by taking $\varepsilon=\frac{1}{2}m$, we can obtain that

$|g(x)|>\frac{1}{2}|m|$ ...(1)

Also, as $l,m$ are the limits of $f(x)$ and $g(x)$ respectively, for $\varepsilon>0$ $\exists$ $\delta_2>0$ such that

$|f(x)-l|<\varepsilon$ and $|g(x)-m|<\varepsilon$ for $0<|x-a|<\delta_2$ ...(2)

Now, consider

$$\left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|=\left|\frac{mf(x)-lg(x)}{mg(x)}\right|=\left|\frac{m(f(x)-l)-l(g(x)-m)}{|m|.|g(x)|}\right|$$

$$\le\frac{|m|\,|f(x)-l|+|l|\,|g(x)-m|}{|m|.|g(x)|}<\frac{|m|.\varepsilon+|l|.\varepsilon}{|m|.\frac{1}{2}|m|}\text{ [Using (1)]}$$

$$=2\left[\frac{|l|+|m|}{|m|^2}\right]\varepsilon=\varepsilon_1 \text{ for } 0<|x-a|<\delta, \text{where } \delta=\min\{\delta_1,\delta_2\}$$

$$\Rightarrow \left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|<\varepsilon \text{ for } 0<|x-a|<\delta$$

$$\Rightarrow \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{l}{m}, \text{ provided } m\neq 0$$

## REMARK

- $\lim_{x\to a}(f\pm g)(x)$, $\lim_{x\to a}(fg)(x)$ and $\lim_{x\to a}\left(\frac{f}{g}\right)(x)$ may exists even if neither of $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists.

  **For example.** Let $f$ and $g$ be defined as follows :

$$f(x)=\begin{cases}-1 & \text{if } x<a\\ 1 & \text{if } x>a\end{cases} \quad \text{and } g(x)=\begin{cases}1 & \text{if } x<a\\ -1 & \text{if } x>a\end{cases}$$

Then $(f+g)(x)=0\ \forall\ x\neq a$ and $(fg)(x)=-1=\left(\frac{f}{g}\right)(x)\ \forall\ x\neq a$

$\Rightarrow$ $\lim_{x\to a}(f+g)(x)=0$, $\lim_{x\to a}(fg)(x)=-1=\lim_{x\to a}\left(\frac{f}{g}\right)(x)$.

Notes

But $\lim_{x\to a} f(x) = -1$ and $\lim_{x\to a+0} f(x) = 1$.

$\Rightarrow$ $\lim_{x\to a-0} f(x)$ does not exist.

Similarly, $\lim_{x\to a} g(x)$ does not exist.

Again, let $f$ and $g$ be defined as follows :

$$f(x) = \begin{cases} 1 & \text{if } x < a \\ -1 & \text{if } x > a \end{cases} \quad \text{and } g(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

Then, $(f-g)(x) = 0 \quad \forall x \neq a$

$\Rightarrow$ $\lim_{x\to a} (f-g)(x) = 0$, but $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist.

**THEOREM 2.** *If* $\lim_{x\to a} f(x) = l$, *then* $\lim_{x\to a} |f(x)| = |l|$.

**Proof.** Given that $\lim_{x\to a} f(x) = l$

Then, by definition, for given $\varepsilon > 0 \; \exists$ a positive number $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ for } 0 < |x - a| < \delta \quad \text{...(1)}$$

Also, we have

$$|f(x) - l| \geq \big|\,|f(x)| - |l|\,\big| \;\; \forall\, x \in R. \quad \text{...(2)}$$

From (1) and (2), we have

$$\big|\,|f(x)| - |l|\,\big| \leq |f(x) - l| < \varepsilon \text{ for } 0 < |x - a| < \delta$$

$\Rightarrow$ $\lim_{x\to a} |f(x)|$ exists and $\lim_{x\to a} |f(x)| = |l|$.

## REMARK

- Converse, of the above theorem need not be true

  For example. Let $f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$

  Then $|f(x)| = 1 \; \forall x \neq a$

  $\lim_{x\to a} |f(x)| = 1$ but $\lim_{x\to a-0} f(x) = -1$ and $\lim_{x\to a+0} f(x) = 1$.

  $\Rightarrow$ $\lim_{x\to a} f(x)$ does not exist.

- Converse of the above theorem is true only if $l = 0$.

**THEOREM 3.** *If* $\lim_{x\to a} f(x) = l$, *then* $\lim_{x\to a} e^{f(x)} = e^l$.

**Proof.** Since $\lim_{x\to a} f(x) = l$, then for $e^l > \varepsilon > 0 \; \exists \; a$ positive number $\delta > 0$

Such that $\log(e^l - \varepsilon) < f(x) < \log(e^l + \varepsilon)$

$\Rightarrow$ $e^l - \varepsilon < e^{f(x)} < e^l + \varepsilon \Rightarrow |e^{f(x)} - e^l| < \varepsilon$

Hence, $\lim_{x\to a} e^{f(x)} = e^l$

**THEOREM 4.** *If* $\lim_{x\to a} f(x) = l$, *then* $\lim_{x\to a} \log f(x) = \log l$.

**Proof.** If $\lim_{x\to a} f(x) = l > 0$

For $\varepsilon > 0 \; \exists \; \delta > 0$ such that

$$0<|x-a|<\delta \Rightarrow le-\varepsilon< f(x)< le+\varepsilon$$

$\Rightarrow$ $-\varepsilon<\log f(x)-\log l< \varepsilon \Rightarrow |\log f(x)-\log l|<\varepsilon$

Hence, $\lim_{x\to a} \log f(x) = \log l$

**THEOREM 5.** *If $f(x)$ is a function defined on a deleted nbd D of a point a such that $f(x)\geq 0$, then $\lim_{x\to a} f(x)\geq 0$ provided it exists.*

**Proof.** Let $\lim_{x\to a} f(x) = l.$

Let if possible $l < 0$.

Setting $\varepsilon=\frac{|l|}{2}$, we can find a number $\delta > 0$ such that

$$|f(x)-l|<\frac{|l|}{2} \text{ for } 0<|x-a|<\delta$$

$\Rightarrow$ $l-\frac{|l|}{2} <f(x)<l+\frac{|l|}{2}$ for $0<|x-a|<\delta$

$\Rightarrow$ $\frac{3l}{2} < f(x)<\frac{l}{2}$ for $0<|x-a|<\delta$ $\qquad \left(\varepsilon=\frac{|l|}{2}=-\frac{l}{2} \text{ as } l<0\right)$

$\Rightarrow$ $f(x)<\frac{l}{2}<0\ \forall x\in D$, which is a contradiction as $f(x)>0$.

Therefore, $\lim_{x\to a} f(x)\geq 0.$

**THEOREM 6.** *If f and g are defined on a deleted nbd D of a point a and $f(x)\geq g(x)\ \forall\ x\in D$, then $\lim_{x\to a} f(x)\geq \lim_{x\to a} g(x)$ provided both limit exist.*

**Proof.** Let us define a function $h$ on $D$ such that

$$h(x)=f(x)-g(x)\ \forall x\in D.$$

Then $h(x)>0$

$[f(x)>g(x)]$

$\Rightarrow$ $\lim_{x\to a} h(x)\geq 0$ ...(1)

Now $\lim_{x\to a} h(x)=\lim_{x\to a} [f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$ ...(2)

Now, from (1) and (2), we have

$$[\lim_{x\to a} f(x)-\lim_{x\to a} g(x)]\geq 0$$

$$\lim_{x\to a} f(x)\geq \lim_{x\to a} g(x)$$

**THEOREM 7.(Squeeze principle)** *If functions f, g and h are defined on a deleted nbd D of a point a such that*

$f(x)\geq g(x)\geq h(x)\ \forall x\in D$ *and* $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=l$

*then $\lim_{x\to a} g(x)$ exists and is equal to l.*

**Proof.** Since $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=l$, then for any $\varepsilon>0\ \exists$ a positive number $\delta>0$ such that

$$|f(x)-l|<\varepsilon \text{ and } |h(x)-l|<\varepsilon \text{ for } 0<|x-a|<\delta$$

or $l-\varepsilon<f(x)<l+\varepsilon$

and $l-\varepsilon<h(x)<l+\varepsilon$ for $0<|x-a|<\delta$.

Therefore, we have

$$l-\varepsilon<h(x)\leq g(x)\leq f(x)<l+\varepsilon \text{ for } 0<|x-a|<\delta$$

Notes

$\Rightarrow \quad l-\varepsilon<g(x)<l+\varepsilon$ for $0<|x-a|<\delta$

$\Rightarrow \quad |g(x)-l|<\varepsilon$ for $0<|x-a|<\delta$

Hence, $\lim_{x\to a} g(x)$ exists and is equal to $l$.

## REMARK

- The Squeeze principle is also known as Sandwitch theorem.

**THEOREM 8.** *If $\lim_{x\to a} f(x) = 0$, $g(x)$ is bounded in some deleted neighbourhood of a, then*

$$\lim_{x\to a} f(x).g(x)=0.$$

**Proof.** Since $g(x)$ is bounded in some deleted *nbd* of $a$, therefore, $\exists$ positive numbers $k$ and $\delta_1$ such that

$$|g(x)|\le k \text{ whenever } 0<|x-a|<\delta_1 \quad ...(1)$$

Let $\varepsilon>0$ since $\lim_{x\to a} f(x)=0$ then $\exists\ \delta_2>0$ such that

$$|f(x)-0|<\varepsilon \text{ or } |f(x)|<\frac{\varepsilon}{k} \text{ whenever } 0<|x-a|<\delta_2 \quad ...(2)$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then $0<|x-a|<\delta\ \forall x$.

Consider $|f(x)g(x)-0|=|f(x)g(x)|=|f(x)||g(x)|<\frac{\varepsilon}{k}.k=\varepsilon$ [Using (1) and (2)]

$\Rightarrow \quad |f(x).g(x)-0|<\varepsilon$

$\Rightarrow \quad \lim_{x\to a} f(x).g(x)=0.$

## CERTAIN LIMITS

(i) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (ii) $\lim_{x\to 0}(1+x)^{1/x}=e$

(iii) $\lim_{x\to\infty}\left(1+\frac{x}{h}\right)^h = \lim_{x\to\infty}\left(1+\frac{1}{h}\right)=e^x$ (iv) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

(v) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a\ \forall a>0$ (vi) $\lim_{x\to 0}\frac{x^p-y^p}{x-a}=pa^{p-1}\ \forall p\neq 0$ and $a\neq 0$ if $p=0$

(vii) $\lim_{x\to 0}\frac{\sin x}{x}=1$ (viii) $\lim_{x\to 0}\cos x=1$

## Solved Examples

**Example 1.** *Evaluate* $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$.

**Solution.** Here we have

$$f(x)=\frac{x^n-a^n}{x-a}$$

$$\Rightarrow f(a+h)=\frac{(a+h)^n-a^n}{a+h-a}=\frac{1}{h}\left[\left\{a^n+na^{n-1}.h+\frac{n(n-1)}{2!}a^{n-2}.h^2+...-a^n\right\}\right]$$

Now, $\quad \text{RHL}=f(a+0)=\lim_{h\to 0}f(a+h)=na^{n-1}$ ...(1)

Similarly we can find

$\text{LHL}=f(a-0)=\lim_{h\to 0}f(a-h)=na^{n-1}$ ...(2)

Now, from (1) and (2) we conclude that

$$f(a+0)=f(a-0)=na^{n-1}$$

Notes

**Example 2.** *Evaluate* $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$.

**Solution.** Here we have

$$f(x)=\frac{(1+x)^n-1}{x}$$

$\therefore$ $$\text{RHL}=f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}[(1+h)^n-1]/h$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\left\{1+nh+\frac{n(n-1)}{2!}h^2+\ldots\right\}-1\right]=n \quad ...(1)$$

Also $$\text{LHL}=f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}[(1-h)^n-1]/-h$$

$$=\lim_{h\to 0}\frac{1}{-h}\left[\left\{1-nh+\frac{n(n-1)}{2!}h^2+\ldots\right\}-1\right]=n \quad ...(2)$$

Now, from (1) and (2) we find that

$$\text{LHL}=\text{RHL}=n\Rightarrow \lim_{x\to 0}f(x)=n.$$

**Example 3.** *Evaluate* . $\lim_{x\to 0}(1+x)^{1/x}$.

**Solution.** Here we have

$$f(x)=(1+x)^{1/x}$$

$$\text{RHL}=f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}(1+h)^{1/h}$$

$$=\lim_{h\to 0}\left[1+\frac{1}{h}.h+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2!}(h^2)+\ldots\right]=1+1+\frac{1}{2!}+\frac{1}{3!}+\ =e \quad ...(1)$$

Similarly, $\text{LHL}=f(0-0)=\lim_{h\to 0}(1-h)^{-1/h}$

$$=\lim_{h\to 0}\left[1-\frac{1}{h}.(-h)+\frac{\left(-\frac{1}{h}\right)\left(-\frac{1}{h}-1\right)}{2!}(-h^2)+\ldots\right]$$

$$=\lim_{h\to 0}\left[1+1+\frac{1(1+h)}{2!}+\frac{1(1+h)(1+2h)}{3!}+\ldots\right]$$

$$=1+1+\frac{1}{2!}+\frac{1}{3!}+\ =e \quad ...(2)$$

From (1) and (2) we find that RHL=LHL=$e$

$$\Rightarrow \quad \lim_{x\to 0}(1+x)^{1/x}.=e$$

**Example 4.** *Evaluate* $\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)$.

**Solution.** Let $$f(x)=x\sin\frac{1}{x}$$

Now, $$\text{RHL}=f(0+0)=\lim_{h\to 0}f(0+h)$$

$$=\lim_{h\to 0}(0+h)\sin\left(\frac{1}{0+h}\right)=\lim_{h\to 0}h\sin\frac{1}{h}$$

=0×a finite quantity lying between – 1 and 1 =0. ...(1)

Notes

Also, $\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(0-h)\sin\left(\frac{1}{0-h}\right)$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0. \quad \text{...(2)}$$

Now, from (1) and (2) we conclude that RHL = LHL = 0

Hence, $\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$

**Example 5.** *Using* $\varepsilon-\delta$ *definition, evaluate* $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

**Solution.** Let $f(x) = x^2\sin\frac{1}{x}$

then $|f(x)-0| = \left|x^2\sin\frac{1}{x}\right| = |x^2|\left|\sin\frac{1}{x}\right|$

Now, since $\left|\sin\frac{1}{x}\right| \le 1$ therefore

$|f(x)-0| \le |x^2|$

$\Rightarrow$ $|f(x)-0| < \varepsilon$ whenever $0 < |x^2| < \varepsilon$

i.e., when $0 < |x| < \sqrt{\varepsilon}$ i.e., when $0 < |x| < \delta(\delta^2 = \varepsilon)$

Hence, by the definition of limit, we have $\lim_{x\to 0} x^2\sin\frac{1}{x} = 0$

**Example 6.** *Evaluate* $\lim_{x\to 0}\left[\left(a^x - b^x\right)/x\right]$

**Solution.** Let $f(x) = \dfrac{a^x - b^x}{x}$

$$\text{RHL} = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{a^{0+h}-b^{0+h}}{(0+h)} = \lim_{h\to 0}\frac{a^h - b^h}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\left\{1 + h\log_e a + \frac{h^2}{2!}(\log_e a)^2 + \ldots\right\} - \left\{1 + h\log_e b + \frac{h^2}{2!}(\log_e b)^2 + \ldots\right\}\right]$$

$$\left(\because\ a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots\right)$$

$$= \lim_{h\to 0}\left[(\log_e a - \log_e b) + \frac{h}{2!}\left\{(\log_e a)^2 - (\log_e b)^2\right\} + \ldots\right]$$

$$= \log_e a - \log_e b = \log_e\frac{a}{b} \quad \text{...(1)}$$

Similarly, we can find

$$\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \log_e\frac{a}{b} \quad \text{...(2)}$$

Thus, we find from (1) and (2) that both RHL and LHL exist and each equal to $\log_e\frac{a}{b}$ hence, $\lim_{x\to 0}\left[\frac{a^x - b^x}{x}\right] = \log_e\left(\frac{a}{b}\right)$

**Example 7.** *Let* $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational}\end{cases}$. *Check the existence of the limit of f(x).*

**Solution.** Here, we have

$$f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational}\end{cases}$$

Now, there are following cases:

**Case (i)** *If a is a non-zero rational number .*

Here, $\quad \text{LHL} = f(a-0) = \lim_{h\to 0} f(a-h)$

$$=\begin{cases} \lim_{h\to 0} (a-h)=a, & \text{if } (a-h) \text{ is rational} \\ \lim_{h\to 0} -(a-h)=-a, & \text{if } (a-h) \text{ is irrational}\end{cases}$$

which is not unique.

$\Rightarrow \quad f(a-0)$ does not exist.

$\Rightarrow \quad \lim_{x\to a} f(x)$ does not exist.

**Case (ii)** If $\quad a = 0.$

Here, $\quad \text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

$$=\begin{cases} \lim_{h\to 0} (-h)=0, & \text{if } -h \text{ is rational} \\ \lim_{h\to 0} h=0, & \text{if } -h \text{ is irrational}\end{cases}$$

Similarly, $f(0+0) = 0$

Hence, $f(0+0) = f(0-0) = 0 \Rightarrow \lim_{x\to 0} f(x)$ exists and is equal to zero.

**Case (iii)** *If a is an irrational number.*

Here, $\quad \text{LHL} = f(a-0) = \lim_{h\to 0} f(a-h)$

$$=\begin{cases} \lim_{h\to 0} (a-h)=a, & \text{if } (a-h) \text{ is rational} \\ \lim_{h\to 0} -(a-h)=-a, & \text{if } (a-h) \text{ is irrational}\end{cases}$$

$\Rightarrow \quad \lim_{x\to a} f(x)$ does not exist.

Hence, we have that $\lim_{x\to 0} f(x)$ exists only when $a=0$.

**Example 8.** *Show that* $f(x) = \lim_{x\to 2} \frac{|x-2|}{x-2}$ *does not exist.*

**Solution.** Let $f(x) = \frac{|x-2|}{x-2}$

Now $\quad \text{RHL} = f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{|2+h-2|}{(2+h-2)} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$

$\text{LHL} = f(2-0) = \lim_{h\to 0} f[2-h] = \lim_{h\to 0} \frac{|2-h-2|}{(2-h-2)} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} -1 = -1.$

Since, $f(2+0) \neq f(2-0)$.

Hence, $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

**Example 9.** *Discuss the existence of the limit of the function f defnied by* $f = \begin{cases} 1 & \text{if} & x<1 \\ 2-x & \text{if} & 1\le x\le 2 \\ 2 & \text{if} & x\ge 2\end{cases}$

**Solution.** Here, we check the existence of the limit at $x=1$ and $x=2$.

**Case (i)** At $x = 1$

$\text{RHL} = f(1+0) = \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} [2-(1+h)] = \lim_{h\to 0} (1-h) = 1$

Notes

$$\text{LHL} = f(1-0) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 1 = 1$$

$\Rightarrow$ $f(1+0) = f(1-0) = 1 \Rightarrow \lim_{x\to 1} f(x)$ exists and is equal to 1.

**Case (ii)** At $x = 2$

$$\text{RHL} = f(2+0) = \lim_{h\to 0} = f(2+h) = \lim_{h\to 0} 2 = 2$$

and $\quad \text{LHL} = f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} f[2-(2-h)] = \lim_{h\to 0} h = 0$

Since $f(2+0) \neq (2-0)$, hence $\lim_{x\to 2} f(x)$ does not exist.

**Example 10.** *Using* $\varepsilon - \delta$ *definition, show that* $\lim_{x\to 2} \frac{1}{x}(x \neq 0) = \frac{1}{2}$

**Solution.** Let $\quad f(x) = \frac{1}{x}$.

In order to show that $\lim_{x\to 2} f(x) = \frac{1}{2}$, we are to prove that for any positive number $\varepsilon$, we can find a positive number $\delta$, when $\delta$ depend upon $\varepsilon$ *i.e.*, $\delta = \delta(\varepsilon)$, such that

$$\left|f(x) - \frac{1}{2}\right| < \varepsilon \text{ when } 0 < |x-2| < \delta.$$

Now, $\quad f(x) - \frac{1}{2} = \frac{1}{x} - \frac{1}{2} = \frac{2-x}{2x}$

$$\Rightarrow \quad \left|f(x) - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} \quad \text{...(1)}$$

Now, choosing $\delta \leq 1$ and $0 < |x-2| < \delta$, we find that $0 < |x-2| < 1$, as $\delta \leq 1$

*i.e.*, $|x-2| < 1$ and $|x-2| > 0$

$\Rightarrow \quad 2-1 < x < 2+1$ and $x \neq 2$

$\Rightarrow \quad 1 < x < 3$ and $x \neq 2$

$\Rightarrow \quad \frac{1}{1} > \frac{1}{x} > \frac{1}{3}$ and $x \neq 2 \quad \Rightarrow \frac{1}{3} < \frac{1}{x} < 1$ and $x \neq 2$

$\Rightarrow \quad \frac{1}{|x|} < 1$ and $x \neq 2 \quad \left(\because \frac{1}{x} > \frac{1}{3} > 0 \Rightarrow \frac{1}{x} = \frac{1}{|x|}\right)$

Therefore, from (1), we have

$$\left|f(x) - \frac{1}{2}\right| = \frac{|x-2|}{2} \cdot \frac{1}{|x|} < \frac{\delta}{2} \cdot 1$$

Now, let us choose $\delta$ such that $\frac{\delta}{2} < \varepsilon$ *i.e.*, $\delta < 2\varepsilon$.

Also $\delta \leq 1$, therefore, if we take $\delta = \min\{1, 2\varepsilon\}$, we have

$$\left|f(x) - \frac{1}{2}\right| < \frac{\delta}{2} < \varepsilon \text{ when } 0 < |x-2| < \delta$$

$$\Rightarrow \quad \lim_{x\to 2} f(x) = \frac{1}{2}$$

**Example 11.** *If* $\lim_{x\to a} f(x)$ *exists and* $\lim_{x\to a} g(x)$ *does not exist, then show that,* $\lim_{x\to a} [f(x) + g(x)]$ *does not exist.*

**Solution.** Let $\lim_{x\to a} f(x) = l$ if exists.

Notes

Then by definition of limit of a function, we have

$$\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = l_1 \quad \ldots(1)$$

Also, given that $\lim_{x\to a} g(x)$ does not exist. So let

$$\lim_{x\to a+0} g(x) = \lambda_1 \text{ and } \lim_{x\to a-0} g(x) = \lambda_2 \text{ such that } \lambda_1 \neq \lambda_2. \quad \ldots(2)$$

Now, $\lim_{x\to a+0} [f(x)+g(x)] = \lim_{x\to a+0} f(x) + \lim_{x\to a+0} g(x) = l_1 + \lambda_1$

and $\lim_{x\to a-0} [f(x)+g(x)] = \lim_{x\to a-0} f(x) + \lim_{x\to a-0} g(x) = l_2 + \lambda_2$

Now, since $\lambda_1 \neq \lambda_2 \Rightarrow l_1 + \lambda_1 \neq l_2 + \lambda_2$

$\Rightarrow \lim_{x\to a+0} [f(x)+g(x)] \neq \lim_{x\to a-0} [f(x)+g(x)]$

Hence, $\lim_{x\to a+0} (f(x)+g(x))$ does not exist.

**Example 12.** *Evaluate* $\lim_{x\to 0} \frac{x-|x|}{x}$.

**Solution.** Let $f(x) = \frac{x-|x|}{x}$

Now, $\text{RHL} = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{h-|h|}{h}$

$= \lim_{h\to 0} \frac{h-h}{h} = \lim_{h\to 0} 0 = 0$

and $\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{-h-|-h|}{-h}$

$= \lim_{h\to 0} \frac{-h-h}{-h} = \lim_{h\to 0} \frac{-2h}{-h} = \lim_{h\to 0} 2 = 2$

Since, $f(0+0) \neq f(0-0)$. Hence, $\lim_{x\to 0} f(x)$ does not exist.

## STUDENT ACTIVITY

1. Find $\lim_{x\to 2} \frac{x^2+3x+2}{x-2}$.

2. Find $\lim_{x\to\infty} \frac{\sin x}{x}$.

3. Evaluate the following limit, if exists. $\lim_{x\to 2} \frac{2x^2-8}{x-2}$.

Notes

## TEST YOURSELF

**1.** Evaluate the following limits:

(i) $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$ (ii) $\lim_{x\to 0} \frac{\sin x}{x}$ (iii) $\lim_{x\to 0} \frac{a^x-1}{x}$ (iv) $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$

(v) $\lim_{x\to 0} \frac{|\sin x|}{x}$ (vi) $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$ (vii) $\lim_{x\to\infty} \left[x\left(a^{1/x}-1\right)\right], a>1$

(viii) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (ix) $\lim_{x\to 0}\left[\frac{2^x-1}{(1+x)^{1/2}-1}\right]$ (x) $\lim_{x\to 1}\left(\frac{\log x}{x-1}\right)$

(xi) $\lim_{x\to 0}\left[\frac{e^x-1}{x}\right]$

**2.** If $f(x)=\frac{\sin[x]}{[x]}, [x]\neq 0$ and $f(x)=0, [x]=0$ where $[x]$ denotes the greatest integer less than or equal to $x$, then find $\lim_{x\to 0} f(x)$.

**3.** Show that $\lim_{x\to 0} f(x) = \lim_{x\to a} f(x-a)$.

**4.** Let $f(x)=\begin{cases} x, & 0\le x\le 1 \\ 3-x, & 1\le x\le 2\end{cases}$. Show that $\lim_{x\to 1+0} f(x)=2$. Does the limit of $f(x)$ at $x=1$ exists.

**5.** If $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$, then prove that $\lim_{x\to a} f(x)=f(a)$.

**6.** Let $f(x)=\begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational}\end{cases}$, then show that $\lim_{x\to a} f(x)$ does not exist for any $a\in R$.

### ANSWERS

**1.** (i) $\frac{3}{2}$ (ii) 1 (iii) $\log a$ (iv) does not exist (v) does not exist RHL=1 LHL=−1

(vi) does not exist RHL=1 LHL=0 (vii) $\log a$ (viii) 1 (ix) 2 log 2 (x) 1 (xi) 1

**2.** does not exist **4.** does not exist

## 1.8 CONTINUITY

A continuous process is one that goes on smoothly without any sudden change. Continuity of a function can also be interpreted in a similar way. For better understanding, consider the following figures. The graph of the function in fig. 3(*a*) has a sudden cut at the point $x = 4$ whereas the graph of the function in fig. 3(*b*) proceeds smoothly. We say that the function of fig. 3(*b*) is continuous, while function of fig. 3(*a*) is not continuous.

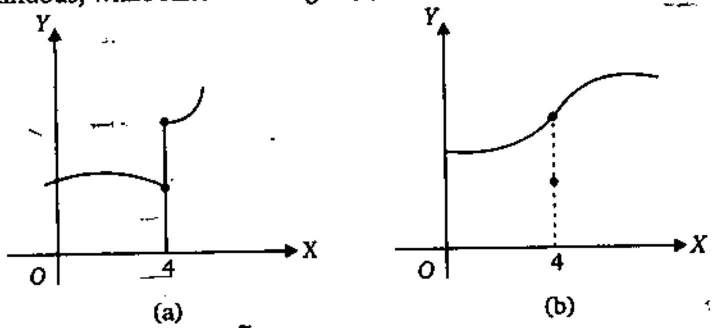


**Fig. 3**

Also, while defining $\lim_{x\to a} f(x)$, the function $f$ may or may not be defined at $x = a$. Even if $f$ is defined at $x = a$, $\lim_{x\to a} f(x)$ may or may not be equal to the value of the function at $x=a$. If

$\lim_{x \to a} f(x) = f(a)$, then we say that $f$ is continuous at $x = a$.

## 1.8.1 CONTINUOUS FUNCTIONS

A function $f$, defined on some nbd of a point $a$, is said to be continuous at $a$ if and only if any one of the following condition is saitsfied.

(i) $\lim_{x \to a} f(x) = f(a)$

(ii) $f(a-0) = f(a+0) = f(a)$

(iii) for $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $0 < |x - a| < \delta$.

The above all conditions are equivalent to each other, and being simple, are of common use.

### REMARKS

- The definition (iii) is known as Cauchy's definition of continuity.
- A function $f$ is said to be continuous in $I$ if it is continuous at every point of the interval $I$.
- From definition (iii), we observe that $|f(x) - f(a)| < \varepsilon$ implies that $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$.
- The interval $I$ may be any one of the following forms $]a,b[$, $]-\infty,\infty[$, $]a,\infty[$, $]-\infty,b[$.
- If a function is not continuous at a point, then it is said to be discontinuous at that point.
- The value of $\delta$ depends upon the values of $\varepsilon$ and $a$.
- Checking the continuity of a function from the smoothness of its graph is not a complete method. Consider the graph of the function $f(x) = x \sin \frac{1}{x}$, then we observe that it has no breaks in the nbd of $x=0$. But this function is not continuous. Observe that the graph oscillate widely near zero.

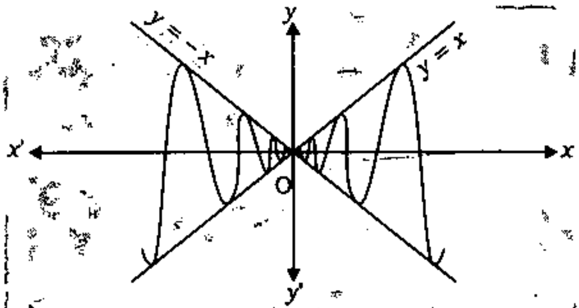


**Fig. 4**

## 1.8.2 MORE DEFINITIONS OF CONTINUITY

(i) If $\lim_{x \to a+0} f(x) = f(a)$, then we say that $f$ is continuous to the right of $a$ (or right continuous at $a$).

(ii) If $\lim_{x \to a-0} f(x) = f(a)$, then we say that $f$ is continuous to the left of $a$ (or left continuous at $a$).

(iii) A function $f$ is said to be continuous in an open interval $]a, b[$ if it is continuous at every point of $]a, b[$.

(iv) A function $f$ is said to be continuous in a closed interval $[a, b]$ if it is

(1) right continuous at $a$

(2) continuous at every point of $]a, b[$

(3) left continuous at $b$.

(v) A function $f$ is said to be continuous in a semi-closed interval $[a,b[$ if it is

(1) right continuous at $a$

(2) continuous at every point of $]a,b[$.

(vi) A function $f$ is continuous in a semi-closed interval $]a,b]$ if it is

(1) continuous at every point of $]a,b[$

(2) left continuous at $b$.

(vii) A function $f$ is said to be continuous at $a \in I$, iff $\lim_{x \to a} f(x)$ exists, finite and is equal to $f(a)$, otherwise the function is said to discontinuous at $x=a$.

Notes

### 1.8.3 SEQUENTIAL CONTINUITY OR HEINE'S DEFINITION OF CONTINUITY

The necessary and sufficient condition for a function $f$ defined on an interval $I \subset R$ to be continuous at a point of interval $I$ is that for each sequence $<a_n>$ in $I$ converges to $a$, the sequence $<f(a_n)>$ converges to $f(a)$. *i.e.*, $f$ is said to be continuous iff

$$\lim_{n\to\infty} f(a_n) = f(a).$$

### 1.8.4 GRAPHICAL MEANING OF CONTINUITY OF A FUNCTION

Continuity of a function $f$ at a point $a$ graphically means that there is no break in the graph of the curve $y = f(x)$ at $x = a$ and given however small $\varepsilon > 0\ \exists\ \delta > 0$ such that the graph of $y = f(x)$ from $x = a - \delta$ to $a+\delta$ lies between the lines $y = f(a) - \varepsilon$ and $y = f(a)+\varepsilon$.

**ILLUSTRATIONS**

(1) Every constant function $f: R\to R$ is continuous on $R$.

For $\varepsilon>0$, $a\in R$, $|x-a|<\varepsilon \Rightarrow |c-c|=0<\varepsilon$

(2) The identity function $f: X\to X \in R$ is continuous on $R$.

For $\varepsilon>0$, $\delta=\varepsilon$ and $|x-a|<\varepsilon \Rightarrow |x-a|<\varepsilon\ \forall a\in R$.

(3) The function $f: X\to X^n$, $n\in N$ is continuous on R.

For any $a\in R$, $\lim_{x\to a} f(x)=a^n=f(a)$.

(4) The polynomial function $f(x)=a_0+a_1x+\ldots+a_nx^n$ is continuous on $R$.

For any $a\in R$, $\lim_{x\to a} f(x)=f(a)$.

## 1.9 DISCONTINUITY

(1) A function $f$ which is not continuous at a point $a$ is said to be discontinuous at the point '$a$', where '$a$' is called the point of discontinuity of $f$ or $f$ is said to have a discontinuity at $a$.

(2) A function which is discontinuous even at a single point of an interval, is said to be discontinuous in that interval.

(3) A function $f$ can be discontinuous at a point $x = a$, because of any one of the following reasons :

(i) $f(x)$ is not defined at $x = a$. (ii) $\lim_{x\to a} f(x)$ does not exist.

(iii) $\lim_{x\to a} f(x)$ and $f(a)$ both exist but are not equal.

## 1.10 TYPE OF DISCONTINUITY

### 1.10.1 REMOVABLE DISCONTINUITY

A function $f$ is said to have a removable discontinuity at a point $a$ if $\lim_{x\to a} f(x)$ exists, but is not equal to the function value at $a$, *i.e.*,

$$f(a-0) = f(a+0) \neq f(a)$$

**REMARK**

- A function $f$ can be made continuous by assigning some suitable value to $a$, such that

$$\lim_{x\to a} f(x) = f(a)$$

For example. Suppose f is a function defined on ]0, 1[ as follows :

$$f(x)=\begin{cases} 2, & 0<x<1, x\neq \frac{1}{2} \\ 1, & x=\frac{1}{2} \end{cases}$$

Then, it is clear that $f$ is continuous in ]0, 1[ except at the point $x = \frac{1}{2}$. At the point $x = \frac{1}{2}$, we have

$$f\left(\frac{1}{2}-0\right) = f\left(\frac{1}{2}+0\right) = 2 \text{ but } f\left(\frac{1}{2}\right) = 1$$

$\Rightarrow$ $f$ has a removable discontinuity at $x = \frac{1}{2}$.

The discontinuity at $x = \frac{1}{2}$ may be removed by choosing $f\left(\frac{1}{2}\right) = 1$.

## 1.10.2 DISCONTINUITY OF FIRST KIND

A function $f$ is said to have a discontinuity of first kind at a point $a$, if both the limits $f(a-0)$ and $f(a+0)$ exist but are not equal. The point $a$ is said to be a point of discontinuity from the left or from right according as

$$f(a-0) \neq f(a) = f(a+0)$$

or

$$f(a-0) = f(a) \neq f(a+0)$$

**For example.** Consider a function $f$ defined on ]0, 1[ as follows

$$f(x) = \begin{cases} 1/2, & 0 < x < 1/2 \\ 0, & x = \frac{1}{2} \\ -1/2, & 1/2 < x < 1 \end{cases}$$

Obviously, $f$ is continuous over the open interval ]0,1/2[ and ]1/2,1[

At the point $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}-0\right) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right) = \frac{1}{2} \neq f(1/2)$$

$$f\left(\frac{1}{2}+0\right) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right) = -\frac{1}{2} \neq f\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad f\left(\frac{1}{2}-0\right) \neq f\left(\frac{1}{2}+0\right)$$

$\Rightarrow$ $f$ has a discontinuity of the first kind at $x = \frac{1}{2}$.

## 1.10.3 DISCONTINUITY OF SECOND KIND

A function $f$ is said to have a discontinuity of second kind at a point $a$ if none of the limit $f(a-0)$ and $f(a+0)$ exist at $a$. The point $a$ is said to be a point of discontinuity of second kind from the left or from the right according as $f(a-0)$ or $f(a+0)$ does not exist.

**For example.** Consider the function $f(x) = \cos\left(\frac{\pi}{x}\right)$ defined on $]-\infty, \infty[$. The graph of the function is given below :

Obviously, at the point $x = 0$, both the limits *i.e.*, $\lim_{x\to 0-} \cos\left(\frac{\pi}{x}\right)$ and $\lim_{x\to 0+} \cos\left(\frac{\pi}{x}\right)$ do not exist. Hence, $x = 0$ is a point of discontinuity of the second kind.

## 1.10.4 MIXED DISCONTINUITY

A function $f$ is said to have a mixed discontinuity at a point $a$ if $f$ has a discontinuity of second kind on one side of $a$ and on the other side, a discontinuity of first kind or may be continuous.

**For example.** For the function $f(x) = e^{1/x} \sin\frac{1}{x}$ then $\lim_{x\to 0-} f(x) = 0$, $\lim_{x\to 0+} f(x)$ does not exist and the function is not defined at $x=0$.

Notes

Therefore, the function has a discontinuity of first kind from the left and a discontinuity of the second kind from the right at $x=0$. Thus, the function has a mixed discontinuity at $x=0$.

### 1.10.5 INFINITE DISCONTINUITY

A function $f$ is said to have an infinite discontinuity at $x = a$ if $f(a+0)$ or $f(a-0)$ is $+\infty$ or $-\infty$. If $f$ has a discontinuity at $a$ and is unbounded in every nbd of $a$, then $f$ is said to have an infinite discontinuity at $a$.

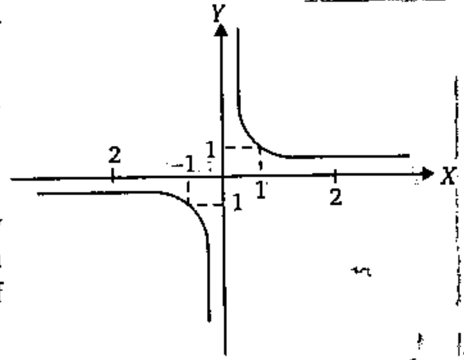

Fig. 7

**For example.** Suppose $f(x)=\frac{1}{x}$ in $]-\infty,\infty[$.

It is clear that $f$ is continuous on $]-\infty,\infty[$ except at $x=0$. At $x=0$, the limits do not exist but tends to infinity. So, $x=0$ is a point of infinite discontinuity. Hence, a rectangular hyperbola is a curve with one point of infinite discontinuity.

### 1.10.6 JUMP OF A FUNCTION AT A POINT

If $f(a+0)$ and $f(a-0)$ both exist, but not equal, then the jump in the function at $x = a$ is defined as the non-negative difference $f(a+0)-f(a-0)$.

**REMARK**

- A function having a finite number of jumps in a given interval is called piecewise continuous or sectionally continuous.

## 1.11 FOR FUNCTIONAL LIMITS

Let us suppose the function $f(x)$ be defined on the closed interval $[a,b]$ and let $x_0 \in [a,b]$.

Let the upper and lower bounds of the function $f(x)$ in the right hand nbd $[x_0, x_0+h]$ of $x_0$ be denoted by $M$ and $m$ respectively where $M=M(h)$ and $m=m(h)$. Let the sequence of diminishing values $h_1, h_2,...$ be assigned to $h$, which converges to zero, then $M(h_1)$, $M(h_2)$, $M(h_3)$ ... is a decreasing sequence and so it possesses a lower limit.

Similarly, the sequence $m(h_1)$, $m(h_2)$, $m(h_3)$... is an increasing sequence and have an upper limit. These lower and upper limits are respectively known as the upper and lower limits of the function $f(x)$ at $x=x_0$ on the right and are denoted by $\overline{f(x_0+0)}$ and $\underline{f(x_0+0)}$.

$$\therefore \quad \overline{f(x_0+0)} = \lim_{h\to 0} M(h) \text{ and } \underline{f(x_0+0)} = \lim_{h\to 0} m(h)$$

If the right hand upper limits $\overline{f(x_0+0)}$ is equal to the right hand lower limit $\underline{f(x_0+0)}$ common value is known as the right hand limit of the function $f(x)$ at $x = x_0$ and is denoted by $f(x_0+0)$

*i.e.,*
$$f(x_0+0) = \overline{f(x_0+0)} = \underline{f(x_0+0)}$$

Similarly, if we consider the left hand nbd $[x_0-h, x_0]$ then the upper limit of $m(h)$ and the lower limit of $M(h)$ are respectively known as the lower and upper limits of the function $f(x)$ at $x=x_0$ on the left and are denoted by $\underline{f(x_0+0)}$ and $\overline{f(x_0+0)}$ respectively.

If the left hand upper limit $\overline{f(x_0-0)}$ is equal to the left hand lower limit $\underline{f(x_0-0)}$, then their common value is known as the left hand limit of the function $f(x)$ at $x = x_0$ and is denoted by $f(x_0-0)$

*i.e.,*
$$f(x_0-0) = \overline{f(x_0+0)} = \underline{f(x_0+0)}$$

Notes

**REMARKS**

- The four numbers $\overline{f(x_0+0)}$, $\underline{f(x_0+0)}$, $\overline{f(x_0-0)}$ and $\underline{f(x_0-0)}$ are known as four functional limits of the function $f(x)$ at $x=x_0$.
- The four functional limits of the function $f(x)$ at $x = x_0$ are independent of the value of the function $f(x)$ at $x = x_0$.
- At $x = 0$, the functional limits are denoted by $\overline{f(+0)}$, $\underline{f(+0)}$, $\overline{f(-0)}$ and $\underline{f(-0)}$.

## Solved Examples

**Example 1.** *Show that* $f(x)=\dfrac{x^2-1}{x-1}$ *is continuous for all values of x except x=1.*

**Solution.** If $x\neq1$, then $f(x)=(x+1)=$ A polynomial $\Rightarrow f(x)$ is continuous for all values of $x\neq1$. If $x=1$, $f(x)$ is of the form $\dfrac{0}{0}$, which is not defined and so the function $f(x)$ is discontinuous at $x=1$.

**Example 2.** *Show that the function f(x) is defined by* $f(x)=\begin{cases}x^2, & x\neq1\\ 2, & x=1\end{cases}$ *is discontinuous at* $x=1$.

**Solution.** Here the value of $f(x)$ at $x = 1$ is 2. $\Rightarrow$ $f(1)=2$

Now, $\quad \text{RHL}=f(1+0)=\lim\limits_{h\to0} f(1+h)=\lim\limits_{h\to0}(1+h)^2=1$

also, $\quad \text{LHL}=f(1-0)=\lim\limits_{h\to0} f(1-h)=\lim\limits_{h\to0}(1-h)^2=1$

Therefore, we have $f(1+0)=f(1-0)\neq f(1)$

$\Rightarrow$ $f(x)$ is not continuous at $x=1$.

**Example 3.** *Examine whether or not the function*

$$f(x)=\begin{cases}\dfrac{2\sin x}{x}, & \text{when } x\neq0\\ 2, & \text{when } x=0\end{cases}$$

*is continuous at* $x = 0$.

**Solution.** Given that $f(x)=2$, when $x=0 \Rightarrow$ $f(0)=2$

Now, $\quad \text{RHL}=f(0+0)=\lim\limits_{h\to0} f(0+h)=\lim\limits_{h\to0}\left[\dfrac{2\sin(0+h)}{(0+h)}\right]=2 \quad \left(\because \lim\limits_{x\to0}\dfrac{\sin x}{x}=1\right)$

and $\quad \text{LHL}=f(0-0)=\lim\limits_{h\to0} f(0-h)=\lim\limits_{h\to0}\left[\dfrac{2\sin(0-h)}{(0-h)}\right]=2$

$[\because \sin(-h)=-\sin(h)]$

Therefrore, we have $f(0+0)=f(0-0)=f(0)=2$

Hence, $f(x)$ is continuous at $x=2$.

**Example 4.** *A function f(x) is defined as follows*

$$f(x)=\begin{cases}(x^2/a)-a, & \text{when } x<a\\ 0, & \text{when } x=0\\ a-(a^2/x), & \text{when } x>a\end{cases}$$

*Prove that f(x) is continuous at x=a.*

**Solution.** Here, we have

$$\text{RHL}=f(a+0)=\lim_{h\to0} f(a+h)=\lim_{h\to0}\left[a-\frac{a^2}{(a+h)}\right]$$

$\left[\text{By using } f(x)=a-\dfrac{a^2}{x} \text{ for } x>a\right]$

$$= \left[a - \frac{a^2}{a}\right] = (a - a) = 0 \quad ...(1)$$

and $\qquad \text{LHL} = f(a-0) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0}\left[\frac{(a-h)^2}{a} - a\right]$

$\left[\text{By using } f(x) = \frac{x^2}{a} - a \text{ for } x<a\right]$

$$= \frac{a^2}{a} - a = 0 \quad ...(2)$$

Also $f(x)=0$ for $x=a$

$\Rightarrow \qquad f(a)=0 \quad ...(3)$

Now, from (1),(2) and (3), we have $f(a+0)=f(a-0)=f(a)=0$

$\Rightarrow \qquad f(x)$ is continuous at $x=a$.

**Example 5.** *A function $f(x)$ is defined as follows*

$$f(x) = \begin{cases} 1+x & \text{if } x \le 2 \\ 5-x & \text{if } x \ge 2 \end{cases}$$

*check the continuity of $f(x)$ at $x=2$.*

**Solution.** Here, we have

$$f(2) = 1 + 2 \text{ or } 5 - 2 = 3 \quad ...(1)$$

Now, $\qquad \text{RHL} = f(2+0) = \lim_{h\to 0} f(2+h)$

$$= \lim_{h\to 0} [5-(2+h)] = \lim_{h\to 0} (3-h) = 3 \quad ...(2)$$

and $\qquad \text{LHL} = f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [1 + (2-h)] = 3. \quad ...(3)$

Now, from (1), (2) and (3), we have

$$f(2+0) = f(2) = f(2-0) = 3$$

Hence, the function $f(x)$ is continuous at $x=2$.

**Example 6.** *Show that the function f defined by*

$$f(x) = \begin{cases} 0, & \text{for } x = 0 \\ \frac{1}{2} - x, & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{2}, & \text{for } x = \frac{1}{2} \\ \frac{3}{2} - x, & \text{for } \frac{1}{2} < x < 1 \\ 1, & \text{for } x = 1 \end{cases}$$

*has three point of discontinuity. Find such points. Also draw the graph of the function.*

**Solution.** Here, we observe that the domain of the function $f(x)$ is closed inverval [0,1] when $0<x<\frac{1}{2}$, the function $f(x) = \frac{1}{2} - x$, which is being the polynomial is continuous at each points of its domain.

$\Rightarrow f(x)$ is continuous at each point of the open interval $]0, \frac{1}{2}[$ when $\frac{1}{2} < x < 1$, $f(x) = \frac{3}{2} - x$, which is also a polynomial in $x$.

$\Rightarrow f(x)$ is continuous in the open interval $]\frac{1}{2}, 1[$.

Now, we check the continuity of $f(x)$ at $x = 0, \frac{1}{2}$ and 1.

**(i)** *At* $x = 0$.

At $x=0$, $\quad f(x)=0$

and $\quad \text{RHL}=f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}\left(\frac{1}{2}-h\right)=\frac{1}{2}$

$\Rightarrow \quad f(0)\neq f(0+0)$

$\Rightarrow f(x)$ is not continuous at $x=0$.

**(ii)** *At* $x=\frac{1}{2}$.

At $x=\frac{1}{2}, f(x)=\frac{1}{2}$

$$\text{LHL}=f\left(\frac{1}{2}-0\right)=\lim_{h\to 0}\left(\frac{1}{2}-h\right)=\lim_{h\to 0}\left[\frac{1}{2}-\left(\frac{1}{2}-h\right)\right]=\lim_{h\to 0} h=0$$

$\Rightarrow f(\frac{1}{2})\neq f\left(\frac{1}{2}-0\right)$

$\Rightarrow f(x)$ is not continuous at $x=\frac{1}{2}$.

**(iii)** *At* $x=1$.

At $x=1, f(x)=1$

$$\text{LHL}=f(1-0)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\left[\frac{3}{2}-(1-h)\right]=\lim_{h\to 0}\left(\frac{1}{2}+h\right)=\frac{1}{2}$$

$\Rightarrow \quad f(1)\neq f(1-0)$

$\Rightarrow \quad f(x)$ is not continuous at $x=1$.

Hence, the function $f(x)$ has three points of discontinuity given by $x=0, \frac{1}{2}$ and 1.

**Graph of f(x).** The graph of the function consists of the point (0, 0), the segment of the line $y=\frac{1}{2}-x$ for $0<x<\frac{1}{2}$, the point $\left(\frac{1}{2},\frac{1}{2}\right)$, the segment of the line $y=\frac{3}{2}-x$ for $\frac{1}{2}<x<1$ and the point (1,1). The graph of $f(x)$ is given as fig. 8.

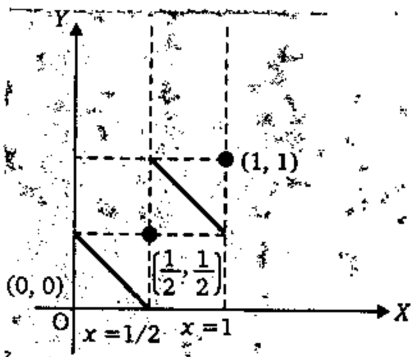


**Fig. 8**

**Example 7.** *Test the following functions for continuity*

(*i*) $f(x)=x\sin\frac{1}{x}, x\neq 0, f(x)=0$ *at* $x=0$.

(*ii*) $f(x)=\dfrac{1}{1-e^{-1/x}}, x\neq 0, f(x)=0$ at $x=0$

**Solution.** (i) Here, we have

$$\text{LHL}=f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$$

$$=\lim_{h\to 0}(-h)\sin\left(\frac{1}{-h}\right)=\lim_{h\to 0} h\sin\frac{1}{h}$$

$=0\times$ (a finite quantity lying between 1 and $-1$)$=0$

and $\text{RHL}=f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0} h\sin\frac{1}{h}=0.$

Also $f(0)=0$ given

$\Rightarrow f(0+0)=f(0-0)=f(0)$.

Hence, the function $f(x)$ is continuous at $x=0$.

Notes

(ii) Here, we have

$$\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{1}{1-e^{1/h}} = 0$$

and $\quad \text{RHL} = f(0+0) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{1}{1-e^{-1/h}} = 1$$

Also $\quad f(0)=0$

$\Rightarrow \; f(0+0) \neq f(0-0) = f(0)$

Hence, $f(x)$ is discontinuous at $x=0$ and this discontinuity is of first kind.

**Example 8.** *Discuss the kind of discontinuity, if any, of the function.*

$$f(x) = \begin{cases} \dfrac{x-|x|}{x}, & \text{if } x \neq 0 \\ 2\;, & \text{if } x = 0 \end{cases}$$

**Solution.** The given function is continuous at all points except possible the origin.

Now at $x = 0$

$$\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{-h-|-h|}{-h} = 2$$

and $\quad \text{RHL} = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \dfrac{h-|h|}{h} = 0.$

Also, $\quad f(0) = 2$(given)

$\Rightarrow \quad f(0-0) = f(0) \neq f(0+0).$

Hence, the given function $f(x)$ is discontinuous at $x = 0$ and this is the discontinuity of first kind.

**Example 9.** *Discuss the continuity of the function $f(x)$ defined by*

$$f(x) = \begin{cases} x^2 & \text{for } \; x < -2 \\ 4 & \text{for } \; -2 \leq x \leq 2 \\ x^2 & \text{for } \; x > 2 \end{cases}$$

**Solution.** Here, we shall check the continuity of $f(x)$ at $x = -2$ and 2.

*At* $x = -2$

We have $\quad f(-2) = 4$

Now $\quad \text{LHL} = f(-2-0) = \lim_{h\to 0} f(-2-h) = \lim_{h\to 0} (-2-h)^2 = 4$

and $\quad \text{RHL} = f(-2+0) = \lim_{h\to 0} f(-2+h) = \lim_{h\to 0} 4 = 4$

$\Rightarrow f(-2-0) = f(-2) = f(-2+0) = 4$

Hence, $f(x)$ is continuous at $x = -2$.

*At* $x = 2$

We have $\quad f(2) = 4$

and $\quad \text{RHL} = f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} (2+h)^2 = 4$

$\quad \text{LHL} = f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} 4 = 4$

$\Rightarrow \quad f(2-0) = f(2) = f(2+0) = 4$

Hence, $f(x)$ is continuous at $x = 2$.

**Example 10.** *Show that the function $f(x)$ defined on R by*

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

*is discontinuous at every point of* R.

**Solution.** Let us first suppose, $x$ be rational. Then $f(x)=1$. For each positive integer $n$, let $x_n$ be an irrational number such that $|x_n-x| < \frac{1}{n}$. Then the sequence $<x_n>$ converges to $x$. Now by definition $f(x_n)= -1 \ \forall\ n$.

$$\Rightarrow \quad \lim_{n\to\infty} f(x_n) = -1 \neq f(x).$$

Hence, $f$ is discontinuous at each rational point.

Now suppose $x$ is an irrational number. Then $f(x)= 1$. For each positive integer $n$, let $x_n$ be the rational number such that $|x_n-x| < \frac{1}{n}$. Then, the sequence $<x_n>$ converges to $x$. Now $f(x_n)= 1 \ \forall\ n$ so that

$$\lim_{n\to\infty} f(x_n)=1\neq f(x).$$

Hence, $f$ is discontinuous at each irrational point.

Therefore, $f$ is discontinuous at every point of R.

## REMARKS

- This function is known as Dirichlet's function.

**Example 11.** *Let a function $f$ : R→R satisfy the equation*

$$f(x+y)=f(x)+f(y) \ \forall x,y\in R$$

*show that if $f$ is continuous at $x=a$, then show that it is continuous for all $x\in$R.*

**Solution.** Since the function $f$ is continuous at $a$, we have

$$f(a)=f(a-0) = \lim_{h\to 0} f(a-h)= \lim_{h\to 0} f(a)+ \lim_{h\to 0} f(-h)$$

$$=f(a)+ \lim_{h\to 0} f(-h)$$

$$\Rightarrow \quad \lim_{h\to 0} f(-h)=0 \qquad ...(1)$$

Similarly $f(a)=f(a+0)= \lim_{h\to 0} f(a+h)= \lim_{h\to 0} f(a)+ \lim_{h\to 0} f(x)$

$$=f(a)+ \lim_{h\to 0} f(h)$$

$$\Rightarrow \quad \lim_{h\to 0} f(h)=0 \qquad ...(2)$$

Now, let $x$ be any arbitrary point of R, then we have

$$f(x-0)= \lim_{h\to 0} f(x-h)= \lim_{h\to 0} f(x)+ \lim_{h\to 0} f(-h)=f(x) \qquad \text{[By using (1)]}$$

and $f(x+0)= \lim_{h\to 0} f(x+h)= \lim_{h\to 0} f(x)+ \lim_{h\to 0} f(h)=f(x)$ [By using (2)]

Thus, $f(x)=f(x-0)=f(x+0)$

$\Rightarrow$ $f$ is continuous at $x\in$R.

Since $x$ is arbitrary. Hence, $f$ is continuous for all $x\in$R.

## STUDENT ACTIVITY

1. Find the function defined below for continuity at $x=0$.

$$f(x)=\frac{\sin^2 ax}{x^2} \text{ for } x\neq 0$$

*and* $f(x)=1$ for $x=0$.

Notes

2. Examine the continuity of the function

$$f(x)=\begin{cases}-x^2 & if \quad x\le 0\\ 5x-4 & if\ 0<x\le 1\\ 4x^2-3x & if\ 1<x<2\\ 3x+4 & if \quad x\ge 2\end{cases}$$

at $x=0,1$ and 2.

3. Test the continuity of the function at $x=0$

$$f(x)=x\cos\left(\frac{1}{x}\right),\ \text{if } x\neq 0,\ f(0)=0$$

## TEST YOURSELF

1. Discuss the continuity of the following functions :

(i) $f(x)=\cos\left(\frac{1}{x}\right)$, when $x\neq 0, f(0)=0$

(ii) $f(x)=\frac{\sin x}{x}, x\neq 0, f(0)=1$

(iii) $f(x)=\frac{1}{1-e^{1/x}}$, when $x\neq 0$, and $f(0)=0$

(iv) $f(x)=\frac{\sin^{-1}x}{x}, x\neq 0, f(0)=1$

(v) $f(x)=\frac{e^{1/x}\sin(1/x)}{1+e^{1/x}}, x\neq 0$, and $f(0)=0$

(vi) $f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}, x\neq 0, f(0)=0$

(vii) $f(x)=3x^2+2x-1$ at $x=2$

(viii) $f(x)=\frac{xe^{1/x}}{1+e^{1/x}}+\sin\frac{1}{x}$, when $x\neq 0, f(0)=0$

(ix) $f(x)=\frac{1}{x-a}\sin\frac{1}{x-a}$, at $x=a$

(x) $f(x)=\sin x\cos\frac{1}{x}$, when $x\neq 0, f(0)=0$

(xi) $f(x)=\frac{e^{1/x}}{1+e^{1/x}}$, when $x\neq 0, f(0)=0$

(xii) $f(x)=\begin{cases}\cos x \text{ for } x\ge 0\\ -\cos x \text{ for } x<0\end{cases}$

(xiii) $f(x)=\begin{cases}\dfrac{x^2-4x+3}{x^2-1} & \text{for } x\neq 1\\ 2 & \text{for } x=1\end{cases}$

(xiv) $f(x)=\frac{1}{x}\cos\frac{1}{x}$

2. Examine the following function for continuity at $x=0$ and $x=1$

$$f(x)=\begin{cases}x^2 & \text{if} \quad x\le 0\\ 1 & \text{if } 0<x\le 1\\ 1/x & \text{if} \quad x>1\end{cases}$$

3. Find out the points of discontinuity of the following functions.

(i) $f(x)=\left(2+e^{1/x}\right)^{-1}+\cos e^{1/x}$ for $x\neq 0$, $f(0)=0$.

(ii) $f(x)=\frac{1}{2^n}$ for $\frac{1}{2^{n+1}}<x\le\frac{1}{2^n}$, $n=0,1,2$, and $f(0)=0$

4. A function $f$ defined on $[0,1]$ is given by $f(x)=\begin{cases} x, & \text{if } x \text{ is rational} \\ \frac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$. Show that $f$ takes every value between 0 and 1, but it is continuous only at the point $x=\frac{1}{2}$.

5. A function $f : R \to R$ is defined as $f(x)=\frac{1}{x-4}$. Discuss the type of discontinuity which the function $f(x)$ has in $]-\infty,\infty[$.

## ANSWERS

1. (i) Discontinuous at $x=0$ (ii) Continuous at $x=0$
   (iii) Discontinuous at $x=0$ (iv) Contnuous at $x=0$
   (v) Discontinuity of the second kind at $x=0$
   (vi) Discontinuous at $x=0$ (vii) Continuous
   (viii) Discontinuous at $x=0$ (ix) Discontinuous
   (x) Continuous for all $x$ (xi) Discontinuous at $x=0$
   (xii) Discontinuous at $x=0$ (xiii) Discontinuous at $x=1$
   (xiv) Continuous for all $x$ except at $x=0$
2. Discontinuous at $x=0$ and continuous at $x=1$
3. (i) Discontinuous at $x=0$, mixed discontinuity
   (ii) Discontinuous at $x=\frac{1}{2^n}$ : $n=1, 2,$, discontinuity of first kind.
5. At $x=4$, function has infinite discontinuity and is continuous at all other points in $R$.

## 1.12 THEOREMS ON CONTINUITY

**THEOREM 1.** *If $f$ and $g$ are two continuous function at a point $a \in I$ then the function*

*(i) $f+g$* *(ii) $cf$*

*(iii) $fg$* *(iv) $f/g[g(a)\neq 0]$ are also continuous.*

**Proof.** Since $f$ and $g$ are continuous at $a$, then

$$\lim_{x\to a} f(x)=f(a) \text{ and } \lim_{x\to a} g(x)=g(a)$$

(i) By definition, we have $(f+g)(x)=f(x)+g(x)\ \forall\ x \in I$

$$\therefore \quad \lim_{x\to a}(f+g)(x)=\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$$

$\Rightarrow$ $(f+g)$ is continuous.

(ii) By definition, we have $(cf)(x)=cf(x)\ \forall\ x \in I$

Therefore, $\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x)=c\lim_{x\to a} f(x)=cf(a)=(cf)(a)$

Hence, $cf$ is continuous at $x=a$.

(iii) By definition, we have $(fg)(x)=f(x).g(x)\ \forall x\in I$.

Therefore,

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a}[f(x).g(x)] = \left[\lim_{x\to a} f(x)\right].\left[\lim_{x\to a} g(x)\right]=f(a).g(a)=(fg)(a).$$

Hence, $fg$ is continuous at $x=a$.

(iv) We have

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}\ \forall x\in I, g(x)\neq 0$$

Therefore, $\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}=\left(\frac{f}{g}\right)(a).$

Hence, $\frac{f}{g}$ is continuous.

Notes

**THEOREM 2.** *If f is continuous at $a \in I$, then $|f|$ is also continuous at a.*

**Proof.** Since $f$ is continuous at $x=a \Rightarrow \lim_{x \to a} f(x)=f(a)$

We know that $|f|(x)=|f(x)|, x \in I$

$$\Rightarrow \lim_{x \to a} |f|(x)=\lim_{x \to a} |f(x)|=\left|\lim_{x \to a} f(x)\right|=|f(a)|=|f|(a)$$

Hence, $|f|$ is continuous.

## REMARK

- The converse of the above theorem need not be true. For example: consider a function $f$ on R defined by

$$f(x)=\begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

then $|f|(x)=1\ \forall x \in R$, therefore $|f|$ is continuous at $x=0$ but $f$ is not continuous at $x=0$.

**THEOREM 3.** *The necessary and sufficient condition for a function f defined on an interval I to be continuous at a point I is that for each sequence $<a_n>$ of I converges to a, the sequence $<f(a_n)>$ converges to f(a).*

**Proof. (i)** **Necessary condition.** Let us first suppose $f$ be continuous at $x=a$ and let the sequence $<a_n>$ in $I$ be such that

$$\lim_{n \to \infty} a_n=a$$

Since, $f$ is continuous at $a$, therefore for a given $\varepsilon>0\ \exists$ a positive integer $m$ such that

$$|f(x)-f(a)|<\varepsilon \text{ whenever } |x-a|<\delta \quad ...(1)$$

Also, $\lim_{n \to \infty} a_n=a$, therefore, $\exists$ a positive integer $m$ such that

$$|a_n-a|<\delta\ \forall n \geq m \quad ...(2)$$

Put $x=a_n$, in (1), we get

$$|f(a_n)-f(a)|<\varepsilon \text{ when } |x-a|<\delta \quad ...(3)$$

Now, from (2) and (3), we get

$$|f(a_n)-f(a)|<\varepsilon\ \forall\ n \geq m.$$

Therefore, $\lim_{n \to \infty} f(a_n)=f(a)$.

**(ii)** **Condition is sufficient.** Let us suppose the sequence $<f(a_n)>$ converges to $f(a)$ if every sequence $<a_n>$ in $I$ converging to $a$. Then, to show that the function $f$ is continuous at $a$.

Let if possible, the function is not continuous at $a$. Then $\exists$ a positive number $\varepsilon>0$ such that for every $\delta>0\ \exists\ a\ x$ such that

$$|a_n-a|<\frac{1}{n}$$

but $|f(a_n)-f(a)|>\varepsilon$ ($\because f$ is not continuous.)

This shows that $\lim_{n \to \infty} a_n=a$. Also $<f(a_n)>$ does not converge to $f(a)$ *i.e.,*

$$\lim_{n \to \infty} f(a_n) \neq f(a)$$

which is a contradiction.

Hence, $f$ must be continuous at $x=a$.

**THEOREM 4.** *A function $f: R \to R$ is continuous on R iff for each open set $A \subset R$, $f^{-1}(A)$ is an open set in R.*

**Proof. (i)** **Necessary Condition.** Let us first suppose $f$ be continuous on R and let $A \subset R$ be open. To show $f^{-1}(A)$ is open. Let $f^{-1}(A)=\phi$, then $f^{-1}(A)$ is open.

($\because \phi$ is an open set)

If $f^{-1}(A) \neq \phi$, let $a \in f^{-1}(A)$, then $f(a) \in A$. Since $A$ is an open subset of $R$ containing

$f(a)$, $\exists\ \delta>0$ such that

$$]f(a)-\varepsilon, f(a)+\varepsilon[\ \subseteq A$$

Now, $f$ is continuous at $x=a$, $\exists\ \delta>0$ such that

$$|f(x)-f(a)|<\varepsilon, \text{ whenever } |x-a|<\delta$$

or $\quad x\in\,]a-\delta, a+\delta[\ \Rightarrow\ f(x)\in\,]f(a)-\varepsilon, f(a)+\varepsilon[$

$\Rightarrow \quad f(a-\delta, a+\delta) \subset\ ]f(a)-\varepsilon, f(a)+\varepsilon[$

$\Rightarrow \quad ]a-\delta, a+\delta[\ \subset f^{-1}]f(a)-\varepsilon, f(a)+\varepsilon[\ \subseteq f^{-1}(A).$

Thus for each $a\in f^{-1}(A)$ $\exists\ \delta>0$ such that $]a-\delta, a+\delta[\subset f^{-1}(A)$

$\Rightarrow \quad f^{-1}(A)$ is open.

**(ii) Condition is sufficient.** Suppose for each open set $A$ in R, $f^{-1}(A)$ is open. To show $f$ is continuous on R.

Let $\quad a\in \text{R} \Rightarrow f(a)\in \text{R}.$

For $\varepsilon>0$, $]f(a-\varepsilon), f(a+\varepsilon)[$ is an open interval and therefore an open set in R. Then, by our assumption $f^{-1}\,]f(a)-\varepsilon, f(a)+\varepsilon[$ is an open set containing $a$.

$\Rightarrow \quad \exists\ \delta>0$ such that $]a-\delta, a+\delta\,[\ \subset f^{-1}\{]f(a)-\varepsilon, f(a)+\varepsilon[\}$

or $\quad f(a-\delta, a+\delta)\subset\ ]f(a)-\varepsilon, f(a)+\varepsilon[.$

Hence, for a given $\varepsilon>0$ $\exists$ a $\delta>0$ such that $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$

$\Rightarrow \quad f$ is continuous at $a$.

Since, $a$ is arbitrary. Hence, $f$ is continuous on R.

**THEOREM 5.** *A function $f: \text{R}\to\text{R}$ is continuous on R iff for every closed set $B$ in R, $f^{-1}(B)$ is closed in R.*

**Proof.** Let us first suppose $f$ is continuous on $B$, where $B$ is a closed subset of R. To show $f^{-1}(B)$ is closed in R.

Since $B$ is closed $\quad \Rightarrow \text{R}-B$ is open.

$\Rightarrow \quad f^{-1}(\text{R}-B)$ is open and $f^{-1}(\text{R}-B)=\text{R}-f^{-1}(B)$

Therefore, we have $\text{R}-f^{-1}(B)$ is an open set in R.

$\Rightarrow \quad f^{-1}(B)$ is a closed set in R.

Conversely, let $f^{-1}(B)$ be closed in R for every closed set $B$ in R. To show, $f$ is continuous.

Now let $A$ be an open set in R

$\Rightarrow \quad \text{R}-A$ is closed $\Rightarrow\ f^{-1}(\text{R}-A)$ is closed

$\Rightarrow \quad \text{R}-f^{-1}(A)$ is closed $\Rightarrow f^{-1}(A)$ is open.

Hence, $f$ is continuous.

**THEOREM 6.** *Let $f$ be a function defined on an interval $I_1$, $a\in I_1$ and let $g$ be a function defined on an interval $I_2$ such that $f(I_1)\subseteq I_2$. If $f$ is continuous at $a$ and $g$ be continuous at $f(a)$, then composite function $g\circ f$ is continuous at $a$.*

**Proof.** Since, $f$ is continuous at $a\in I_1$

$$\Rightarrow \quad \lim_{x\to a} f(x) = f(a)$$

Also, $g$ is continuous at $f(a)\in I_2$ $\quad \Rightarrow \lim\limits_{x\to f(a)} g(y)=g[f(a)]$

By definition, $(g\circ f)(x)=g[f(x)]\,x\in I_1 \quad \Rightarrow \quad (g\circ f)(a)=g[f(a)]$

Now, suppose the sequence $<a_n>$ in $I_1$ converges to $a$.

Since, $\quad \lim\limits_{x\to a} f(x)=f(a) \quad \Rightarrow \quad \lim\limits_{x\to a} f(a_n)=f(a)$

Also $f(I_1)\subseteq I_2$ and $<f(a_n)>$ is a sequence in $I_2$, and

$\lim\limits_{y\to f(a)} g(y)=g[f(a)]$. Therefore $\lim\limits_{n\to\infty} g[f(a_n)]=g[f(a)]$

Notes

$$\Rightarrow \qquad \lim_{n\to\infty} (g \circ f)(a_n) = (g \circ f)(a).$$

Since, this is true for every sequence $<a_n>$ in $I_1$ converging to $a$, therefore,

$$\lim_{n\to\infty} (g \circ f)(x) = (g \circ f)(a).$$

Hence, the composite function $g \circ f$ is continuous at $a$.

## REMARKS

- **Borel's theorem**. If $f$ is continuous function on the closed interval $[a,b]$, then the interval can always be divided up into a finite number of subintervals such that $\varepsilon > 0$. $|f(x_1) - f(x_2)| < \varepsilon$ where, $x_1$ and $x_2$ are any two points in the same subinterval.

**THEOREM 7. (Boundedness theorem).** *If a function f is continuous in a closed interval $[a,b]$, then it is bounded in $[a,b]$.*

**Proof.** Let if possible $f$ be unbounded on $I$. Then for each $n \in \mathrm{N}$ $\exists\, x_n \in I$ such that $|f(x_n)| > n$. The bounded sequence $<x_n>$ in $I$ has a subsequence $\langle x_{n_k} \rangle$ such that it converges to a point $x_0 \in I$

($\because$ every subsequence of a convergent sequence is convergent.)

$$\Rightarrow \qquad \langle x_{n_k} \rangle \to x_0 \text{ and } \left|f\left(x_{n_k}\right)\right| > n_k \ \forall\ n_k \in \mathrm{N}$$

$$\Rightarrow \qquad \left\langle f\left(x_{n_k}\right)\right\rangle \text{ cannot converge to } f(x_0).$$

$\Rightarrow$ $f$ is not continuous at $x_0$ which is a contradiction.

This contradiction leads to the result that $f$ is bounded on $I$.

## REMARK

- The converse of the above theorem need not be true. For example, the function

$$f(x) = \begin{cases} \sin\dfrac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is bounded on $[0,1]$ but not continuous in $[0,1]$. ($\because$ It is discontinuous at $x=0$)

**THEOREM 8.** *If a function f is continuous on a closed and bounded interval $[a,b]$, then, it attains its bounds on $[a,b]$.*

**Proof.** Since, the function $f$ is continuous on the closed and bounded interval $[a,b]$, therefore, it is bounded.

$\Rightarrow$ supremum $M$ and infimum $m$ of $f$ exist in $[a,b]$.

To show, there exist two points $x_1, x_2 \in [a,b]$ such that $f(x_1) = m, f(x_2) = M$

Then, by definition of supremum $f(x) \le M \quad \forall x \in [a,b]$.

Let if possible $f(x) \neq M$ for any $x \in [a,b]$, then $f(x) < M\ \forall x \in [a,b]$. Therefore,

$$M - f(x) > 0\ \forall x \in [a,b].$$

Since, $f(x)$ is continuous on $[a,b]$ and $M$ is constant, therefore $M - f(x)$ is continuous on $[a,b]$.

Also $M - f(x) \neq 0$ for any $x \in [a,b]$

$$\Rightarrow \qquad \frac{1}{M - f(x)} \text{ is continuous on } [a, b]$$

$$\Rightarrow \qquad \frac{1}{M - f(x)} \text{ is bounded on } [a, b]$$

$\Rightarrow$ $\exists$ a number $k > 0$ such that

$$\frac{1}{M - f(x)} \le k \ \ \forall\, x \in [a,b]$$

$\Rightarrow \quad M-f(x) \geq \frac{1}{k} \ \forall x \in [a,b]$

$\Rightarrow \quad f(x) \leq M - \frac{1}{k} \ \forall x \in [a,b]$

$\Rightarrow M - \frac{1}{k}$ is an upper bound if $f$ on $[a,b]$ such that $M - \frac{1}{k} < M = \sup f(x)$ which is a contradiction

$\Rightarrow \quad \exists$ a point $x_2 \in [a,b]$ such that $M = f(x_2)$.

Similarly, we can show that if $m = \inf f(x)$ $\exists$ a point $x_1$ such that

$$m = f(x_1)$$

**THEOREM 9.** *If a function $f(x)$ is continuous at $x = a$ and $f(a) \neq 0$ then $\exists$ a number $\delta > 0$ such that $f(x)$ has same sign as $f(a)$ for all values of $x$ in $]a-\delta, a+\delta[$.*

**Proof.** Since, $f$ is continuous at $x = a$, for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x-a| < \delta$$

$\Rightarrow \quad f(a) - \varepsilon < f(x) < f(a) + \varepsilon$ whenever $a - \delta < x < a + \delta$.

Now $f(a) \neq 0 \Rightarrow \quad |f(a)| > 0$. Let us choose $0 < \varepsilon < |f(a)|$, then we have $f(a) - \varepsilon$ and $f(a) + \varepsilon$ having the same sign as $f(a)$

$\Rightarrow \quad f(x)$ has the same sign as $f(a)$ for all $x$ in the interval $]a-\delta, a+\delta[$.

**THEOREM 10.** *If a function $f$ is continuous in $[a,b]$ and $f(a)$, $f(b)$ have opposite signs, then there is at least one value of $x$ for which $f(x)$ vanishes.*

**Proof.** Since, the function $f(x)$ have opposite signs for $a$ and $b$ *i.e.*, $f(a) < 0$ and $f(b) > 0$.

Let us define $\quad S = [x : x \in [a, b], f(x) < 0]$.

Now, since $f(a) < 0$, therefore $a \in S \quad \Rightarrow \quad S \neq \phi$.

Let $\quad u = \sup S$.

Now, to show $a < u < b$ and $f(u) = 0$.

First, we shall show that $u \neq a$. Since $f(a) < 0$ and $f$ is continuous at $a$,

$\Rightarrow \quad \exists$ a number $\delta_1$ such that $f(x) < 0 \ \forall x \in ]a, a+\delta_1[$.

$\Rightarrow \quad [a, a+\delta_1] \subset S$

$\Rightarrow \quad \sup S$ must be greater than or equal to $a + \delta_1$. Therefore, $u \geq a + \delta_1 \Rightarrow u \neq a$.

Now, to show $u \neq b$

Since $f(b) > 0 \Rightarrow \exists \ \delta_2$ such that $f(x) > 0 \ \forall x \in [b - \delta_2, b]$

$\Rightarrow \quad ]b-\delta_2, b[ \subset S$

$\Rightarrow \quad u = \sup S \leq b - \delta_2 < b \quad \Rightarrow \quad u \neq b$

Now, we shall show that $f(u) \not> 0$. Since $a < u < b$. Therefore, if $f(u) > 0$. Then we can find a number $\delta_3 > 0$ such that $f(x) > 0$ for $u - \delta_3 < x < u + \delta_3$.

Also, $u = \sup S$. Therefore, $\exists \ x_1 \in S : u - \delta_3 < x_1 < u \Rightarrow \quad f(x) > 0$.

Also $\quad x_1 \in S \Rightarrow \quad f(x_1) < 0$ ; which is a contradiction

$\Rightarrow \quad f(u) \not> 0$.

Now, we shall show that $f(u) \not< 0$. If $f(u) < 0$, then we can find a positive number $\delta_4$ such that

$$u + \delta_4 < b \text{ and } f(x) < 0 \text{ for } u - \delta_4 < x < u + \delta_4.$$

If $x_2$ is any other point such that $u < x_2 < u + \delta_4$. Then $f(x_2) < 0$. But this is a contradiction to the fact that $u$ is the supremum of $S$ consequently $f(u) \not< 0$.

Hence, $\quad f(u) = 0$.

Notes

**THEOREM 11. (Intermediate value theorem).** *Let f be a function continuous on the closed and bounded interval* $[a,b]$.*If k be any real number between* $f(a)$ *and* $f(b)$, *then there exist a real number c between a and b* $(a<c<b)$ *such that* $f(c)=k$

**Proof.** Let us suppose

$$f(a)<k<f(b). \quad ...(1)$$

Define a fucntion $g$ such that

$$g(x)=f(x)-k;\ x\in[a,b]. \quad ...(2)$$

Now, since $f$ is continuous on $[a,b]$ and $k$ is constant, $g$ is continuous on $[a,b]$. ...(3)

From (1), we say that $k$ lies between $f(a)$ and $f(b)$. Therefore, either

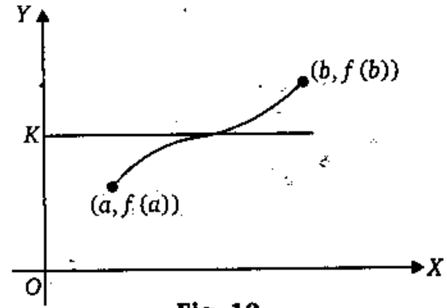


**Fig. 10**

$$f(a)<k<f(b) \text{ or } f(b)<k<f(a).$$

From (2), $\quad g(a)=f(a)-k<0 \quad \Rightarrow \quad g(b)=f(b)-k>0$

$\Rightarrow \quad g(a).\,g(b)<0 \quad ...(4)$

Now, from (3) and (4) there exists a point $c\in]a,b[$ such that $g(c)=0$

$\Rightarrow \quad f(c)-k=0 \quad \Rightarrow \quad f(c)=k$

Hence, these exist a point $c$ such that $a<c<b$ and $f(c)=k$.

## REMARKS

- The above theorem can be restated as:
  *If a function f is continuous in the closed interval* $[a,b]$, *then* $f(x)$ *must take at least once of all values between* $f(a)$ *and* $f(b)$.
- This theorem guarantees only the existence of the number $c$. It does not tell us how to find it.Also the number $c$ need not be unique.
- If $f$ is continuous on $[a,b]$ and let $k\in[m,M]$ where $m=\inf. f$ and $M=\sup. f$ on $[a,b]$ then there exists $c\in[a,b]$ such that $f(c)=k$.
- If $f$ is continuous on $[a,b]$, then $f([a,b])=[m, M]$.Also, $f([a,b])$ is a closed set.
- If $f$ is a continuous, one to one function on a finite closed interval $[a, b]$,then $f$ is also continuous on its domain.

## 1.13 UNIFORM CONTINUITY

We know that if a function $f(x)$ is continuous in the closed interval $I$, then for a given positive number $\varepsilon$, $\exists$ a positive number $\delta>0$ such that

$$|f(x)-f(a)|<\varepsilon \text{ for } |x-a|<\delta, a\in I.$$

Here, we observe that the number $\delta$ depends on, besides $\varepsilon$, on the point $a$ as it is a function of $a$. In general, $\delta$ is different at different points in $I$.

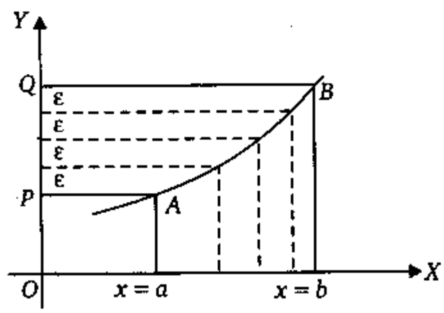


**Fig. 11**

For this, let us consider the figure, where $PQ$ is divided into equal parts, each of length $\varepsilon$.

The corresponding subdivision of $I=[a,b]$ is such that $\delta$ is not the same for all points $x$ in $[a,b]$.

Therefore, if we can find a positive number $\delta_0$ such that for a chosen $\varepsilon$, $|f(x)-f(a)|<\varepsilon$ for $|x-a|<\delta_0$ where the number $\delta_0$ is independent of the point $a$, then the function $f(x)$ is said to be uniformly continuous on $[a,b]$.

**Definition.** *A function* $f(x)$ *defined on an interval I is said to be uniformly continuous in*

*I if to each* $\varepsilon>0$ $\exists$ *a positive number* $\delta>0$, *(depending upon* $\varepsilon$*) but independent of* $x\in I$ *such that*

$$|f(x_2)-f(x_1)|<\varepsilon \text{ whenever } |x_2-x_1|<\delta$$

*where* $x_1, x_2 \in I$.

**REMARK**

- A function $f$ is not uniformly continuous on $I$, if there exist some $\varepsilon>0$ for which no $\delta>0$ works *i.e.*, for any $\delta>0$ $\exists$ $x_1,x_2 \in I$ such that $|f(x_2)-f(x_1)|\geq\varepsilon$ for $|x_2-x_1|<\delta$.
- The uniform continuity of $f$ on an arbitrary set $S$ can be defined by replacing the interval $I$ by $S$ in the above definition.

**THEOREM 1.** *If a function f is uniformly continuous on an interval I, then it is continuous on I.*

**Proof.** Let us suppose that $f$ is uniformly continuous on $I$

$\Rightarrow$ given $\varepsilon>0$ $\exists$ $\delta>0$ such that

$$|f(x_2)-f(x_1)|<\varepsilon, \text{ whenever } |x_2-x_1|<\delta \ \forall x_1, x_2\in I$$

In particular, let us take $x_2\in I$, then we have

$$|f(x)-f(x_1)|<\varepsilon, \text{ whenever } 0<|x-x_1|<\delta$$

$\Rightarrow f(x)$ is continuous at $x_1\in I$.

Since, $x_1$ is arbitrary, consequently $f(x)$ is continuous on $I$.

**REMARKS**

- The converse of the above theorem is not true as can be seen in the example, given below: Consider the function $f(x)=x^2$ $\forall x\in R$ which is continuous for all $x\in R$ but not uniformly continuous.
- The uniform continuity is a property associated with an interval and not with a single point *i.e.*, the concept of continuity is local in character, while the uniform continuity is global in character.

**THEOREM 2.** *If a function f(x) is continuous on an closed and bounded interval* $I=[a,b]$, *then it is uniformly continuous on* $[a,b]$.

**Proof.** Since $f$ is given to be continuous in the interval $[a,b]$.

Let $\varepsilon > 0$ be given $\Rightarrow [a,b]$ can be divided into a finite number of subintervals such that $|f(x_2)-f(x_1)|<\frac{\varepsilon}{2}$, where $x_1, x_2$ are any two points of the same subinterval.

Let us divide the whole interval $[a,b]$ into $n$ sub intervals, say

$$[x_0=a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n=b]$$

$$\Rightarrow \quad |f(x')-f(x'')|<\frac{\varepsilon}{2}, \text{ where } x', x'' \text{ belongs to the same subinterval} \qquad \ldots(1)$$

Let $\delta=\min\{\delta_1, \delta_2 \ldots\delta_r, \ldots\delta_n\}$ where $\delta_r$ denotes the length of the $r^{th}$ subinterval *i.e.*,

$$\delta_r=|x_r-x_{r-1}|$$

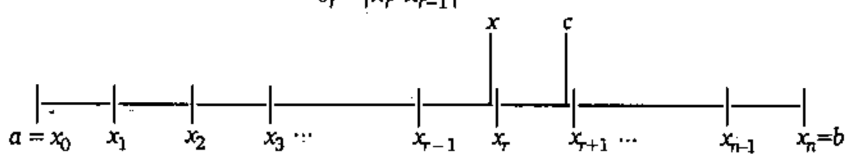


**Fig. 12**

Let $x$ and $c$ be any two points of $[a,b]$ such that $|x-c|<\delta$.

Since $\delta>0$, less than the length of each subinterval. Therefore, following two cases may arise:

Notes

Case (i) When $x$ and $c$ belongs to same interval:

$\Rightarrow$ $$|f(x)-f(c)| < \frac{\varepsilon}{2}, \text{ when } |x-c|<\delta; \text{ where } x, c \in [a, b]$$

$\Rightarrow$ function $f$ is uniformly continuous in $[a,b]$.

Case (ii) When $x$ and $c$ belongs to the two consecutive sub intervals say

$$x_{r-1}<x<x_r<c<x_{r+1}.$$

Consider
$$|f(x)-f(c)| = |f(x)-f(x_r)+f(x_r)-f(c)|$$
$$\leq |f(x)-f(x_r)| + |f(x_r)-f(c)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ when } |x-c|<\delta<\varepsilon \text{ when } |x-c|<\delta.$$

$\therefore$ Given $\varepsilon>0 \ \exists \ \delta>0$ such that $|f(x) - f(c)|<\varepsilon$ where $x$ and $c$ are any two points of $[a, b]$ such that $|x-c|<\delta$

$\Rightarrow$ $f$ is uniformly continuous on $[a,b]$.

Hence, $f$ is continuous on a closed and bounded interval $[a, b]$

$\Rightarrow$ $f$ is uniformly continuous on $[a, b]$.

## Solved Examples

**Example 1.** *Show that the function $f(x) = x^2+3x$, $x\in[-1,1]$ is uniformly continuous in* $[-1,1]$.

**Solution.** Let $\varepsilon>0$ be given

Let $x_1, x_2\in[-1,1]$

$\Rightarrow |f(x_2)-f(x_1)| = |(x_2^2+3x_2)-(x_1^2+3x_1)| = |(x_2^2-x_1^2)+3(x_2-x_1)|$

$= |(x_2-x_1)(x_2+x_1+3)| = |x_2-x_1||x_2+x_1+3|$

$\leq |x_2-x_1|(|x_2|+|x_1|+3) \leq 5|x_2-x_1|$ $\quad (x_1, x_2\in[-1,1])$

$\Rightarrow |x_1|\leq 1$ and $|x_2|\leq 1$

$\Rightarrow$ $|f(x_2)-f(x_1)|<\varepsilon$ for $|x_2-x_1|<\dfrac{\varepsilon}{5}$

Thus for any $\varepsilon<0$, $\exists \ \delta=\dfrac{\varepsilon}{5}>0$ such that

$$|f(x_2) - f(x_1)|<\varepsilon \text{ whenever } |x_2-x_1|<\delta \ \forall \ x_1, x_2\in[-1,1].$$

Hence, $f(x)$ is uniformly continuous in $[-1, 1]$.

**Example 2.** *Show that the function $f$ defined by $f(x)=x^3$ is uniformly continuous on* $[-2,2]$.

**Solution.** In order to show that the function $f$ is uniformly continuous we have to prove that for a given $\varepsilon>0 \ \exists \ \delta>0$ such that

$$|f(x_2)-f(x_1)|<\varepsilon \text{ when } 0<|x_2-x_1|<\delta \text{ where } x_1, x_2\in[-2,2].$$

Consider $|f(x_2)-f(x_1)| = |x_2^3-x_1^3|$

$= |(x_2-x_1)(x_2^2+x_1^2+x_1x_2)|$

$\leq |(x_2-x_1)|[|x_1^2|+|x_2^2|+|x_1x_2|]$

$\leq 12|x_2-x_1|$ $(\because x_1, x_2\in[-2,2] \Rightarrow |x_1|\leq 2$ and $|x_2|\leq 2)$

$\therefore$ $|f(x_2)-f(x_1)|<\varepsilon$ whenever $|x_2-x_1|<\dfrac{\varepsilon}{12}$.

Therefore, given $\varepsilon>0 \ \exists \ \delta(=\varepsilon/12)$ such that

$$|f(x_2) - f(x_1)|<\varepsilon \text{ whenever } |x_2-x_1|<\delta, x_1, x_2\in[-2, 2].$$

Hence, $f$ is uniformly continuous on $[-2, 2]$.

**Example 3.** *Show that the function $f$ defined by $f(x)=\dfrac{1}{x}, \forall x\in]0, 1]$ is not uniformly continuous in* $]0, 1]$.

Notes

**Solution.** In order to show that the function $f$ is uniformly continuous in $]0,1]$ we have to prove that for a given $\varepsilon>0 \; \exists \; \delta>0$, independent of the choice of $x$, $x\in]0,1]$ such that

$$|f(x)-f(c)|=\left|\frac{1}{x}-\frac{1}{c}\right| <\varepsilon \text{ whenever } 0<|x-c|<\delta$$

*i.e.,* $$|x-c|<\delta \Rightarrow \left|\frac{c-x}{cx}\right|<\varepsilon$$

*i.e.,* $$x\in]c-\delta,c+\delta[ \Rightarrow \left|\frac{c-x}{cx}\right|<\varepsilon \qquad \ldots(1)$$

Let us take $c=\delta$, then $]c-\delta, c+\delta[ = ]0, 2\delta[$.

Since, the condition (1) must hold for all $x\in]0,2\delta[$

$\therefore$ $$\text{as } x\to 0, \frac{\delta-x}{\delta x}\to\infty \text{ and } x\in]0, 2\delta[$$

*i.e.*, if we choose $x$ close to zero, then condition (1) does not hold.

$\Rightarrow$ $f(x)=\frac{1}{x}$ is not uniformly continuous in $]0,1]$.

**Example 4.** *Show that the function f defined on* $R^+$ *as*

$$f(x)=\sin\frac{1}{x}, \; \forall \; x>0$$

*is continuous, but not uniformly continuous on* $R^+$.

**Solution.** Let $a\in R^+$.

We have $\text{LHL}=f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\sin\frac{1}{a-h}=\sin\frac{1}{a}$

$$\text{RHL}=f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\sin\frac{1}{a+h}=\sin\frac{1}{a}$$

$\Rightarrow$ $$f(a)=\sin\frac{1}{a}$$

$\Rightarrow$ $$f(a+0)=f(a)=f(a-0)$$

$\Rightarrow$ $f$ is continuous at $a$.

Since, $a$ is arbitrary point in $R^+$. Therefore, $f$ is continuous on $R^+$.

Now, to show $f$ is not uniformly continuous on $R^+$.

Let $\delta$ be any positive number. Take

$$x_1=\frac{1}{n\pi}, x_2=\frac{1}{n\pi+\pi/2}=\frac{2}{(2n+1)(\pi)} \text{ where } n\in Z^+$$

such that $x_1-x_2=\frac{1}{n\pi}-\frac{2}{(2n+1)\pi}<\delta$

Now, $|x_1-x_2|<\delta$ but $|f(x_1)-f(x_2)|=\left|\sin n\pi-\sin\frac{1}{2}(2n+1)\pi\right|=1>\varepsilon$

which shows that for this choice of $\varepsilon$, we cannot find a $\delta>0$ such that

$$|f(x_1)-f(x_2)|<\varepsilon \text{ for } |x_1-x_2|<\delta \;\; \forall x_1, x_2 \in R^+$$

Hence, $f$ is not uniformly continuous on $R^+$.

## STUDENT ACTIVITY

**1.** Show that $f(x)=\sqrt{x}$ is uniformly continuous in $[0,1]$.

Notes

2. Show that $f(x)=\sin x^2$ is not uniformly continuous on $[0,\infty[$.

## TEST YOURSELF

1. Let $f: R\to R$ given $f(x)=x^2$. Show that $f$ is not uniformly continuous on R.
2. Show that the function $x^2$ and $x^3$ are not uniformly continuous on $[0,\infty[$.
3. In each of the following cases, show that $f$ is continuous but not uniformly continuous on their respective intervals.

(i) $f(x)=\sin\frac{1}{x}\ \forall x\in]0,1[$ (ii) $f(x)=\frac{1}{2x}\ \forall x\in[-1,0[$

(iii) $f(x)=\frac{1}{1-x}\ \forall x\in]0,1[$ (iv) $f(x)=e^x\ \forall x\in[0,\infty[$

4. If $f(x+y)=f(x).f(y)\ \forall x,y\in R$, show that $f$ is continuous on R if and only if $f$ is continuous at least one point of R. If $f$ is continuous at some point $a\in R$, prove that $f$ is uniformly continuous on every bounded subset of R.
5. Show that the function $f$ defined by $f(x)=\begin{cases}x^2\sin\frac{1}{x^2} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$ is uniformly continuous in $[-1,1]$.
6. Show that if $f$ and $g$ are bounded and uniformly continuous on an interval $I$, then the product function $fg$ is also uniformly continuous on $I$.

## Summary

- If $f$ and $g$ are two continuous functions at $x=a$ then $f+g, f-g, fg, \frac{f}{g}\ (g\neq 0)$ are also continuous at $x=a$.
- If $f$ is continuous at $x=a$ then $|f|$ is also continuous at $x=a$. Converse is not necessarily true.
- Composite of two continuous functions is a continuous function.
- Every continuous function is bounded. Converse is not true.
- If a function $f$ is continuous on $[a, b]$ and $c\in]a, b[$ such that $f(c)\neq 0$ then there exists some $\delta>0$ such that $f(x)$ has the same sign as $f(c)\ \forall x\in]c-\delta, c+\delta[$.
- If a function $f$ is continuous on $[a, b]$ then
  (i) $f(a)>0\Rightarrow\exists\,\delta>0$ such that $f(x)>0\quad\forall x\in[a, a+\delta[$
  (ii) $f(a)<0\Rightarrow\exists\,\delta>0$ such that $f(x)<0\quad\forall\in[a, a+\delta[$
  (iii) $f(b)>0\Rightarrow\exists\,\delta>0$ such that $f(x)>0\quad\forall x\in]b-\delta, b]$
  (iv) $f(b)<0\Rightarrow\exists\,\delta>0$ such that $f(x)<0\quad\forall x\in]b-\delta, b]$.
- A function $f: R\to R$ is continuous on R if and only if for every open set $A$ in R, $f^{-1}(A)$ is open in R.
- A function $f: R\to R$ is continuous on R if and only if for each closed set $A$ in R, $f^{-1}(A)$ is closed in R.
- If a function $f$ is continuous on a closed interval $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite sign that there exists at least one point $c\in]a, b[$ such that $f(c)=0$
- If a function $f$ is continuous on a closed interval $[a, b]$ and $f(a)\neq f(b)$ then $f$ assume every value between $f(a)$ and $f(b)$.
- A function $f$ which is continuous on a closed interval $[a, b]$ assumes every value between its bounds.

Notes

- Every uniformly continuous function is continuous.
- If a function $f$ is continuous on a closed and bounded interval $[a, b]$ then it is uniformly continuous on $[a, b]$.

## Objective Evaluation

### FILL IN THE BLANKS

1. Limit of a function, if exist is ________.
2. If $\lim f(x) = l$ then $\lim \frac{1}{f(x)} = \frac{1}{l}$ provided ________.
3. The limit of the quotient is equal to the ________ of the limits.
4. A function $f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) =$ ________.
5. $\lim_{x \to 0} (1+x)^{1/x} =$ ________.
6. $\lim_{x \to a} \frac{x^m - a^m}{x - a} =$ ________
7. In the definition of continuity, the value of $\delta$ depends upon the value of ________.
8. A polynomial function is always ________.
9. $\lim_{x \to 0} \frac{a^x - 1}{x} =$ ________
10. A function is said is have ________ if $f(a+0) = f(a-0) \neq f(a)$.

### TRUE/ FALSE

**Write 'T' for true and 'F' for false statement.**

1. Every continuous function in closed interval is bounded. **(T/F)**
2. Every continuous function in open interval is bounded. **(T/F)**
3. For $\lim_{x \to a} f(x)$ to exist, the function $f(x)$ must be defined at $x = a$. **(T/F)**
4. The limit of a products is equal to the product of the limits. **(T/F)**
5. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{3}{2}$. **(T/F)**
6. For a function $f(x)$ to be continuous at $x = a$, it is necessary that $\lim_{x \to a} f(x)$ must exist. **(T/F)**
7. The function must be defined at the point of continuity. **(T/F)**
8. If a function having a finite number of jumps in a given interval then function is called piecewise continuous. **(T/F)**
9. Sum of two continuous functions is not necessarily continuous. **(T/F)**
10. If a function $f$ is continuous in the closed interval $[a, b]$, then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$. **(T/F)**

### MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one**

1. If $\lim_{x \to a} f(x) = l$ and $f(x) \geq 0$, then
   (a) $l=0$ (b) $l \leq 0$ (c) $l \geq 0$ (d) none of these
2. If $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| =$:
   (a) $l$ (b) $|l|$ (c) 0 (d) 1
3. If $\lim_{x \to \infty} f(x) = l$ and $\lim_{x \to \infty} g(x)$ does not exist, then:
   (a) $\lim_{x \to \infty} f(x).\, g(x)$ does not exist
   (b) $\lim_{x \to \infty} f(x).\, g(x)$ exist necessarily
   (c) $\lim_{x \to \infty} f(x).\, g(x)$ may or may not exist
   (d) none of these

Notes

4. $\lim_{x \to 2} \frac{|x-2|}{x-2} =$ :

(a) 0 (b) 1 (c) 2 (d) does not exist

5. The value of $\lim_{x \to 0} \frac{\sin x}{x}$ is:

(a) 1 (b) 0 (c) $\infty$ (d) does not exist.

6. The value of $\lim_{x \to \infty} \frac{\sin x}{x}$ is:

(a) 1 (b) 0 (c) $\infty$ (d) 2

7. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, then $\lim_{x \to a} [f(x)+g(x)]$ :

(a) does not exist (b) necessarily exist
(c) may or may not exist (d) none of the above

8. The equation $\lim_{x \to 0} f(x) = \lim_{x \to a} f(x-a)$ is:

(a) always true (b) may or may not be true
(c) always false (d) depend upon the value of $a$.

9. The value of $k$ for which $f(x) = \begin{cases} \frac{\sin 5x}{3x} & , \text{if } x \neq 0 \\ k & , \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ is:

(a) $\frac{1}{3}$ (b) $\frac{3}{5}$ (c) 0 (d) $\frac{5}{3}$

10. Let $f(x) = \begin{cases} \lambda x^2 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$, then the value of $\lambda$ is:

(a) 2 (b) 3 (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

## ANSWERS

### FILL IN THE BLANKS

1. Unique 2. $l \neq 0$ 3. quotient 4. $f(a)$
5. $e$ 6. $ma^{m-1}$ 7. $\varepsilon$ 8. continuous
9. $\log_e a$ 10. removable discontinuity

### TRUE OR FALSE

1. T 2. F 3. F 4. T 5. T 6. T 7. T
8. T 9. F 10. T

### MULTIPLE CHOICE QUESTIONS

1. (c) 2. (b) 3. (c) 4. (d) 5. (a) 6. (b) 7. (c)
8. (a) 9. (d) 10. (d)

□□□□

Chapter

# 2 Successive Differentiations

## STRUCTURE

- Inroduction
- $n^{th}$ differentiation of some Standard functions
- Use of partial fractions
- Leibnitz's theorem
- Determination of the value of derivative of a function at x = 0
  - Summary
  - Objective Evaluation

## LEARNING OBJECTIVES

After going through this unit you will learn:

- How to differentiate the given functions upto finite number of times
- Leibnitz's rule which is applicable for the product of two or more functions

## 2.1 INTRODUCTION

Let $y = f(x)$ be a function, then the differential coefficient of $f(x)$ denoted by $f'(x)$ is defined as follows

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{dy}{dx}$$

If the limit exists (*i.e.*, limit is finite and unique), then $f'(x)$ is called *first differential coefficient of* $f(x)$ *with respect to* $x$. Similarly, if $f(x)$ is differentiable twice, it is denoted by $f''(x)$, if it is differentiable thrice, it is denoted by $f'''(x)$, *i.e.*,

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

If $y = f(x)$ be a function of $x$, then we adopt the following notations.

$$y_1 = f'(x) = \frac{dy}{dx} = Df(x) = \frac{d}{dx}(f(x))$$

$$y_2 = f''(x) = \frac{d^2y}{dx^2} = D^2f(x) = \frac{d^2}{dx^2}(f(x))$$

$$y_3 = f'''(x) = \frac{d^3y}{dx^3} = D^3f(x) = \frac{d^3}{dx^3}(f(x))$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_n = f^n(x) = \frac{d^ny}{dx^n} = D^nf(x) = \frac{d^n}{dx^n}(f(x))$$

**Definition.** *This process of finding the differential coefficients of a function is called successive differentiation.*

Notes

## 2.2 $n^{th}$ DIFFERENTIATION OF SOME STANDARD FUNCTIONS

**(i) $y = f(x) = x^n$.**

We have $y = f(x) = x^n$

$$y_1 = f'(x) = nx^{n-1}$$

$$y_2 = f''(x) = n(n-1)x^{n-2}$$

$$y_3 = f'''(x) = n(n-1)(n-2)x^{n-3}$$

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$$y_n = f^n(x) = n(n-1)(n-2)\ldots 3.2.1.x^0$$

$$\Rightarrow \quad \frac{d^n}{dx^n}(x^n) = y_n = n!$$

**(ii) $y = f(x) = x^m$.**

We have $y_1 = f'(x) = mx^{m-1}$, $y_2 = f''(x) = m(m-1)x^{m-2}$,

$$y_3 = f'''(x) = m(m-1)(m-2)x^{m-3}, \ldots,$$

$$y_n = f^n(x) = m(m-1)(m-2)\ldots(m-n+1).x^{m-n}$$

$$= \left[\frac{m(m-1)(m-2)\ldots(m-n+1)(m-n)\ldots 3.2.1}{(m-n)(m-n-1)\ldots 3.2.1}\right]x^{m-n}$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$$

**(iii) $y = f(x) = \frac{1}{(ax+b)}$.**

We have $y_1 = f'(x) = -\frac{a}{(ax+b)^2}$, $\quad y_2 = f''(x) = \frac{a^2.2}{(ax+b)^3}$,

$$y_3 = f'''(x) = -\frac{a^3.2.3}{(ax+b)^4}, \ldots\ldots, y_n = f^n(x) = \frac{(-1)^n a^n.2.3.4\ldots n}{(ax+b)^{n+1}}$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n.a^n.n!}{(ax+b)^{n+1}}$$

**(iv) $y = f(x) = \frac{1}{(ax+b)^m}$.**

We have $y_1 = f'(x) = -\frac{a.m}{(ax+b)^{m+1}}$; $\quad y_2 = f''(x) = \frac{a^2.m(m+1)}{(ax+b)^{m+2}}$,

$$y_3 = f'''(x) = -\frac{a^3.m(m+1)(m+2)}{(ax+b)^{m+3}}, \ldots,$$

$$y_n = f^n(x) = (-1)^n\frac{a^n.m(m+1)(m+2)\ldots(m+n-1)}{(ax+b)^{m+n}}$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}\left(\frac{1}{(ax+b)^m}\right) = (-1)^n\frac{a^n.(m+n-1)!}{(m-1)!(ax+b)^{m+n}}$$

**(v) $y = f(x) = \sin(ax+b)$.**

We have $y_1 = f'(x) = a\cos(ax+b) = a\sin\left(\frac{\pi}{2} + ax + b\right)$,

$$y_2 = f''(x) = a^2\cos\left(\frac{\pi}{2} + ax + b\right) = a^2\sin\left(2.\frac{\pi}{2} + ax + b\right)$$

Notes

$$y_3 = f'''(x) = a^3 \cos\left(2.\frac{\pi}{2} + ax + b\right) = a^3 \sin\left(3.\frac{\pi}{2} + ax + b\right)$$

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$$y_n = f^n(x) = a^n \cos\left((n-1)\frac{\pi}{2} + ax + b\right) = a^n \sin\left(n.\frac{\pi}{2} + ax + b\right)$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

**(vi) y = f(x) = cos (ax + b).**

We have $y_1 = f'(x) = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2} + ax + b\right)$,

$$y_2 = f''(x) = -a^2 \sin(\frac{\pi}{2} + ax + b) = a^2 \cos\left(\frac{2\pi}{2} + ax + b\right)$$

$$y_3 = f'''(x) = -a^3 \sin(2.\frac{\pi}{2} + ax + b) = a^3 \cos\left(3.\frac{\pi}{2} + ax + b\right)$$

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$$y_n = f^n(x) = -a^n \sin\left((n-1)\frac{\pi}{2} + ax + b\right) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

**(vii) y = f(x) = $e^{ax+b}$.**

We have $y_1 = f'(x) = a.e^{ax+b}$

$$y_2 = f''(x) = a^2.e^{ax+b}$$

$$y_3 = f'''(x) = a^3.e^{ax+b}$$

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$$y_n = f^n(x) = a^n.e^{ax+b}$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}(e^{ax+b}) = a^n e^{ax+b}$$

**(viii) y = f(x) = log (ax + b).**

We have $y_1 = f'(x) = \dfrac{a}{ax+b}$

Now using result (iii), we get

$$y_n = f^n(x) = (-1)^{n-1} \frac{a^n (n-1)!}{(ax+b)^n}$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[\log(ax+b)] = (-1)^{n-1} \frac{a^n (n-1)!}{(ax+b)^n}$$

**(ix) y = f(x) = $e^{ax}\sin(bx+c)$.**

We have $y_1 = f'(x) = ae^{ax}.\sin(bx+c) + be^{ax}\cos(bx+c)$

$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

Put $a = r\cos\theta, b = r\sin\theta \Rightarrow r^2 = a^2 + b^2$ and $\tan\theta = b/a$ *i.e.*, $\theta = \tan^{-1} b/a$

Therefore, $y_1 = f'(x) = r.e^{ax}\sin(bx + c + \theta)$

$$= (a^2+b^2)^{1/2}.e^{ax} \sin\left(bx + c + \tan^{-1}\frac{b}{a}\right)$$

Notes

Similarly,

$$y_2 = f''(x) = (a^2+b^2)^{1/2}(a^2+b^2)^{1/2}.e^{ax}\sin(bx+c+\tan^{-1}b/a+\tan^{-1}b/a)$$

$$= (a^2+b^2)^{2/2}.e^{ax}\sin(bx+c+2\tan^{-1}b/a)$$

$$y_3 = f'''(x) = (a^2+b^2)^{3/2}.e^{ax}\sin(bx+c+3\tan^{-1}b/a)$$

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$$y_n = f^n(x) = (a^2+b^2)^{n/2}.e^{ax}\sin(bx+c+n\tan^{-1}b/a)$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}.e^{ax}\sin(bx+c+n\tan^{-1}b/a)$$

**(x) $\mathbf{y = f(x) = e^{ax}\cos(bx+c)}$.**

We have $\quad y_1 = f'(x) = ae^{ax}.\cos(bx+c) - be^{ax}\sin(bx+c)$

$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$$

Put $a = r\cos\theta$, $b = r\sin\theta \Rightarrow \theta = \tan^{-1}b/a$ and $r = (a^2+b^2)^{1/2}$

$$\therefore \quad y_1 = f'(x) = r.e^{ax}[\cos\theta\cos(bx+c) - \sin\theta\sin(bx+c)]$$

$$= re^{ax}\cos(bx+c+\theta) = (a^2+b^2)^{1/2}.e^{ax}\cos(bx+c+\tan^{-1}b/a)$$

Similarly, $\quad y_2 = f''(x) = (a^2+b^2)^{2/2}.e^{ax}\cos(bx+c+2\tan^{-1}b/a)$

$$y_3 = f'''(x) = (a^2+b^2)^{3/2}.e^{ax}\cos(bx+c+3\tan^{-1}b/a)$$

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$$y_n = f^n(x) = (a^2+b^2)^{n/2}.e^{ax}\cos(bx+c+n\tan^{-1}b/a)$$

$$\Rightarrow \quad y_n = \frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = (a^2+b^2)^{n/2}.e^{ax}\cos(bx+c+n\tan^{-1}b/a)$$

## Solved Examples

**Example 1.** *Find the $n^{th}$ differential coefficient of* $\tan^{-1}\frac{x}{a}$.

**Solution.** We have $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow \quad y_1 = \frac{a}{x^2+a^2} = \frac{a}{(x+ia)(x-ia)}$$

Let us suppose

$$\frac{a}{(x+ia)(x-ia)} = \frac{A}{(x+ia)} + \frac{B}{(x-ia)} \qquad \text{(Using partial fractions)}$$

$\Rightarrow a = A(x-ia) + B(x+ia)$

To find the value of $A$, put $x = -ia$

We get $\quad A = -\frac{1}{2i}$

and for $B$, put $x = ia$, which gives $B = \frac{1}{2i}$ therefore, we have

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right] = \frac{1}{2i}[(x-ia)^{-1} - (x+ia)^{-1}]$$

Differentiating $(n-1)$ times, we get

$$y_n = \frac{1}{2i}[(-1)^{n-1}(n-1)!(x-ia)^{-n} - (-1)^{n-1}(n-1)!(x+ia)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}[(x-ia)^{-n}-(x+ia)^{-n}]$$

Put $x = r\cos\theta$, $a = r\sin\theta$, we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}.2i\sin n\theta \qquad [\because \sin(-n\theta) = -\sin n\theta]$$

$$= (-1)^{n-1}(n-1)!r^{-n}.\sin n\theta$$

$$= (-1)^{n-1}(n-1)!\left(\frac{a}{\sin\theta}\right)^{-n}\sin n\theta \qquad \left[\text{since } r = \frac{a}{\sin\theta}\right]$$

$$= (-1)^{n-1}(n-1)!a^{-n}\sin^n\theta.\sin n\theta$$

**Example 2.** *Find the $n^{th}$ differential coefficient of* $\log(ax + x^2)$.

**Solution.** Let $y = \log(ax + x^2) = \log[x(a + x)] = \log x + \log(a + x)$

Differentiating $n$ times, we get

$$y_n = \frac{d^n}{dx^n}(\log x) + \frac{d^n}{dx^n}\log(a+x)$$

$$= \frac{(-1)^{n-1}(n-1)!.1^n}{x^n} + \frac{(-1)^{n-1}(n-1)!.1^n}{(x+a)^n} = (-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$$

**Example 3.** *Find the $n^{th}$ differential coefficients of*

(i) $e^{ax}\sin bx\cos cx$

(ii) $e^{2x}\sin^3 x$

**Solution.** (i) Let $y = e^{ax}\sin bx\cos cx$

$$= \frac{1}{2}e^{ax}[2\sin bx\cos cx] = \frac{1}{2}e^{ax}[\sin(bx+cx) + \sin(bx-cx)]$$

$$= \frac{1}{2}[e^{ax}\sin(b+c)x + e^{ax}\sin(b-c)x] \qquad \ldots(1)$$

Differentiating (1) $n$ times, we get

$$\frac{d^n}{dx^n}[y] = y_n = \frac{1}{2}[\{a^2+(b+c)^2\}^{n/2}e^{ax}\sin\{(b+c)x + n\tan^{-1}(b+c)/a\}$$
$$+ \{a^2+(b-c)^2\}^{n/2}e^{ax}\sin\{(b-c)x + n\tan^{-1}(b-c)/a\}]$$

(ii) Let $y = e^{2x}\sin^3 x$.

Now using the result

$\text{Sin}3x = 3\sin x - 4\sin^3 x$

We have

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

Therefore,

$$y = \frac{1}{4}e^{2x}[3\sin x - \sin 3x] = \frac{3}{4}e^{2x}\sin x - \frac{1}{4}e^{2x}\sin 3x.$$

Now, differentiating $n$ times, we get

$$y_n = \frac{3}{4}[(2^2+1^2)^{1/2}]^n e^{2x}\sin[x + n\tan^{-1}1/2]$$
$$- \frac{1}{4}[(2^2+3^2)^{1/2}]^n e^{2x}\sin[3x + n\tan^{-1}3/2].$$

**Example 4.** *Find the $n^{th}$ differential coefficients of* $\sin^5 x\cos^3 x$.

Notes

**Solution.** First we reduce $\sin^5 x \cos^3 x$ into a function consisting sine function of multiple of $x$.

Let $z = \cos x + i \sin x$.

The $z^{-1} = \cos x - i \sin x$

$\therefore z + z^{-1} = 2\cos x$ and $z - z^{-1} = 2i\sin x$

Also, by De-Moivre's theorem, we have

$$z^m + z^{-m} = 2\cos mx$$

and $z^m - z^{-m} = 2i\sin mx$

Now $(2i\sin x)^5 (2\cos x)^3 = (z - z^{-1})^5 + (z + z^{-1})^3$

$\Rightarrow \quad 2^8 i \sin^5 x \cos^3 x = (z^8 - z^{-8}) - 2(z^6 - z^{-6}) - 2(z^4 - z^{-4}) + 6(z^2 - z^{-2})$

$= 2i \sin 8x - 4i \sin 6x - 4i \sin 4x + 12 i \sin 2x$

$\Rightarrow \quad \sin^5 x \cos^3 x = 2^{-7}[\sin 8x - 2\sin 6x - 2\sin 4x + 6\sin 2x]$.

Dfferentiating both sides $n$ times w.r.t. $x$, we get

$$D^n(\sin^5 x \cos^3 x) = 2^{-7}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right) - 2.6^n \sin\left(6x + \frac{n\pi}{2}\right) - 2.4^n \sin\left(4x + \frac{n\pi}{2}\right) + 6.2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

## 2.3 USE OF PARTIAL FRACTIONS

To determine the $n^{th}$ derivative of any rational function, we have to split it into partial fractions.

Partial fractions for

(i) $\dfrac{f(x)}{(x-a)(x-b)(x-c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$

(ii) $\dfrac{f(x)}{(x-a)^2(x-b)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$

(iii) $\dfrac{f(x)}{(x-a)^3(x-b)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3} + \dfrac{D}{(x-b)}$

(iv) $\dfrac{f(x)}{(x-a)(x-b)(px^2+qx+r)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{Cx+D}{(px^2+qx+r)}$

To find $A$, $B$, $C$, $D$ etc., we put each linear factor of LCM equal to zero. The remaining constants are obtained by comparing coefficients of like powers on both sides.

**REMARK**

- Forming partial fractions is converse process of taking LCM.
- To resolve a fraction into partial fractions, the degree of the numerator must be less than the degree of denominator.

### Solved Examples

**Example 1.** *Find the $n^{th}$ differential coefficients of*

(i) $\dfrac{1}{1-5x+6x^2}$ (ii) $\dfrac{x^2}{[(x+2)(2x+3)]}$

**Solution.** (i) Let $y = \dfrac{1}{1-5x+6x^2} = \dfrac{1}{(3x-1)(2x-1)} = \dfrac{2}{2x-1} - \dfrac{3}{3x-1}$

(By resolving into partial fractions)

$= 2(2x-1)^{-1} - 3(3x-1)^{-1}$

Differentaittng, $n$ times, we get

$y_n = 2(-1)^n n! 2^n (2x-1)^{-n-1} - 3(-1)^n n! 3^n (3x-1)^{-n-1}$

$= (-1)^n . n! [2^{n+1}(2x-1)^{-n-1} - 3^{n+1} (3x-1)^{-n-1}]$

Notes

(ii) Let $y = \frac{x^2}{[(x+2)(2x+3)]}$

Since, the given fraction is not a proper one so, divide the Nr. by Dr., we observe that the quotient will be 1/2.

So let $\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} + \frac{A}{x+2} + \frac{B}{2x+3}$

which gives $A = -4, B = 9/2$

Therefore,

$$y = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2(2x+3)} = \frac{1}{2} - 4(x+2)^{-1} + \frac{9}{2}(2x+3)^{-1}$$

Differentiating $n$ times, we get

$$y_n = -4(-1)^n n!(x+2)^{-n-1} + \frac{9}{2}(-1)^n . n! 2^n (2x+3)^{-n-1}$$

$$= (-1)^n n! \left[\frac{9.2^{n-1}}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}}\right]$$

**REMARK**

- If none of the standard formulae is applicable to find $y_n$ in any problem, then find $y_1, y_2, y_3$ and then generalise.

## More Solved Examples

**Example 1.** *If* $y = \sqrt{x+a}$, *find* $y_n$.

**Solution.** We have

$$y = \sqrt{x+a} = (x+a)^{1/2}$$

$$y_1 = \frac{1}{2}(x+a)^{-1/2}$$

$$y_2 = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x+a)^{-3/2}$$

$$y_3 = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+a)^{-5/2} = (-1)^2 \frac{1.3}{2^3}(x+a)^{-5/2}$$

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$$y_n = (-1)^{n-1} \frac{1.3.5\ldots\text{upto } (n-1) \text{ times}}{2^n} (x+a)^{-\frac{(2n-1)}{2}}$$

$$y_n = (-1)^{n-1} \frac{1.3\ldots(2n-3)}{2^n} (x+a)^{-\left(\frac{2n-1}{2}\right)}$$

**Example 2.** *If* $y = \tan^{-1}\left\{\frac{\sqrt{(1+x^2)}-1}{x}\right\}$, *show that*

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta,$$

*where* $\theta = \cot^{-1} x$.

**Solution.** We have $y = \tan^{-1}\left\{\frac{\sqrt{(1+x^2)}-1}{x}\right\}$.

Put $x = \tan\phi$, then

$$y = \tan^{-1}\left\{\frac{\sqrt{(1+\tan^2\phi)}-1}{\tan\phi}\right\} = \tan^{-1}\left[\frac{\sec\phi - 1}{\tan\phi}\right]$$

Notes

$$= \tan^{-1}\left(\frac{1-\cos\phi}{\sin\phi}\right) = \tan^{-1}\left(\frac{2\sin^2(\phi/2)}{2\sin(\phi/2)\cos(\phi/2)}\right)$$

$$= \tan^{-1}\tan(\phi/2) = \phi/2 = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow y_1 = \frac{1}{2(1+x^2)} = \frac{1}{2(x-i)(x+i)} = \frac{1}{4i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

Differentiating $(n-1)$ times, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{4i}[(x-i)^{-n} - (x+i)^{-n}]$$

Now putting $x = r\cos\theta$, $1 = r\sin\theta$, we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{4i}\left[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{4i}r^{-n}\left[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)\right]$$

$$= \frac{1}{2}(-1)^{n-1}(n-1)!\,r^{-n}\sin n\theta = \frac{1}{2}(-1)^{n-1}(n-1)!\left(\frac{1}{\sin\theta}\right)^{-n}\sin n\theta$$

$$\left[\because r = \frac{1}{\sin\theta}\right]$$

$$= \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta \text{ where } \theta = \tan^{-1}\frac{1}{x} = \cot^{-1}x.$$

**Example 3.** *If $y = \sin mx + \cos mx$, prove that* $y_n = m^n[1 + (-1)^n \sin 2mx]^{1/2}$.

**Solution.** We have

$$y_n = \frac{d^n}{dx^n}(\sin mx) + \frac{d^n}{dx^n}(\cos mx) = m^n\sin\left(mx + n\frac{\pi}{2}\right) + m^n\cos\left(mx + n\frac{\pi}{2}\right)$$

$$= m^n\left[\left\{\sin\left(mx + n\frac{\pi}{2}\right) + \cos\left(mx + n\frac{\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= m^n\left[1 + 2\sin\left(mx + n\frac{\pi}{2}\right).\cos\left(mx + n\frac{\pi}{2}\right)\right]^{1/2}$$

$$= m^n\left[1 + \sin(2mx + n\pi)\right]^{1/2} = m^n[1 \pm \sin 2mx]^{1/2}$$

$$= m^n[1 + (-1)^n\sin 2mx]^{1/2}.$$

**Example 4.** *If* $y = x\log\frac{x-1}{x+1}$, *show that* $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

**Solution.** Let $y = x\log\frac{x-1}{x+1}$

$\Rightarrow\ y = x\log(x-1) - x\log(x+1)$ ...(1)

Differentiating (1) w.r.t. $x$ we get

$$y_1 = \frac{x}{x-1} + \log(x-1) - \frac{x}{x+1} - \log(x+1)$$

$$= 1 + \frac{1}{x-1} + \log(x-1) - 1 + \frac{1}{x+1} - \log(x+1)$$

$$= \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1) \quad \text{...(2)}$$

Differentiating both sides of (2) w.r.t. $x$, $(n-1)$ times we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} + \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}$$

Notes

$$= (-1)^{n-2}(n-2)!\left\{-\frac{(n-1)+x-1}{(x-1)^n}\right\} + (-1)^{n-2}(n-2)!\left\{\frac{-(n-1)-(x+1)}{(x+1)^n}\right\}$$

$$\Rightarrow y_n = (-1)^{n-2}(n-2)!\left\{\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right\}$$

**Example 5.** *Find the $n^{th}$ derivative of $\frac{1}{x^2+a^2}$.*

**Solution.** Let $y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$ ...(1)

Differentiating (1) $n$ times w.r.t. $x$ we get

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right] = \frac{(-1)^n.n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right] \quad ...(2)$$

Let $x = r\cos\theta$ and $a = r\sin\theta$ *i.e.*, $\theta = \tan^{-1}\frac{a}{x}$ in (2), we get

$$y_n = \frac{(-1)^n.n!}{2iar^{n+1}}\left[\frac{1}{(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n.n!}{2iar^{n+1}}\left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}\right]$$

$$= \frac{(-1)^n.n!}{2iar^{n+1}}\left[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\}\right]$$

$$= \frac{(-1)^n.n!}{2iar^{n+1}}\left[2i\sin(n+1)\theta\right] = \frac{(-1)^n.n!\sin(n+1)\theta}{a\left(\frac{a^{n+1}}{\sin^{n+1}\theta}\right)} \qquad [\because a = r\sin\theta]$$

$$= \frac{(-1)^n.n!\sin(n+1)\theta\sin^{n+1}\theta}{a^{n+2}},$$

$\theta = \tan^{-1}\frac{a}{x}$.

## STUDENT ACTIVITY

1. Find the $n^{th}$ differential coefficient of $\log[(ax+b)(cx+d)]$.

2. Find the $n^{th}$ derivative of $y = \cos^4 x$.

3. If $y = \sin(\sin x)$, show that $\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\tan x + y\cos^2 x = 0$.

Notes

4. If $y = A \sin mx + B \cos mx$, show that $\frac{d^2y}{dx^2} + m^2 y = 0$.

5. If $y = e^{ax} \sin bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$.

## TEST YOURSELF

1. Find the $n^{th}$ derivatives of
   (i) $\sin 3x$ (ii) $\cos x \cos 2x \cos 3x$ (iii) $eax\cos 2x \sin x$
   (iv) $\sin 5x \cos 3x$ (v) $\sin ax \cos bx$ (vi) $\sin 2x \sin 2x$
2. Find the $n^{th}$ derivatives of
   (i) $\frac{x^4}{(x-1)(x-2)}$ (ii) $\frac{x}{1+3x+2x^2}$ (iii) $\frac{1}{(x-2)(x-1)^3}$ (iv) $\frac{1}{x^2-a^2}$
   (v) $\frac{x^2}{(x-a)(x-b)}$ (vi) $\frac{17x^2+26x-42}{6x^3-25x^2-29x+20}$
3. Find the $n^{th}$ derivatives of
   (i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ (ii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$
4. Show that the value of the $n^{th}$ differential coefficients of $\frac{x^3}{x^2-1}$ for $x = 0$, is zero if $n$ is even and is $-n!$, if $n$ is odd and greater than 1.
5. If $y = x(a^2 + x^2)^{-1}$, show that $y_n = (-1)^n n! a^{-n-1} \sin^{n+1}\theta \cos(n+1)\theta$ where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$. [MUMBAI–2007]
6. (i) If $x = a(t - \sin t)$ and $y = a(1 + \cos t)$, prove that $\frac{d^2y}{dx^2} = \frac{1}{4a}\text{cosec}^4\left(\frac{t}{2}\right)$. (MADURAI–1990, 2004)
   (ii) If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, find $\frac{d^2y}{dx^2}$.
7. If $p^2 = a^2\cos 2\theta + b^2 \sin 2\theta$, prove that $p + \frac{d^2p}{d\theta^2} = \frac{a^2.b^2}{p^3}$.
8. Prove that the value when $x = 0$ of $\frac{d^n}{dx^n}(\tan^{-1} x)$ is 0, $(n-1)!$ or $-(n-1)!$ according as $n$ is of the form $2p$, $4p+1$ or $4p+3$ respectively.

## ANSWERS

1.(i) $y_n = \frac{3}{4}\sin\left(x + \frac{n\pi}{2}\right) - \frac{1}{4}.3^n \sin\left(3x + \frac{n\pi}{2}\right)$

(ii) $y_n = \frac{1}{4}\left\{6^n \cos\left(6x + \frac{1}{2}n\pi\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right\}$

(iii) $y_n = \frac{1}{4}[(a^2+1)^{n/2} e^{ax} \sin\{x + n\tan^{-1} 1/a\} + (a^2+9)^{n/2} e^{ax} \sin(3x + n\tan^{-1} 3/a)]$

(iv) $y_n = 2^{-7}\left[8^n \sin\left(8x + \frac{1}{2}n\pi\right) - 2.6^n \sin\left(6x + \frac{1}{2}n\pi\right) - 2.4^n \sin\left(4x + \frac{1}{2}n\pi\right) + 6.2^n \sin\left(2x + \frac{1}{2}n\pi\right)\right]$

(v) $y_n = \frac{1}{2}\left[(a+b)^n \sin\left\{(a+b)x + \frac{1}{2}n\pi\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{1}{2}n\pi\right\}\right]$

(vi) $y_n = 2^{n-1} \sin\left(2x + \frac{1}{2}n\pi\right) - 4^{n-1} \sin\left(4x + \frac{1}{2}n\pi\right)$

**2.** (i) $y_n = (-1)^n n![16(x-2)^{-n-1} - (x-1)^{-n-1}]$ (ii) $y_n = (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]$

(iii) $y_n = (-1)^{n+1}.n!\left[\frac{(n+2)(n+1)}{2(x-1)^{n+3}} + \frac{(n+1)}{(x-1)^{n+2}} + \frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right]$

(iv) $y_n = \frac{1}{2a}(-1)^n.n!\{(x-a)^{-n-1} - (x+a)^{-n-1}\}$

(v) $y_n = \frac{(-1)^n.n!}{(a-b)}\left[\frac{a^2}{(x-a)^{n+1}} - \frac{b^2}{(x-b)^{n+1}}\right]$

(vi) $y_n = (-1)^n.n!\left[\frac{2^n}{(2x-1)^{n+1}} - \frac{2.3^n}{(3x+4)^{n+1}} + \frac{3}{(x-5)^{n+1}}\right]$.

**3.** (i) $y_n = (-1)^{n-1}.(n-1)!\sin^n\theta \sin n\theta$, where $\theta = \tan^{-1}\frac{1}{x}$

(ii) $y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$, where $\theta = \tan^{-1}\frac{1}{x}$ **6.** (ii) $y_2 = \frac{1}{a}\frac{\sec^3\theta}{\theta}$

## 2.4 LEIBNITZ'S THEOREM

This theorem help us to find the $n$th differential coefficient of the product of two functions in terms of the successive derivatives of the functions.

**Statement.** *If u, v be two functions of x, having derivative of $n^{th}$ order, then*

$$D^n(nv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n$$

*where suffixes of u and v denote differentiations w.r.t. x.*

**Step 1.** Let $y = uv$

$\Rightarrow$ $y_1 = u_1 v + u v_1$

and $y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2 = u_2 v + 2u_1 v_1 + u v_2$

$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2.$

Thus the theorem is true for $n = 1, 2$.

**Step 2.** Let us assume that the theorem is true for a particular value of $n$ say $m$, then we have

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \ldots + {}^mC_m u v_m. \quad \ldots(1)$$

**Step 3.** Now, differentiating (1), we have

$$y_{m+1} = u_{m+1} v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2 + {}^mC_2 u_{m-1} v_2 + {}^mC_2 u_{m-2} v_3 + \ldots$$
$$+ {}^mC_{r-1} u_{m-r+2} v_{r-1} + {}^mC_{r-1} u_{m-r+1} v_r + {}^mC_r u_{m-r+1} v_r$$
$$+ {}^mC_r u_{m-r} v_{r+1} + \ldots + {}^mC_m u_1 v_m + {}^mC_m u v_{m+1}.$$

$$= u_{m+1}.v + ({}^mC_1 + 1) u_m v_1 + ({}^mC_2 + {}^mC_1) u_{m-1} v_2 + \ldots + ({}^mC_r + {}^mC_{r-1}) u_{m-r+1} v_r$$
$$+ \ldots + {}^mC_m u v_{m+1}.$$

Notes

Now using Pascal's law, given by ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

For $r = 1, 2, 3, \ldots$

We have $${}^mC_0 + {}^mC_1 = {}^{m+1}C_1 \Rightarrow 1 + {}^mC_1 = {}^{m+1}C_1$$
$${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$
..................................................
..................................................

and $${}^mC_m = 1 = {}^{m+1}C_{m+1}$$

Therefore,

$$y_{m+1} = u_{m+1}.v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \ldots + {}^{m+1}C_r u_{m-r+1} v_r + \ldots + {}^{m+1}C_{m+1} u v_{m+1}$$

$\Rightarrow$ If the theorem is true for $n = m$, then it is also true for the next higher value $n = m + 1$.

Then, by the principle of Mathematical induction, we can say that theorem is true for any positive integer $n$.

## Solved Examples

**Example 1.** *Find the $n^{th}$ derivative of $x^2 \sin x$.*

**Solution .** Let $u = \sin x$ and $v = x^2$.

Then, $u_n = \sin\left[x + \frac{n\pi}{2}\right]$.

$$u_{n-1} = \sin\left(x + (n-1)\frac{\pi}{2}\right)$$

$$u_{n-2} = \sin\left[x + (n-2)\frac{\pi}{2}\right]$$

Also, $v_1 = 2x$, $v_2 = 2$, $v_3 = 0$

Now, by Leibnitz's theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n.v + {}^nC_1 u_{n-1}.v_1 + {}^nC_2 u_{n-2}.v_2$$

$$\Rightarrow \frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + \frac{n\pi}{2}\right)x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right]2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right]2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1)\sin\left[x + (n-2)\frac{\pi}{2}\right]$$

**Example 2.** *Find the nth derivative of $x^{n-1}\log x$.*

**Solution .** Let $y = x^{n-1}\log x$ ...(1)

Differentiating (1) w.r.t. $x$ we get

$$y_1 = x^{n-1}.\frac{1}{x} + (n-1)x^{n-2}\log x = x^{n-1}.\frac{1}{x} + (n-1)\frac{x^{n-1}}{x}\log x$$

$\Rightarrow$ $xy_1 = x^{n-1} + (n-1)y$ ...(2)

Finally, differentiating (2) both the sides $(n-1)$ times w.r.t. $x$, we get

$$y_n x + (n-1)y_{n-1}.1 = (n-1)! + (n-1)y_{n-1}$$

Hence, $y_n = \frac{(n-1)!}{x}$.

**Example 3.** *If* $y = a\cos(\log x) + b\sin(\log x)$, *show that* $x^2y_2 + xy_1 + y = 0$ *and* $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

**Solution.** We have

$$y = a\cos(\log x) + b\sin(\log x) \quad \ldots(1)$$

Differentiating (1) with respect to $x$, we have

$$y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Again, differentiating w.r.t. $x$, we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x) = -y$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0 \quad \ldots(2)$$

Now, differentiating (2) both sides $n$ times by Leibnitz's theorem, we get

$$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$$

$$\Rightarrow (D^ny_2)x^2 + {}^nC_1(D^{n-1}y_2)(Dx^2) + {}^nC_2(D^{n-2}y_2)(D^2x^2) + (D^ny_1)x + {}^nC_1(D^{n-1}y_1)(Dx) + D^ny = 0$$

$$\Rightarrow x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}2y_n + xy_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

**Example 4.** *If* $y = e^{a\sin^{-1}x}$, *show that* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.

**Solution.** We have

$$y = e^{a\sin^{-1}x} \Rightarrow y_1 = e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2} = ae^{a\sin^{-1}x} = ay$$

$$\Rightarrow y_1^2(1-x^2) = a^2y^2 \quad \ldots(1)$$

Now differentiating (1) with respect to $x$, we get

$$2y_1y_2(1-x^2) + y_1^2(-2x) = 2a^2yy_1$$

$$\Rightarrow 2y_1[y_2(1-x^2) - xy_1 - a^2y] = 0 \quad [\because 2y_1 \neq 0]$$

$$\Rightarrow [y_2(1-x^2) - xy_1 - a^2y] = 0 \quad \ldots(2)$$

Using Leibnitz's theorem, differentiating (2), $n$ times, we get

$$D^n[y_2(1-x^2)] - D^n(y_1x) - a^2D^ny = 0$$

$$\Rightarrow \left[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)\right] - [y_{n+1}x + ny_n] - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

**Example 5.** *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$. *Prove that* $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

**Solution.** We have

$$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n = n\log\frac{x}{n} = n(\log x - \log n)$$

Notes

Now, differentiating with respect to $x$, we get

$$-\frac{1}{\sqrt{\left[1-\frac{y^2}{b^2}\right]}}\frac{y_1}{b}=\frac{n}{x}$$

or $$-\frac{y_1}{\sqrt{b^2-y^2}}=\frac{n}{x}$$

or $$y_1{}^2x^2=n^2(b^2-y^2)$$

Again, differentiating, with respect to $x$, we get

$$2x^2y_1y_2+2xy_1{}^2=-2n^2yy_1$$

or $y_2x^2+y_1x+n^2y=0.$ $[\because 2y_1\neq 0]$

Using Leibnitz's theorem, differentiating $n$ times, we get

$$y_{n+2}x^2+{}^nC_1y_{n+1}(2x)+{}^nC_2y_n(2)+y_{n+1}x+{}^nC_1y_n+n^2y_n=0$$

$$\Rightarrow x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0.$$

**Example 6.** *If* $y=(x^2-1)^n$, *Prove that* $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$

*Hence if* $P_n=\frac{d^n}{dx^n}(x^2-1)^n$ *show that* $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0$

**Solution.** We have $y=(x^2-1)^n$ ...(1)

Therefore $y_1=n(x^2-1)^{n-1}.2x$

or $(x^2-1)y_1=n(x^2-1)^n.2x$

$$\Rightarrow (x^2-1)y_1=2nxy. \quad ...(2)$$

Differentiating (2), $(n+1)$ times by Leibnitz's theorem, we get

$$D^{n+1}[y_1(x^2-1)]-2nD^{n+1}(yx)=0$$

or $y_{n+2}(x^2-1)+(n+1)y_{n+1}.2x+\frac{n(n+1)}{2}.y_n.2-2ny_{n+1}.x-2n(n+1)y_n.1=0$

or $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

Hence, the first result. From (2), we get

$$(x^2-1)D^2y_n+2xDy_n-n(n+1)y_n=0. \quad ...(3)$$

Putting $y_n=\frac{d^n}{dx^n}(x^2-1)^n=P_n;$

equation (3) becomes

$$(x^2-1)D^2P_n+2xDP_n-n(n+1)P_n=0$$

or $-(1-x^2)D^2P_n+2xD(P_n)-n(n+1)P_n=0$

or $-\frac{d}{dx}\{(1-x^2)DP_n\}-n(n+1)P_n=0$

or $\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n\right\}+n(n+1)P_n=0$

**Example 7.** *If* $y=\sin(m\sin^{-1}x)$, *Prove that* $(1-x^2)y_2-xy_1+m^2y=0$ *and* $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0.$

**Solution.** Let $y=\sin(m\sin^{-1}x)$ ...(1)

Differentiating w.r.t. $x$ we get

$$y_1=\cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1\sqrt{1-x^2}=m\cos(m\sin^{-1}x)$$

$$\Rightarrow y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x) = m^2[1-\sin^2(m\sin^{-1}x)]$$
$$= m^2(1-y^2) \quad ...(2)$$

Again, differentiating both sides of (2) w.r.t. $x$ we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = -2m^2yy_1$$
$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = -m^2y$$
$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0 \quad ...(3)$$

Finally, differentiating (3) $n$ times, by Leibnitz's theorem, we get

$$\left[y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2)\right] - \left[y_{n+1}x + {}^nC_1y_n\right] + m^2y_n = 0$$
$$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n+1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$
$$\text{or} \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

**Example 8.** *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$, *Show that* $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+m^2)y_n = 0.$

**Solution.** We have $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$

$$\Rightarrow \quad y = b\cos\left(m\log\left(\frac{x}{m}\right)\right)$$
$$\therefore\ y_1 = -b\sin\left(m\log\left(\frac{x}{m}\right)\right).m\frac{1}{(x/m)}.\frac{1}{m}$$
$$\Rightarrow\ xy_1 = -bm\sin\left(m\log\left(\frac{x}{m}\right)\right)$$

Again differentiating, we get

$$xy_2 + y_1 = -bm\cos\left\{m\log\left(\frac{x}{m}\right)\right\}.m.\frac{1}{(x/m)}.\frac{1}{m}$$
$$\Rightarrow\ x^2y_2 + xy_1 = -m^2b\cos\left\{m\log\left(\frac{x}{m}\right)\right\} = -m^2y$$
$$\therefore\quad x^2y_2 + xy_1 + m^2y = 0$$

Differentiating both sides of the above equation, $n$ times by Leibnitz's theorem, we get

$$[y_{n+2}.x^2 + {}^nC_1y_{n+1}(2x) + {}^nC_2y_n(2)] + [y_{n+1}(x) + {}^nC_1y_n(1)] + m^2y_n = 0$$
$$\Rightarrow \quad x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+m^2)y_n = 0$$

**Example 9.** *If* $x = \tan(\log y)$, *prove that*

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0.$$

**Solution.** Let $x = \tan(\log y)$

$$\Rightarrow \quad y = e^{\tan^{-1}x} \quad ...(1)$$
$$\Rightarrow\ y_1 = e^{\tan^{-1}x}.\frac{1}{(1+x^2)}$$
$$\therefore \quad (1+x^2)y_1 = y \quad ...(2)$$

Differentiating (2) $n$ times by Leibnitz's theorem, we get

$$y_{n+1}(1+x^2) + {}^nC_1y_n(2x) + {}^nC_2y_{n-1}(2) = y_n$$
$$\Rightarrow (1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$$

**Example 10.** *If* $y = (1-x)^{-\alpha}e^{-\alpha x}$, *prove that* $(1-x)y_{n+1} - (n+\alpha x)y_n - n\alpha y_{n-1} = 0.$

Notes

**Solution.** We have $y=(1-x)^{-\alpha}e^{-\alpha x}$ ...(1)

$$\Rightarrow \quad y_1=(1-x)^{-\alpha}(-\alpha e^{-\alpha x})+e^{-\alpha x}(-\alpha)(1-x)^{-\alpha-1}(-1)$$

$$=e^{-\alpha x}(1-x)^{-\alpha}\left(-\alpha+\frac{\alpha}{1-x}\right)$$

$$\Rightarrow \quad y_1(1-x)=\alpha xy \quad \text{...(2)}$$

Differentiating (2) $n$ times by Leibnitz's theorem, we get

$$y_{n+1}(1-x)+{}^nC_1 y_n(-1)=\alpha[y_n(x)+{}^nC_1 y_{n-1}(1)]$$

$$\therefore \quad (1-x)y_{n+1}+(-n-\alpha x)y_n-n\alpha y_{n-1}=0$$

$$\Rightarrow \quad (1-x)y_{n+1}-(n+\alpha x)y_n-n\alpha y_{n-1}=0$$

## STUDENT ACTIVITY

**1.** Find the $n^{th}$ derivative of $x^3\cos x$.

**2.** If $x=\cosh\left(\frac{1}{m}\log y\right)$, prove that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.

**3.** If $y=\sin\log(x^2+2x+1)$, prove that $(1+x^2)y_{n+2}+(2n+1)(1+x)y_{n+1}-(n^2+4)y_n=0$.

**4.** If $y=\sinh[m\log(x+\sqrt{x^2+1})]$, prove that $(x^2+1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.

**5.** If $\sin^{-1}y=2\log(x+1)$, prove that $(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(x^2+4)y_n=0$.

**6.** Prove the following $\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right]=\frac{(-1)^n.n!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{n}\right)$.

Notes

## TEST YOURSELF

1. Use Leibnitz's theorem, to find $y_n$ in the following cases :
   (i) $x^3e^{ax}$ (ii) $x^2e^x$ (iii) $x^3\sin ax$ (iv) $x^3\log x$
   (v) $x^2e^x\cos x$ (vi) $e^x\log x$ (vii) $x^n\log x$ (viii) $x^2\tan^{-1}x$
2. If $I_n = \frac{d^n}{dx^n}(x^n\log x)$, prove that $I_n = nI_{n-1}+(n-1)!$ and hence show that $I_n = n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)$
3. If $y = e^{\tan^{-1}x}$, prove that $(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$.
4. If $y=(\sin^{-1}x)^2$, provethat $(1-x^2)y_2-xy_1-2=0$ and $(1-x^2)y_{n+2}-x(2n+1)y_{n+1}-n^2y_n=0$.
5. If $y = \frac{\sin^{-1}x}{\sqrt{(1-x^2)}}$, prove that $(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0$.
6. If $y = [\log\{x+\sqrt{(1+x^2)}\}]^2$, prove that $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$.
7. Differentiating $n$ times the equation :
   (i) $(1+x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+a^2y=0$. (ii) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$.
8. If $y=[x+\sqrt{(1+x^2)}]^m$, prove that $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.
9. If $y^{1/m}+y^{-1/m}=2x$, prove that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.
10. If $y=\cos(\log x)$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.
11. If $x+y=1$, prove that $\frac{d^n}{dx^n}(x^ny^n)=n![y^n-({}^nC_1)^2y^{n-1}x+({}^nC_2)^2y^{n-2}x^2...+(-1)^nx^n]$.
12. If $y=x\cos(\log x)$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-2n+2)y_n=0$.
13. If $y=\left(\frac{1+x}{1-x}\right)^{1/2}$, prove that $(1-x^2)y_n-[2(n-1)x+1]y_{n-1}-(n-1)(n-2)y_{n-2}=0$.
14. If $y=\frac{\sinh^{-1}x}{\sqrt{1+x^2}}$, prove that $(1+x^2)y_{n+2}+(2n+3)xy_{n+1}+(n+1)^2y_n=0$.
15. If $x=\sin t$, $y=\cos pt$, prove that $(1-x^2)y_2-xy_1+p^2y=0$.
    Hence, show that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-p^2)y_n=0$.

## ANSWERS

1. (i) $e^{ax}a^{n-3}[a^3x^3+3na^2x^2+3n(n-1)ax+n(n-1)(n-2)]$ (ii) $e^x[x^2+2nx+n(n-1)]$

   (iii) $a^{n-3}\left[a^3x^3\sin\left(ax+\frac{n\pi}{2}\right)+3na^2x^2\sin\left(ax+(n-1)\frac{\pi}{2}\right)+3n(n-1)ax\sin\left\{ax+(n-2)\frac{\pi}{2}\right\}\right.$
   $\left.+n(n-1)(n-2)\sin\left(ax+(n-3)\frac{\pi}{2}\right)\right]$

   (iv) $\frac{(-1)^{n-1}n!}{x^{n-3}}\left[\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}\right]$

Notes

(v) $e^x\left[2^{n/2}x^2\cos\left(x+\frac{n\pi}{4}\right)+2^{(n-1)/2}2nx\cos\left(x+(n-1)\frac{\pi}{4}\right)+2^{(n-2)/2}n(n-1)\cos\left(x+(n-2)\frac{\pi}{4}\right)\right]$

(vi) $e^x[\log x+{}^nC_1x^{-1}-{}^nC_2x^{-2}+{}^nC_3 2!x^{-3}-\ldots+{}^nC_n(-1)^{n-1}(n-1)!x^{-n}]$

(vii) $y_{n+1}=\frac{n!}{x}$

(viii) $(-1)^{n-1}(n-3)![(n-1)(n-2)x^2\sin^n\phi\sin n\phi-2n(n-1)\sin^{n-1}\phi\sin(n-1)\phi$
$+n(n-1)\sin^{n-2}\phi\sin(n-2)\phi]$ where $\phi=\tan^{-1}\frac{1}{x}$

7. (i) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-a^2)y_n=0$

(ii) $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$

## 2.5 DETERMINATION OF THE VALUE OF $n^{th}$ DERIVATIVE OF A FUNCTION AT X = 0

### WORKING PROCEDURE

**Step 1.** Put the given function equal to $y$.

**Step 2.** Find $y_1=\frac{dy}{dx}$. Then

(i) Take L.C.M. (if required).

(ii) Square both sides, if square roots are there.

(iii) Try to get y in R.H.S. (if possible).

**Step 3.** Again differentiating both sides w.r.t. $x$ and get an equation in $y_2$, $y_1$ and $y$.

**Step 4.** Differentiate both sides $n$ times w.r.t. $x$ by Leibnitz's theorem.

**Step 5.** Put $x=0$ in equations of step 1, 2, 3, 4.

**Step 6.** Put $n=1, 2, 3, 4, \ldots$ in last equation of step 5.

**Step 7.** Discuss the two cases, when n is even and when $n$ is odd.

### Solved Examples

**Example 1.** *If $y=e^{a\cos^{-1}x}$, show that*

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$$

*and hence calculate $y_n$ at $x=0$.*

**Solution.** We have $y=e^{a\cos^{-1}x}$ ...(1)

$$\therefore\quad y_1=e^{a\cos^{-1}x}.\frac{-a}{\sqrt{1-x^2}}=-\frac{ya}{\sqrt{1-x^2}} \quad ...(2)$$

$$\Rightarrow\quad y_1\sqrt{1-x^2}=-ya$$

Now squaring both sides we get

$$y_1^2(1-x^2)=y^2a^2$$

Differentiating w.r.t. $x$, we have

$$(1-x^2)2y_1y_2-2xy_1^2=2a^2yy_1$$

$$\Rightarrow\quad (1-x^2)y_2-xy_1=a^2y \quad ...(3)$$

Now, using Leibnitz's theorem, differentiating (3), $n$ times, we get

$$(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=a^2y_n$$

$$\Rightarrow\quad (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0 \quad ...(4)$$

By putting $x=0$ in (1), (2), (3) and (4), we get

$$y(0)=e^{a.\pi/2}$$

Notes

$$y_1(0) = -ae^{a.\pi/2}$$
$$y_2(0) = a^2 y(0) = a^2.e^{a.\pi/2}$$

$\Rightarrow y_{n+2}(0) = (n^2+a^2)y_n(0)$ ...(5)

Put $n-2$ for $n$ in (5), we get

$$y_n(0) = [(n-2)^2 + a^2]y_{n-2}(0) \quad ...(6)$$

Again put $n-4$ for $n$ in (5), we get

$$y_{n-2}(0) = [(n-4)^2 + a^2]y_{n-4}(0) \quad ...(7)$$

From (6) and (7), we get

$$y_n(0) = [(n-2)^2+a^2][(n-4)^2+a^2]y_{n-4}(0) \quad ...(8)$$

Again put $n-6$ for $n$ in (5), we get

$$y_{n-4}(0) = [(n-6)^2 + a^2]y_{n-6}(0) \quad ...(9)$$

From (8) and (9), we get

$$y_n(0) = [(n-2)^2+a^2][(n-4)^2+a^2][(n-6)^2+a^2]y_{n-6}(0) \quad ...(10)$$

Now there are following two cases :

**Case I.** When $n$ is even.

$$y_n(0) = [(n-2)^2+a^2][(n-4)^2+a^2][(n-6)^2+a^2]...[2^2+a^2]a^2e^{a\pi/2}$$

**Case II.** When $n$ is odd.

$$y_n(0) = [(n-2)^2+a^2][(n-4)^2+a^2][(n-6)^2+a^2]...[1^2+a^2](-ae^{a\pi/2})$$

**Example 2.** *If $y = \tan^{-1} x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$. Hence, determine the values of all the derivatives of y with respect to x when x = 0.*

**Solution.** We have $y = \tan^{-1} x$. ...(1)

$$\therefore \quad y_1 = \frac{1}{1+x^2} \quad ...(2)$$

$\Rightarrow \quad y_1(1+x^2) = 1.$

Differentiating, $n$ times by Leibnitz's theorem, we have

$$y_{n+1}(1+x^2) + ny_n.2x + \frac{n(n-1)}{2}y_{n-1}.2 = 0$$

$$\Rightarrow \quad (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0 \quad ...(3)$$

Putting $x = 0$ in (1), (2) and (3), we get

$$y(0) = 0$$
$$y_1(0) = 1$$

.......................................

$$y_{n+1}(0) = -n(n-1)y_{n-1}(0) \quad ...(4)$$

Put $n = 1$ in (4), we get $y_2(0) = 0$.

Put $n-1$ for $n$ in (4), we get $y_n(0) = -(n-1)(n-2)y_{n-2}(0)$ ...(5)

Put $n-3$ for $n$ in (4), we get $y_{n-2}(0) = -(n-3)(n-4)y_{n-4}(0)$ ...(6)

From (5) and (6), we get

$$y_n(0) = (n-1)(n-2)(n-3)(n-4)y_{n-4}(0) \quad ...(7)$$

There arise following two cases :

**Case I.** When $n$ is even.

$$y_n(0) = (-1)^{(n-2)/2}(n-1)(n-2)(n-3)(n-4)...4.2y_2(0)$$
$$= (-1)^{(n-2)/2}(n-1)(n-2)(n-3)(n-4)...3.2.0 = 0 \quad [\because y_2(0) = 0]$$

**Case II.** When $n$ is odd.

$$y_n(0) = (-1)^{(n-1)/2}(n-1)(n-2)(n-3)...3.2.1y_1(0)$$
$$= (-1)^{(n-1)/2}(n-1)!\,y_1(0) = (-1)^{(n-1)/2}(n-1)! \quad [\because y_1(0) = 1]$$

Notes

**Example 3.** *If* $y = [x+\sqrt{1+x^2}]^m$, *find* $(y_n)_0$.

**Solution.** We have $y = [x+\sqrt{1+x^2}]^m$. ...(1)

Differentiating both sides w.r.t. $x$, we get

$$y_1 = m[x+\sqrt{1+x^2}]^{m-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

or $$y_1 = \frac{m}{\sqrt{1+x^2}}[x+\sqrt{1+x^2}]^m$$

or $$\sqrt{1+x^2}\cdot y_1 = m[x+\sqrt{1+x^2}]^m$$

or $$\sqrt{1+x^2}\cdot y_1 = my.$$

Squaring both sides, we get

$$y_1^2(1+x^2) = m^2y^2. \qquad ...(2)$$

Again differentiating both sides, we get

$$2y_1(1+x^2)y_2 + 2xy_1^2 = 2m^2yy_1.$$

or $$(1+x^2)y_2 + xy_1 - m^2y = 0. \qquad ...(3)$$

Applying Leibnitz's theorem to differentiate $n$ times, we get

$$D^n[(1+x^2)y_2] + D^n(xy_1) - m^2D^2y = 0.$$

$$(1+x^2)y_{n+2} + {}^nC_1y_{n+1}D(1+x^2) + {}^nC_2y_nD^2(1+x^2) + xy_{n+1} + {}^nC_1y_nD(x) - m^2y_n = 0$$

or $$(1+x^2)y_{n+2} + ny_{n+1}2x + \frac{n(n-1)}{2}y_n.2 + xy_{n+1} + ny_n - m^2y_n = 0$$

or $$(1+x^2)y_{n+2} + x(2n+1)y_{n+1} + (n^2-m^2)y_n = 0. \qquad ...(4)$$

Putting $x = 0$ in (1), (2), (3) and (4), we get

$$(y)_0 = 1$$
$$(y_1)_0 = m(y_0) = m$$
$$(y_2)_0 = m^2(y)_0 = m^2$$

and $$(y_{n+2})_0 = (m^2-n^2)(y_n)_0. \qquad ...(5)$$

Put $n-2$ for $n$ in (5), we get

$$(y_n)_0 = [m^2-(n-2)^2](y_{n-2})_0 \qquad ...(6)$$

Put $n-4$ for $n$ in (5), we get

$$(y_{n-2})_0 = [m^2-(n-4)^2](y_{n-4})_0 \qquad ...(7)$$

From (6) and (7), we get

$$(y_n)_0 = [m^2-(n-2)^2][m^2-(n-4)^2](y_{n-4})_0. \qquad ...(8)$$

There arise two cases :

**Case I.** When $n$ is even.

$$(y_n)_0 = [m^2-(n-2)^2][m^2-(n-4)^2]...(m^2-2^2)(y_2)_0$$
$$= [m^2-(n-2)^2][m^2-(n-4)^2]...[m^2-2^2]m^2 \qquad [\because (y_2)_0 = m^2]$$

**Case II.** When $n$ is odd.

$$(y_n)_0 = [m^2-(n-2)^2][m^2-(n-4)^2]...(m^2-1^2)(y_1)_0$$
$$= [m^2-(n-2)^2][m^2-(n-4)^2]...(m^2-1^2)m \qquad [\because (y_1)_0 = m]$$

**Example 4.** *If* $y = \sin(a\sin^{-1}x)$, *then, prove that* $(1-x^2)y_2 - xy_1 + a^2y = 0$ *and* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (a^2-n^2)y_n = 0$. *Hence, find* $y_n(0)$.

Notes

**Solution.** We have $y = \sin(a\sin^{-1}x)$ ...(1)

Differentiating (1) w.r.t. $x$ we get

$$y_1 = \cos(a\sin^{-1}x).\frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1 = \frac{a}{\sqrt{1-x^2}}\cos(a\sin^{-1}x)$$

$$\Rightarrow (\sqrt{1-x^2})y_1 = a\cos(a\sin^{-1}x)$$

$$\Rightarrow (1-x^2){y_1}^2 = a^2\cos^2(a\sin^{-1}x) = a^2(1-\sin^2(a\sin^{-1}x))$$

$$\Rightarrow (1-x^2){y_1}^2 = a^2(1-y^2) \quad ...(2)$$

(Using (1))

Differentiating (2) w.r.t. $x$, we get

$$(1-x^2)2y_1y_2 - 2x{y_1}^2 = a^2(-2yy_1)$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + a^2y = 0 \quad ...(3)$$

Now differentiating (3) $n$ times by Leibnitz's theorem, we get

$$[(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n] - \left[xy_{n+1} + {}^nC_1(1)y_n\right] + a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - n.1.y_n + a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (a^2 - n^2 - n + n)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (a^2-n^2)y_n = 0 \quad ...(4)$$

From (1),

$$y(0) = \sin(a\sin^{-1}0) = 0$$

From (2),

$$y_1(0) = \frac{a}{\sqrt{1-0}}\cos(a\sin^{-1}0) = a\cos 0 = a$$

From (3),

$$(1-0^2)y_2(0) - 0.y_1(0) + a^2y(0) = 0$$

$$\Rightarrow y_2(0) = 0$$

Form (4),

$$(1-0^2)y_{n+2}(0) - (2n+1).0 + (a^2-n^2)y_n(0) = 0$$

$$\Rightarrow y_{n+2}(0) = (n^2-a^2)y_n(0) \quad ...(5)$$

**Case I.** If $n$ is even.

Put $n = 2$ in equation (5), we get

$$y_4(0) = (2^2-a^2)y_2(0) = 0$$

Put $n = 4$ in equation (5), we get

$$y_6(0) = (4^2-a^2)y_4(0) = 0$$

Put $n = 6$ in equation (5), we get

$$y_8(0) = (6^2-a^2)y_6(0) = 0$$

$$\Rightarrow y_n(0) = 0, \text{ if } n \text{ is even}$$

**Case II.** If $n$ is odd.

Put $n = 1$ in equation (5), we get

$$y_3(0) = (1^2-a^2)y_1(0) = (1^2-a^2).a$$

Put $n = 3$ in equation (5), we get

$$y_5(0) = (3^2-a^2)y_3(0) = (1^2-a^2)(3^2-a^2).a$$

Notes

Put $n = 5$ in equation (5), we get

$$y_7(0) = (5^2 - a^2)y_5(0) = (1^2 - a^2)(3^2 - a^2)(5^2 - a^2).a$$

$$\Rightarrow y_n(0) = (1^2 - a^2)(3^2 - a^2)(5^2 - a^2)\ldots[(n-2)^2 - a^2)]a$$

if $n$ is odd and $n \neq 1$

$$\text{Hence, } y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (1^2 - a^2)(3^2 - a^2)(5^2 - a^2)\ldots[(n-2)^2 - a^2]a & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

## Student Activity

1. If $y = \sin(m \sin^{-1} x)$, then prove that $y_{n+2}(0) = (n^2 - m^2)(y_n)_0$ and find $y_n(0)$.

2. If $y = e^{a \sin^{-1} x}$, show that $(1 - x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2 + a^2)y_n = 0$ and hence, find the value of $y_n(0)$.

3. If $x = \sin\left(\frac{1}{a}\log y\right)$, find $(y_n)_0$.

## Test Yourself

1. If $y = \sin^{-1} x$, prove that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$ and also find the value of $y_n(0)$. *(SVTU-2009)*

2. (i) If $y = [\log\{x + \sqrt{(1 + x^2)}\}]^2$, find all the derivatives of $y$ w.r.t. $x$ when $x = 0$.

   (ii) If $y = (\sinh^{-1} x)^2$, prove that

   $(1 + x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$ Hence, find $y_n(0)$.

3. If $y = [x + \sqrt{1 + x^2}]^m$, find $y_n(0)$.

## Answers

1. When $n$ is even, $y_n(0) = 0$; When $n$ is odd $y_n(0) = 1^2.3^2.5^2\ldots(n-2)^2$
2. (i),(ii) when $n$ is even, $y_n(0) = (-1)^{n/2-1}.2.2^2.4^2\ldots(n-2)^2$, when $n$ is odd $y_n(0) = 0$
3. When $n$ is even, $y_n(0) = [m^2 - (n-2)^2][m^2 - (n-4)^2]\ldots(m^2 - 2^2)m^2$

   When $n$ is odd, $y_n(0) = [m^2 - (n-2)^2][m^2 - (n-4)^2]\ldots(m^2 - 1^2)m$

Notes

## Summary

- $\frac{d^n}{dx^n}(x^n) = n!$
- $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$
- $\frac{d^n}{dx^n}\left(\frac{1}{(ax+b)^m}\right) = (-1)^n \frac{a^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}$
- $\frac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(\frac{n\pi}{2}+ax+b\right)$
- $\frac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(\frac{n\pi}{2}+ax+b\right)$
- $\frac{d^n}{dx^n}(e^{ax+b}) = a^n e^{ax+b}$
- $\frac{d^n}{dx^n}[\log(ax+b)] = (-1)^{n-1}\frac{a^n(n-1)!}{(ax+b)^n}$
- $\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}.e^{ax}.\sin(bx+c+n\tan^{-1}b/a)$
- $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = (a^2+b^2)^{n/2}.e^{ax}.\cos(bx+c+n\tan^{-1}b/a)$

## Objective Evaluation

### FILL IN THE BLANKS

1. $D^n(\log x)$ is equal to ________ .
2. To find the $n^{th}$ derivative of the product of two functions we use ________ theorem.
3. If $y = \sin(ax+b)$, then $D^n(ax+b) =$ ________ .
4. If $y = (ax+b)^{-1}$, then $D^n(ax+b)^{-1} =$ ________ .
5. If $y = e^{ax}\sin bx$, then $y_2 - 2ay_1 =$ ________ .
6. If $y = e^x \sin^2 x$, then $D^n(y) =$ ________ .
7. $D^3(x^3) =$ ________ .
8. $D^n(x^{n-1}) =$ ________ .
9. $D^n(\sin^3 x) =$ ________ .
10. If $y = \tan^{-1}x$, then $(y_5)_0$ is equal to ________ .

### TRUE/FALSE

**Write 'T' for True and 'F' for False statement.**

1. To find the $n^{th}$ derivative of the product of two functions we use Leibnitz's theorem. **(T/F)**
2. If we observe that one of the two functions is such that all its differential coefficients after a certain steps, become zero, then we should take this function as second function. **(T/F)**
3. If $y = a\cos(\log x) + b\sin(\log x)$, then $x^2y_2 + xy_1 = y$. **(T/F)**
4. $D^n(\log x) = \frac{(n-1)!}{x^{n+1}}$. **(T/F)**
5. The $n^{th}$ differential coefficient of $y_k$ is the $(n+k)^{th}$ differential coefficient of $y$. **(T/F)**

### MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. $D^n(e^{ax+b})$ is equal to :

   (a) $a^n e^{ax}$ (b) $e^{ax+b}$ (c) $a^n b^n e^{ax+b}$ (d) $a^n e^{ax+b}$

Notes

2. $D^n \log x$ is equal to :

(a) $\frac{(n-1)!}{x^n}$ (b) $\frac{(-1)^n(n-1)!}{x^n}$ (c) $\frac{(-1)^{n-1}(n-1)!}{x^n}$ (d) $\frac{(-1)^n n!}{x^n}$

3. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$ then $p + \frac{d^2p}{d\theta^2}$ is equal to :

(a) $\frac{a^2b^2}{p^2}$ (b) $\frac{a^2b^2}{p^3}$ (c) $\frac{a^2b^2}{p}$ (d) $\frac{a^2b^2}{p^4}$

4. If $y = A\sin mx + B\cos mx$ then $y_2 + m^2y$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 3

5. If $y = e^{ax}\sin bx$ then $y_2 - 2ay_1$ is equal to :

(a) $(a^2 + b^2)y$ (b) $-(a^2 + b^2)y$ (c) 0 (d) 1

6. If $y = \sin^{-1}x$ then $(1-x^2)\frac{d^2y}{dx^2}$ is equal to :

(a) $\frac{dy}{dx}$ (b) $x^2\frac{dy}{dx}$ (c) $x\frac{dy}{dx}$ (d) $\frac{1}{x}\frac{dy}{dx}$

7. If $x = a(t - \sin t)$ and $y = a(1 + \cos t)$, then $\frac{d^2y}{dx^2}$ is equal to :

(a) $4a\ \text{cosec}^4 t$ (b) $\frac{1}{4a}\text{cosec}^4(t/2)$ (c) $\frac{1}{4a}\sin^4(t/2)$ (d) $4a\sin^4 t$

8. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ then $\frac{d^2y}{dx^2}$ is equal to :

(a) $\frac{1}{a}\sec^3\theta$ (b) $a\sec^3\theta$ (c) $\frac{1}{a\theta\cos^3\theta}$ (d) $a\theta\sec^3\theta$

9. $D^n(\sin^3x)$ is equal to :

(a) $\sin\left(x+\frac{n\pi}{2}\right)$ (b) $\frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right)-\frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right)$

(c) $\frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right)$ (d) none of these

10. $[\cos^2x\sin^3x]$ is equal to :

(a) $\frac{1}{4}[2\sin x+\sin 3x-\sin 5x]$ (b) $\frac{1}{16}[2\sin x+\sin 3x-\sin 5x]$

(c) $\frac{1}{16}[\sin x+\sin 3x-\sin 5x]$ (d) none of these

## ANSWERS

### FILL IN THE BLANKS

1. $\frac{(-1)^{n-1}(n-1)!}{x^n}$ 2. Leibnitz's 3. $a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$

4. $(-1)^n . n! a^n (ax+b)^{-n-1}$ 5. $-(a^2+b^2)y$ 6. $\frac{1}{2}[e^x-(5)^{x/2}e^x\cos(2x+n\tan^{-1}x)]$

7. 3! 8. 0 9. $\frac{3}{4}\sin\left(x+\frac{n\pi}{4}\right)-\frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right)$ 10. 4!

### TRUE/FALSE

1. T 2. T 3. F 4. F 5. T

### MULTIPLE CHOICE QUESTIONS

1. (d) 2. (c) 3. (b) 4. (a) 5. (b) 6. (c) 7. (b)
8. (c) 9. (b) 10. (b)

□□□□

*Chapter*

# 3 Partial Differentiation



## STRUCTURE

- Introduction
- Rules of partial differentiation
- Partial derivatives of the higher order
- Homogeneous functions
- Total differential
- Implicit relation of x and y
- Differentiation of implicit functions
- Change of variables
  - Summary
  - Objective evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:
- The partial differentiations and its rules
- The concept of homogeneous functions
- Euler's theorem on homogeneous function
- The concept of total differentiations
- The differentiations of implicit functions

## 3.1 INTRODUCTION

We know that the differential coefficient of $f(x)$ with respect to $x$ is $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$, provided this limit exists, and it is denoted by

$$f'(x) \quad \text{or} \quad \frac{d}{dx}[f(x)]$$

If $u = f(x, y)$ be a continuous function of two independent variables $x$ and $y$, then the differential coefficient of $u$ w.r.t. $x$ (regarding $y$ as constant) is called the partial derivative or partial differential co-efficient of $u$ w.r.t. $x$ and is denoted by various symbols such as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x$$

Symbolically, if $u = f(x, y)$, then $\lim_{\delta x \to 0} \frac{f(x+\delta x, y)-f(x, y)}{\delta x}$

if it exists, is called the partial derivative or partial differential co-efficient of $u$ w.r.t. $x$ and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x \quad \text{or} \quad u_x.$$

Similarly, by keeping $x$ constant and allowing $y$ alone to vary, we can define the partial derivative or partial differential coefficient of $u$ w.r.t. $y$. It is denoted by any one of the symbols $\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_y$.

Symbolically, $$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y)-f(x, y)}{\delta y}$$

provided this limit exists.

**For Example :**

If $u = ax^2 + 2hxy + by^2$ then $\frac{\partial u}{\partial x} = 2ax + 2hy$ and $\frac{\partial u}{\partial y} = 2hx + 2by$.

Notes

## 3.2 RULES OF PARTIAL DIFFERENTIATION

**Rule (1) :**

(a) If $u$ is a function of $x, y$ and we are to differentiate partially w.r.t. $x$ then, $y$ is treated as constant.

(b) Similarly, if we are to differentiate $u$ partially w.r.t. $y$ then $x$ is treated as constant.

(c) If $u$ is a function of $x, y, z$ and we are to differentiate partially w.r.t. $x$, then $y$ and $z$ are treated as constant.

**Rule (2) :** If $z = u \pm v$, where $u$ and $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}.$$

**Rule (3) :** If $z = uv$, where $u$ and $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

**Rule (4) :** If $z = \frac{u}{v}$, where $u, v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}.$$

**Rule (5) :** If $z = f(u)$, where $u$ is a function of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}.$$

**REMARKS**

- Partial means a 'part of'.
- If $z$ is a function of one variable $x$, then $\frac{\partial z}{\partial x} = \frac{dz}{dx}$.
- If $z$ is a function of two variables $x_1$ and $x_2$, we get $\frac{\partial z}{\partial x_1}$ and $\frac{\partial z}{\partial x_2}$.
- If $z$ is a function of $n$ variables $x_1, x_2, ..., x_n$ we can find $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, ..., \frac{\partial z}{\partial x_n}$.

### 3.2.1 SYMMETRIC FUNCTION OF X AND Y

A function $u = u(x, y)$ is said to be symmetric if, on interchanging $x$ and $y$, $u$ remains unchanged.

## 3.3 PARTIAL DERIVATIVES OF THE HIGHER ORDER

We can find partial derivative of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ just as we found those of $u$ for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are itself functions of $x$ and $y$.

The four derivatives, thus obtained, called the second order partial derivatives of $u$ or $f(x, y)$ are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

and are denoted as

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$$

or $$f_{xx}, f_{yx}, f_{xy}, f_{yy}.$$

**REMARKS**

- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$
- $\frac{\partial^2 u}{\partial x \partial y} \neq \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}$
- The partial derivatives $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are distinguished by the order in which $u$ is successively differentiated by the order in which $u$ is successively differntiated w.r.t. $x$ and $y$, but it will be seen that, in general, that are equal.

## Solved Examples

**Example 1.** *Verify that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, *where* $u = x \sin y + y \sin x$.

**Solution.** We have $u = x \sin y + y \sin x$. ...(1)

Differentiating partially both sides of (1) *w.r.t.* $x$ and $y$ respectively, we get

$$\frac{\partial u}{\partial x} = \sin y + y \cos x \quad ...(2)$$

and $$\frac{\partial u}{\partial y} = x \cos y + \sin x. \quad ...(3)$$

Again differentiating (2) partially *w.r.t.* $y$ and (3) *w.r.t.* $x$, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \cos y + \cos x \quad ...(4)$$

and $$\frac{\partial^2 u}{\partial x \partial y} = \cos y + \cos x. \quad ...(5)$$

Form (4) and (5), we obtain

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

**Example 2.** *If* $u = x^2 y + y^2 z + z^2 x$, *then show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$.

**Solution.** Given that $u = x^2 y + y^2 z + z^2 x$. ...(1)

Differentiating partially both sides of (1) *w.r.t.* $x, y$ and $z$ respectively, we get

$$\frac{\partial u}{\partial x} = 2xy + z^2 \quad ...(2)$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz \quad ...(3)$$

and $$\frac{\partial u}{\partial z} = y^2 + 2zx. \quad ...(4)$$

Adding (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2.$$

**Example 3.** *If* $u = f\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

**Solution.** We have $u = f\left(\frac{y}{x}\right)$ ...(1)

Differentiating (1) partially w.r.t. $x$ and $y$ respectively, we get

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$$

Notes

$$\Rightarrow \qquad x\frac{\partial u}{\partial x} = -\frac{y}{x}f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

and $$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$\Rightarrow \qquad y\frac{\partial u}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right) \qquad \ldots(3)$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

**Example 4.** *If* $z = f(x + ay) + \phi(x - ay)$, *prove that* $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

**Solution.** Given that $z = f(x + ay) + \phi(x - ay)$. ...(1)

Differentiating partially both sides of (1) w.r.t. $x$ and $y$ respectively, we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay) \qquad \ldots(2)$$

and $$\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay). \qquad \ldots(3)$$

Again differentiating partially both sides of (2) w.r.t. $x$ and (3) *w.r.t.* $y$, we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \qquad \ldots(4)$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay). \qquad \ldots(5)$$

Form (4) and (5), we get

$$\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$

**Example 5.** *If* $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *show that* $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

**Solution.** We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Differentiating partially with respect to $x$, we have

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}. \qquad \ldots(1)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz} \qquad \ldots(2)$$

and $$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz} \qquad \ldots(3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} = \frac{3}{(x+y+z)}.$$

Also,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right]$$

$$=-\frac{9}{(x+y+z)^2}$$

**Example 6.** *If* $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, *show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

**Solution.** We have $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

$$\Rightarrow \quad \frac{\partial u}{\partial x}=\frac{1}{\sqrt{\left(1-\left(\frac{x}{y}\right)^2\right)}}.\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(-\frac{y}{x^2}\right)$$

$$=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{(x^2+y^2)}$$

$$\Rightarrow x\frac{\partial u}{\partial x}=\frac{x}{\sqrt{(y^2-x^2)}}-\frac{xy}{x^2+y^2} \quad ...(1)$$

Also, $\frac{\partial u}{\partial y}=\frac{1}{\sqrt{\left(1-\left(\frac{x}{y}\right)^2\right)}}.\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{1}{x}\right)=-\frac{x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$

$$\Rightarrow y\frac{\partial u}{\partial y}=-\frac{x}{\sqrt{(y^2-x^2)}}+\frac{xy}{x^2+y^2} \quad ..(2)$$

On adding (1) and (2), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

**Example 7.** *If* $u=f(r)$, *where* $r^2=x^2+y^2$, *show that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

**Solution.** We have $r^2=x^2+y^2$

$$\left.\begin{aligned}\Rightarrow \quad & 2r\frac{\partial r}{\partial x}=2x \text{ or } \frac{\partial r}{\partial x}=\frac{x}{r}\\ \text{and} \quad & 2r\frac{\partial r}{\partial y}=2y \text{ or } \frac{\partial r}{\partial y}=\frac{y}{r}\end{aligned}\right] \quad ...(1)$$

Since, $u=f(r)$

$$\Rightarrow \quad \frac{\partial u}{\partial x}=[f'(r)].\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)$$

and $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left[x.\frac{1}{r}f'(r)\right]$

Notes

$$= 1.\frac{1}{r}.f'(r) + [xf'(r)]\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right] + \frac{x}{r}[f''(r)]\frac{\partial r}{\partial x}$$

$$= \frac{1}{r}.f'(r) - \frac{x}{r^2}.\frac{x}{r}f'(r) + \frac{x^2}{r^2}f''(r)$$

$$= \frac{1}{r}.f'(r) - \frac{x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r). \quad \text{...(2)}$$

Similarly, we may get

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r}.f'(r) - \frac{y^2}{r^3}f'(r) + \frac{y^2}{r^2}f''(r) \quad \text{...(3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r}.f'(r) - \frac{x^2+y^2}{r^3}f'(r) + \frac{x^2+y^2}{r^2}f''(r)$$

$$= \frac{2}{r}.f'(r) - \frac{r^2}{r^3}f'(r) + \frac{r^2}{r^2}f''(r)$$

$$= \frac{2}{r}.f'(r) - \frac{1}{r}f'(r) + f''(r) = f''(r) + \frac{1}{r}.f'(r).$$

**Example 8.** *If $x^x y^y z^z = c$. Show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -[x \log ex]^{-1}$*

**Solution .** We have $x^x y^y z^z = c.$ ...(1)

Here $z$ can be regarding as a function of two independent variables $x$ and $y$.

Taking log of both sides of (1), we have

$$x \log x + y \log y + z \log z = \log c. \quad \text{...(2)}$$

Differentiating (2) partially w.r.t. $x$, we get

$$x.\frac{1}{x} + 1.\log x + \left[z.\frac{1}{z} + 1.\log z\right]\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}. \quad \text{...(3)}$$

Similarly differentiating (2), w.r.t. $y$, we get

$$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}. \quad \text{...(4)}$$

Also, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right] = -(1+\log y)\frac{\partial}{\partial x}[(1+\log z)^{-1}]$

$$= -(1+\log y).\left[-(1+\log z)^{-2}\frac{1}{z}.\frac{\partial z}{\partial x}\right] = \frac{(1+\log y)}{z(1+\log z)^2}.\left[-\left(\frac{1+\log x}{1+\log z}\right)\right].$$

For $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)} = \frac{-1}{x[\log e + \log x]} \quad [\because \log e = 1]$$

$$= \frac{-1}{x\log(ex)} = -[x\log(ex)]^{-1}.$$

**Example 9.** *If $u = (1 - 2xy + y^2)^{-1/2}$, prove that* $\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$

**Solution .** We have $u = (1-2xy+y^2)^{-1/2}$ ...(1)

Differentiating (1) partially with respect to $x$, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

or $\dfrac{\partial u}{\partial x} = y(1-2xy+y^2)^{-3/2}$

$\Rightarrow (1-x^2)\dfrac{\partial u}{\partial x} = y(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}$

Again differentiating partially *w.r.t.* $x$, we get

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}$$

$$= y\left[-2x(1-2xy+y^2)^{-3/2}+(1-x^2)\left(-\frac{3}{2}\right)(-2y)(1-2xy+y^2)^{-5/2}\right]$$

$$= -2xy(1-2xy+y^2)^{-3/2}+3y^2(1-x^2)(1-2xy+y^2)^{-5/2}$$

$$\therefore \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = -2xyu^3+3y^2(1-x^2)u^5 \qquad \text{[Using (1)]} \ \ldots(2)$$

Differentiating (1) partially *w.r.t.* $y$, we get $\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$

or $\dfrac{\partial u}{\partial y} = (x-y)(1-2xy+y^2)^{-3/2}$

$\Rightarrow y^2\dfrac{\partial u}{\partial y} = (x-y)y^2(1-2xy+y^2)^{-3/2}$

Again differentiating partially *w.r.t.* $y$, we get

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = (2xy-3y^2)(1-2xy+y^2)^{-\frac{3}{2}}+(xy^2-y^3)\left(-\frac{3}{2}\right)(-2x+2y)(1-2xy+y^2)^{-5/2}$$

$$= 2xy(1-2xy+y^2)^{-3/2}-3y^2(1-2xy+y^2)^{-3/2}+3y^2(x-y)^2(1-2xy+y^2)^{-5/2}$$

$$= 2xy(1-2xy+y^2)^{-3/2}-3y^2(1-2xy+y^2)^{-5/2}\{(1-2xy+y^2)-(x-y)^2\}$$

$$= 2xy(1-2xy+y^2)^{-3/2}-3y^2(1-x^2)(1-2xy+y^2)^{-5/2}$$

$$\therefore \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 2xyu^3-3y^2(1-x^2)u^5 \qquad \text{[Using (1)]} \ \ldots(3)$$

Adding (2) and (3), we get

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$$

**Example 10.** *If* $u = (x^2+y^2+z^2)^{-1/2}$, *show that*

(*i*) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z} = -u$ (*ii*) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2} = 0$

**Solution.** (i) We have $u = (x^2+y^2+z^2)^{-1/2}$ ...(1)

Differentiating (1) partially *w.r.t.* $x$, $y$ and $z$ respectively, we get

$$\frac{\partial u}{\partial x} = \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

or $\dfrac{\partial u}{\partial x} = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} \qquad \ldots(2)$$

$$\text{Similarly, } y\frac{\partial u}{\partial y} = \frac{-y^2}{(x^2+y^2+z^2)^{3/2}} \qquad \ldots(3)$$

$$\text{and } z\frac{\partial u}{\partial z} = \frac{-z^2}{(x^2+y^2+z^2)^{3/2}} \qquad \ldots(4)$$

Notes

Adding (2), (3) and (4), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=-(x^2+y^2+z^2)^{-1/2}$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u$$

(ii) We have $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial x}\left\{\dfrac{-x}{(x^2+y^2+z^2)^{3/2}}\right\}$

$$=-\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}+x\left\{\left(-\frac{3}{2}\right)(2x)(x^2+y^2+z^2)^{-5/2}\right\}\right]$$

$$=-\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right]=-\frac{(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \qquad \ldots(5)$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ ...(6)

and $\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$ ...(7)

Adding (5), (6) and (7), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

**Example 11.** *If* $\theta=t^n e^{-r^2/4t}$, *find the value of n for which* $\dfrac{1}{r^2}\cdot\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$.

**Solution.** We have $\theta=t^n e^{-r^2/4t}$ ...(1)

Then $\dfrac{\partial\theta}{\partial r}=t^n\left[e^{-r^2/4t}\left(-\dfrac{2r}{4t}\right)\right]=-\dfrac{r}{2}t^{n-1}e^{-r^2/4t}$

$$\Rightarrow r^2\frac{\partial\theta}{\partial r}=-\frac{r^3}{2}t^{n-1}e^{-r^2/4t}$$

Now $\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=-\dfrac{1}{2}t^{n-1}\left[3r^2e^{-r^2/4t}+r^3e^{-r^2/4t}\left(\dfrac{-2r}{4t}\right)\right]$

$$=-\frac{3}{2}r^2t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^4t^{n-2}e^{-r^2/4t}$$

$$\therefore \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3}{2}t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2t^{n-2}e^{-r^2/4t} \qquad \ldots(2)$$

Again from (1), we get

$$\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-r^2/4t}+t^n.e^{-r^2/4t}.\left(\frac{r^2}{4t^2}\right)$$

or $\dfrac{\partial\theta}{\partial t}=nt^{n-1}e^{-r^2/4t}+\dfrac{1}{4}r^2t^{n-2}.e^{-r^2/4t}$ ...(3)

Since, $\dfrac{1}{r^2}\cdot\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$

Then from (2) and (3), we have

$$-\frac{3}{2}t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2t^{n-2}e^{-r^2/4t}=nt^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2t^{n-2}.e^{-r^2/4t}$$

$$\Rightarrow n=-\frac{3}{2}.$$

Notes

## STUDENT ACTIVITY

1. If $u(x+y) = x^2 + y^2$, show that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

2. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, if (i) $u = e^{my} \cos mx$ (ii) $u = \tan^{-1} \frac{y}{x}$.

3. If $z = e^{ax+by} f(ax - by)$, show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

## TEST YOURSELF

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when:

   (i) $u = \log(x^2 + y^2)$ (ii) $u = \cos^{-1}\left(\frac{x}{y}\right)$ (iii) $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

   (iv) $u = \tan^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$

2. Find the second order partial derivatives of $\log(e^x + e^y)$.

3. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, where

   (i) $u = \log(y \sin x + x \sin y)$ (ii) $u = \log\left(\frac{x^2 + y^2}{xy}\right)$

   (iii) $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ (iv) $u = \sin^{-1}\frac{x}{y}$ (v) $u = x^y$ (vi) $u = \log\tan\left(\frac{y}{x}\right)$

   (vii) $u = x^4 + x^2y^2 + y^4$ (viii) $u = \log\left(\frac{xy}{x^2 + y^2}\right)$ (ix) $u = x \log y$

4. If $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}, \frac{\partial x}{r\partial\theta} = r\frac{\partial\theta}{\partial x}$.

5. If $u = \log(\tan x + \tan y)$, prove that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2$.

Notes

6. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

7. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

8. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)^{-1}$ (ii) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x+y)^{-2}$

9. If $u = f(x + 2y) + g(x - 2y)$, show that $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$.

10. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

## ANSWERS

1. (i) $\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}$ (ii) $-\frac{1}{\sqrt{y^2-x^2}}, \frac{x}{y\sqrt{y^2-x^2}}$ (iii) $\frac{2x}{a^2}, \frac{2y}{b^2}$

(iv) $\frac{(x^2+2xy-y^2)}{(x+y)^2+(x^2+y^2)^2}, \frac{(y^2+2xy-x^2)}{(x+y)^2+(x^2+y^2)^2}$

2. (i) $\frac{e^{x+y}}{(e^x+e^y)^2}, -\frac{e^{x+y}}{(e^x+e^y)^2}, \frac{e^{x+y}}{(e^x+e^y)^2}$

## 3.4 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous function of degree $n$, if the degree of each of its terms in $x$ and $y$ is equal to $n$. Thus

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n \quad ...(1)$$

is homogeneous function in $x$ and $y$ of order $n$.

### REMARKS

- This definition of homogeneity applies to polynomial functions only. To widen the concept of homogeneity so as to bring even transcendental functions within its scope, we define u as a homogeneous function in x and y of order or degree n, if it can be expressed in the form of $x^n f\left(\frac{y}{x}\right)$.
- This definition also covers the polynomial function (1), which can be written as

$$x^n\left[a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + ... + a_n\left(\frac{y}{x}\right)^n\right] = x^n f\left(\frac{y}{x}\right).$$

$\therefore$ It is a homogeneous function of order $n$.

- To test whether a given function $f(x, y)$, is homogeneous or not we put $x = hx$ and $y = hy$ in it. If we get $f(hx, hy) = h^n f(x, y)$, the function $f(x, y)$ is homogeneous of degree $n$, otherwise $f(x, y)$ is not a homogeneous function.
- A homogeneous function in $x$ and $y$ of degree $n$ can also be written as $y^n f\left(\frac{x}{y}\right)$.
- A function $u$ of three variables $x, y, z$ is said to be homogeneous function of degree $n$, if it can be expressed in the form

$$u = x^n f_1\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad y^n f_2\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad z^n f_3\left(\frac{x}{z}, \frac{y}{z}\right).$$

In general, a function $u$ of several variables $x_1, x_2, ..., x_n$ is said to be homogeneous function of degree $m$ if it can be expressed in the form $u = x_1^m f_1\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_n}{x_1}\right)$ or $x_2^m f_2\left(\frac{x_1}{x_2}, \frac{x_3}{x_2}, ..., \frac{x_n}{x_2}\right)$ or etc.

Notes

**THEOREM 1.** *If u is a homogeneous function of x and y of degree n, then* $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *are homogeneous function of degree (n – 1) each.*

**Proof.** Since, $u$ is a homogeneous function of $x$ and $y$ of degree $n$ therefore, $u$ can be expressed as $u = x^n f\left(\frac{y}{x}\right)$. ...(1)

Now from (1)

$$\frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = x^{n-1}\left[nf\left(\frac{y}{x}\right) + f'\left(\frac{y}{x}\right)\left(-\frac{y}{x}\right)\right]$$

$$= x^{n-1} \times \text{a function of } \frac{y}{x} = x^{n-1}g\left(\frac{y}{x}\right) \text{(say)}.$$

which is a homogeneous function of degree $(n - 1)$.

Also, $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right) = x^{n-1} \times \text{a function of } \frac{y}{x}$$

$$= x^{n-1}g\left(\frac{y}{x}\right) \text{(say)}.$$

which is a homogeneous function of $x$ and $y$ of degree $(n - 1)$.

**THEOREM 2. [Euler's Theorem on Homogeneous Functions].**

*If u be a homogeneous function of x and y of degree n, then* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

**Proof.** Since, $u$ is a homogeneous function of $x$ and $y$ of degree $n$ therefore, $u$ can be expressed as

$$u = x^n f\left(\frac{y}{x}\right).$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right).$$

Also, $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right).$$

Now, $$\text{L.H.S.} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)\right] + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) - yx^{n-1}f'\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right) = nx^n f\left(\frac{y}{x}\right) = nu = \text{R.H.S.}$$

## REMARK

- Euler's theorem can be extended to a homogeneous functions of several variables. Thus, if $u$ be the function of $m$ independent variables $x_1, x_2, ..., x_m$ of degree $n$ then, Euler's theorem states that $x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + ... + x_m\frac{\partial u}{\partial x_m} = nu$.

**THEOREM 3.** *If u is a homogeneous function in x and y of degree n, then*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

**Proof.** Since, $u$ is a homogeneous function in $x$ and $y$ of degree $n$ therefore, by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad ...(1)$$

Differentiating (1) partially *w.r.t.* $x$, we get

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(nu)$$

$$\left(\because \text{Each of } \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} \text{ is a function of both } x \text{ and } y\right)$$

Notes

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \quad \ldots(2)$$

Again differentiating (2) partially w.r.t. $y$, we get

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial y} \quad \ldots(3)$$

Now, multiply (2) by $x$, (3) by $y$ and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial y}\right]=(n-1)nu=n(n-1)u.$$

**REMARK**

- If $z$ is a homogeneous function of $x$ and $y$ of degree $n$ and if $z = f(u)$, then we have the following results :

  (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=G(u)$

  (ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=G(u)[G'(u)-1]$

## Solved Examples

**Example 1.** *Verify the Euler's theorem for the function $u = axy + byz + czx$.*

**Solution.** We have $u = axy + byz + czx$. ...(1)

which is a homogeneous function of $x, y$ and $z$ of degree 2.

To verify the Euler's theorem, we must show $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$

Now, $\frac{\partial u}{\partial x}=ay+cz, \frac{\partial u}{\partial y}=ax+bz, \frac{\partial u}{\partial z}=by+cx.$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x(ay+cz)+y(ax+bz)+z(by+cx).$$

$$=2(axy+byz+czx)=2u.$$

Hence, Euler's theorem is verified.

**Example 2.** *If $u=\sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.*

**Solution.** We have $\sin u=\left[\frac{x^2+y^2}{x+y}\right]$

Let $v=\frac{x^2+y^2}{x+y}$

$\Rightarrow v$ is a homogeneous of $x$ and $y$ of degree 1.

Then, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v \quad \ldots(1)$$

$$v=\sin u \Rightarrow \frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x}$$

and $$\frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}.$$

Put these values in (1), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=v$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{v}{\cos u}=\frac{\sin u}{\cos u}=\tan u.$$

**Example 3.** *If* $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u.$$

**Solution.** We have $u=\tan^{-1}\frac{x^3+y^3}{x-y}$

$$\therefore\ \tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\frac{y}{x}\right]}=x^2 f\left(\frac{y}{x}\right)$$

tan $u$ is of the form $x^n f\left(\frac{y}{x}\right)$ with $n=2$.

$\therefore$ tan $u$ is a homogeneous function in $x, y$ of degree 2. Then, by Euler's theorem

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=2\sin u\cos u=\sin 2u \qquad \ldots(1)$$

Differentiate (1) partially *w.r.t.* $x$, we get

$$\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

$$\therefore\ x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)\frac{\partial u}{\partial x} \qquad \ldots(2)$$

Interchanging $x$ and $y$ in (2), we get

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)\frac{\partial u}{\partial y} \qquad \ldots(3)$$

Now multiplying (2) by $x$, (3) by $y$ and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=(2\cos 2u-1).\sin 2u$$

$$=[2(1-2\sin^2 u)-1]\sin 2u=(1-4\sin^2 u)\sin 2u.$$

**Example 4.** *If* $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0\times u=0$

**Solution.** We have $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=x^0\left[\sin^{-1}\left(\frac{1}{y/x}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right]$

$\Rightarrow u$ is a homogeneous function of order 0.

Then, by Euler's theorem,we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0\times u=0$$

Notes

**Example 5.** *If* $u = \left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)$.

*Apply Euler's theorem to find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

**Solution.** Here, we have

$$u(x,y) = \left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)$$

$$\Rightarrow u(tx,ty) = t^{\frac{1}{4}}(x^{\frac{1}{4}} + y^{\frac{1}{4}})t^{\frac{1}{5}}(x^{\frac{1}{5}} + y^{\frac{1}{5}})$$

$$= t^{9/20}(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5}) = t^{9/20}u(x,y)$$

Clearly, $u$ is a homogeneous function of degree $\frac{9}{20}$.

Hence, by Euler's theorem we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9}{20}u$.

**Example 6.** *Verify Euler's theoerm for* $f(x,y,z) = 3x^2yz + 5xy^2z + 4z^4$.

**Solution.** Let $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$.

$$\therefore \frac{\partial f}{\partial x} = 6xyz + 5y^2z; \frac{\partial f}{\partial y} = 3x^2z + 10xyz$$

and $\frac{\partial f}{\partial z} = 3x^2y + 5xy^2 + 16z^3$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(6xyz + 5y^2z) + y(3x^2z + 10xyz) + z(3x^2y + 5xy^2 + 16z^3)$$

$$= 4(3x^2yz + 5xy^2z + 4z^4) = 4f \quad \ldots(1)$$

Also, $f(x,y,z) = x^4\left[3.\frac{y}{x}.\frac{z}{x} + 5\left(\frac{y}{x}\right)^2\left(\frac{z}{x}\right) + 4\left(\frac{z}{x}\right)^4\right]$

is a homogeneous function of $x, y, z$ of degree 4.

Hence, by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 4f. \quad \ldots(2)$$

From (1) and (2) we conclude that Euler's theorem is verified.

**Example 7.** *If* $u = f\left(\frac{y}{x}\right) + \sqrt{x^2 + y^2}$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$.

**Solution.** Let us write $u = v + w$

where $v = f\left(\frac{y}{x}\right) = x^0 f\left(\frac{y}{x}\right)$

and $w = \sqrt{x^2 + y^2} = x\sqrt{1 + \left(\frac{y}{x}\right)^2}$

Therefore, $v$ and $w$ are homogeneous function of degree 0 and 1 in $x$ and $y$ respectively. Hence, by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.v = 0 \quad \ldots(1)$$

and $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 1.w = \sqrt{x^2 + y^2} \quad \ldots(2)$

On adding (1) and (2), we get

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = \sqrt{x^2 + y^2} \quad \ldots(3)$$

Notes

Now, since $u = v + w$, then using (3) we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$.

**Example 8.** *If* $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, *then show that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

**Solution.** Let $u = x^n f_1\left(\frac{y}{x}\right), v = y^{-n} f_2\left(\frac{x}{y}\right)$ ...(1)

$\therefore \quad z = u + v$ ...(2)

Clearly, $u$ and $v$ are homogeneous functions of degree $n$ and $-n$ respectively. Then by Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad ...(3)$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = (-n).v \quad ...(4)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \quad ...(5)$$

and $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = (-n)(-n-1)v = n(n+1)v \quad ...(6)$$

Since $z = u + v \quad \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$

and $$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad ...(7)$$

Adding (3) and (4) and using (7) we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n(u - v) \quad ...(8)$$

Similarly, adding (5) and (6) and using (7) we get

$$x^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right] + 2xy\left[\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x\partial y}\right] + y^2\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$= n(n-1)u + n(n+1)v$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n^2(u+v) - n(u-v)$$

$$= n^2 z - \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) \quad \text{(Using 8)}$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

**Example9.** *If* $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, *find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

**Solution.** Let $v = \frac{x^3y^3z^3}{x^3+y^3+z^3}$ and $w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Notes

Clearly, $v = x^6 \left[ \dfrac{\left(\frac{y}{x}\right)^3 \left(\frac{z}{x}\right)^3}{1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3} \right]$ is a homogeneous function of degree 6.

$\therefore$ By Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=6v \qquad \ldots(1)$$

Further, $w = \log \left[ \dfrac{\frac{y}{x}+\frac{y}{x}\cdot\frac{z}{x}+\frac{z}{x}}{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} \right]$ is a homogeneous function of degree zero.

Then, by Euler's theorem

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=0 \qquad \ldots(2)$$

Adding (1) and (2), we get

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=6v$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}\right)+z\left(\frac{\partial u}{\partial z}\right)=6.\frac{x^3y^3z^3}{x^3+y^3+z^3}$$

## STUDENT ACTIVITY

**1.** If $\sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-3\tan u$.

**2.** If $u = \dfrac{x}{y+z}+\dfrac{y}{z+x}+\dfrac{z}{x+y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0$.

**3.** If $\log u = \dfrac{x^3+y^3}{3x+4y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2u\log u$.

Notes

4. If $u = x^3 + y^3 + z^3 + 3xyz$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$.

## TEST YOURSELF

1. Verify the Euler's theorem for the following functions :

(i) $u = \frac{x(x^3 - y^3)}{x^3 + y^3}$ (ii) $u = x^n \sin\left(\frac{y}{x}\right)$ (iii) $u = x^n \log\left(\frac{y}{x}\right)$ (iv) $u = \frac{1}{\sqrt{x^2 + y^2}}$

(v) $u = x^n \sin\frac{y}{x}$ (vi) $x^4 \log\frac{y}{x}$ (vii) $u = \log\left(\frac{x^2 + y^2}{xy}\right)$

(viii) $u = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

2. (i) If $u = xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

(ii) If $u = f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

(iii) If $u = xyf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

(iv) If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, show by Euler's theorem : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

3. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \text{ and } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$$

4. If $u = \tan^{-1}\frac{y}{x}$, show that(using Euler's theorem) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

5. If $u = \sin^{-1}\frac{x + y}{\sqrt{x} + \sqrt{y}}$, show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$ (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

6. If $u = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

7. (i) If $u = \log\frac{x^4 + y^4}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

(ii) If $u = \log\frac{x^3 + y^3}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

8. If $\sin u = \frac{x^2 y^2}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

9. (i) If $u = \frac{x^2 y^2}{x + y}$, show that $y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial y}$.

(ii) If $u = \frac{xy}{x + y}$, show that $x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Notes

(iii) If $u = \dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x} = 2\dfrac{\partial u}{\partial x}$.

**10.** If $u = xf_1\left(\dfrac{y}{x}\right) + f_2\left(\dfrac{y}{x}\right)$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

**11.** (i) If $u = \log\left(\sqrt{x} + \sqrt{y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}$.

(ii) If $u = \log\dfrac{x^4 + y^4 + x^2y^2}{x + y + \sqrt{xy}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

**12.** If $z$ be a homogeneous function of degree $n$, show that $x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x \partial y} = (n-1)\dfrac{\partial z}{\partial x}$.

## 3.5 TOTAL DIFFERENTIAL

Let
$$u = f(x, y) \quad \ldots(1)$$
be the given function of $x$ and $y$, which have continuous partial derivatives of first order *w.r.t.* $x$ and $y$.

Let $\delta x$ and $\delta y$ be the increments in $x$ and $y$ respectively and let $\delta u$ be the consequent change in $u$, then we have

$$u + \delta u = f(x + \delta x, y + \delta y)$$
$$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - f(x, y) \quad \ldots(2)$$
$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$$
$$\Rightarrow \quad \frac{\delta u}{\delta t} = \frac{[f(x + \delta x, y + \delta y) - f(x, y + \delta y)]}{\delta t} + \frac{[f(x, y + \delta y) - f(x, y)]}{\delta t}$$

Now,
$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t}$$
$$= \lim_{\delta t \to 0}\left[\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}\frac{\delta y}{\delta t}\right] \quad \ldots(3)$$

Since $\delta x$ and $\delta y$ tends to zero, when $\delta t \to 0$ so we have

$$\lim_{\delta x \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}.$$

Similarly, $$\lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} \text{ and } \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Therefore, from (3), we get $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

**REMARKS**

- This result can be extended as follows :
If $u = f(x_1, x_2, \ldots, x_m)$ and $x_1, x_2, \ldots, x_m$ all are functions of $t$, then
$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt} + \ldots + \frac{\partial u}{\partial x_m}.\frac{dx_m}{dt}.$$
- The differentials $dx$ and $dy$ of the independent variables $x$ and $y$ are the actual changes $\delta x$ and $\delta y$ but the differential $du$ of the dependent variable $u$ is not the same as the change $\delta u$, it being the principal part of the increment $\delta u$.

## 3.6 IMPLICIT RELATION OF x AND y

In most of the cases, we are mainly concerned with the case in which $y$ is expressed explicity *i.e.*, directly in terms of $x$. There are so many cases in which $y$ is not expreesed directly in terms of $x$, but functionally it is implied by an algebraic relation $f(x, y) = 0$ connecting $x$ and $y$.

The relation of the type $f(x, y) = c$, where $y$ is not explicity in terms of $x$ are called implicit function.

## 3.7 DIFFERENTIATION OF IMPLICIT FUNCTIONS

*To find $\frac{dy}{dx}$ for an implicit function $f(x, y) = 0$ or $f(x, y) = c$:*

Let $f(x, y)$ be a function of two variables $x$ and $y$ and $y$ itself is a function of $x$ *i.e.*, $f(x, y)$ may be consider as a composite function of $x$. Then, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \quad \Rightarrow \quad \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \qquad \ldots(1)$$

Since $f(x, y) = 0$, therefore $\frac{df}{dx} = 0$.

Now from (1), we have $\quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = -\frac{f_x}{f_y}, \text{ provided } f_y \neq 0.$$

### Solved Examples

**Example 1.** *If* $x^y + y^x = a^b$. *Find* $\frac{dy}{dx}$.

**Solution.** Let $f(x, y) = x^y + y^x - a^b \quad \Rightarrow \quad f(x, y) = 0$

Therefore $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$.

**Example 2.** *If* $u = \log [(x^2 + y^2)/xy]$, *find* $du$.

**Solution.** Let $u = \log (x^2 + y^2) - \log x - \log y$.

$$\therefore \frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x} = \frac{2x^2 - x^2 - y^2}{x(x^2 + y^2)} = \frac{x^2 - y^2}{x(x^2 + y^2)}$$

and $\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y} = \frac{2y^2 - x^2 - y^2}{y(x^2 + y^2)} = \frac{y^2 - x^2}{y(x^2 + y^2)}$

Now, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{(x^2 - y^2)}{x(x^2 + y^2)}dx + \frac{(y^2 - x^2)}{y(x^2 + y^2)}dy$

$$= \frac{(x^2 - y^2)}{xy(x^2 + y^2)}(ydx - xdy).$$

**Example 3.** *If* $f(x, y) = 0$ *and* $g(y, z) = 0$, *show that* $\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y}$.

**Solution.** Let $f(x, y) = 0$, then we have

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}. \qquad \ldots(1)$$

Also, let $g(y, z) = 0$

$$\Rightarrow \quad \frac{dz}{dy} = -\frac{\partial g/\partial y}{\partial g/\partial z}. \qquad \ldots(2)$$

Now, from (1) and (2), we have

$$\frac{dy}{dx}\cdot\frac{dz}{dy} = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y}\right) \Big/ \left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial z}\right)$$

$$\Rightarrow \quad \frac{dz}{dx}\cdot\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y}$$

Notes

**Example 4.** *If $u = x^2y$, where $x^2 + xy + y^2 = 1$. Find $\frac{du}{dx}$.*

**Solution.** We know that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$. ...(1)

Given that $u = x^2y$

$$\frac{\partial u}{\partial x} = 2xy \text{ and } \frac{\partial u}{\partial y} = x^2$$

$$\therefore\ f(x,y) = x^2 + xy + y^2 - 1$$

Then $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x+y}{x+2y}$

Putting all these values in (1), we get

$$\frac{du}{dx} = 2xy + x^2.\left(-\frac{2x+y}{x+2y}\right) = 2xy - \frac{x^2(2x+y)}{x+2y}$$

**Example 5.** *If $u = x \log (xy)$, where $x^3 + y^3 + 3xy = 1$. Find $\frac{du}{dx}$.*

**Solution.** We have $u = x \log (xy)$. ...(1)

$$\Rightarrow \frac{\partial u}{\partial x} = x\left(\frac{1}{xy}.y\right) + \log xy = 1 + \log xy$$

and $\frac{\partial u}{\partial y} = x\left(\frac{1}{xy}.x\right) = \frac{x}{y}$

Also it is given that

$$x^3 + y^3 + 3xy = 1 \qquad ...(2)$$

Differentiating (2) we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x^2+y}{x+y^2}\right)$$

Now, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} = 1 + \log(xy) + \frac{x}{y}\left\{-\frac{(x^2+y)}{(y^2+x)}\right\}$

$$= 1 + \log(xy) - \frac{x(x^2+y)}{y(y^2+x)}$$

**Example 6.** *If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.*

**Solution.** Suppose $v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$ ...(1)

Then clearly, $u = u(v, w)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \qquad ...(2)$$

Further, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0)$

$$\Rightarrow \quad y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \qquad ...(3)$$

Similarly, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}.\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right)$

$$\Rightarrow \quad z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \qquad \ldots(4)$$

Finally, adding (2), (3) and (4), we get

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

**Example 7.** *If $f(x, y) = 0$, show that* $\dfrac{\partial^2 y}{\partial x^2} = -\dfrac{q^2 r - 2pqs + p^2 t}{q^3}$

**Solution.** We have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = \dfrac{-p}{q}$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = \frac{-q\dfrac{dp}{dx} + p\dfrac{dq}{dx}}{q^2} \qquad \ldots(1)$$

Now, $\dfrac{dp}{dx} = \dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y}.\dfrac{dy}{dx} = r + s\left(-\dfrac{p}{q}\right) = \dfrac{qr - ps}{q}$

and $\dfrac{dq}{dx} = \dfrac{\partial q}{\partial x} + \dfrac{\partial q}{\partial y}.\dfrac{dy}{dx} = s + t\left(-\dfrac{p}{q}\right) = \dfrac{qs - pt}{q}$

Putting all these value in (1), we get

$$\frac{d^2 y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr - ps}{q}\right) - p\left(\frac{qs - pt}{q}\right)\right] = -\frac{q^2 r - 2pqs + p^2 t}{q^3}$$

Here, $p = \dfrac{\partial f}{\partial x}, q = \dfrac{\partial f}{\partial y}, r = \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial p}{\partial x}$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}, \; t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$$

## STUDENT ACTIVITY

1. If $u = \sin^{-1}(x - y), x = 3t, y = 4t^3$, show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

2. Show that $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$ where $x = u \cos \alpha - v \sin \alpha, y = u \sin \alpha + v \cos \alpha$.

## TEST YOURSELF

1. If $(\tan x)^y + (y)^{\cot x} = a$. Find the value of $\dfrac{dy}{dx}$.
2. If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$. Find the value of $\dfrac{du}{dx}$.

Notes

3. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

4. If $z$ is a function of $x$ and $y$; where $x = e^{u}+e^{-v}$ and $y = e^{-u}-e^{v}$, show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

5. Find the total derivative of u with respect to $t$, when

   (i) $u=\cosh\left(\frac{y}{x}\right)$, where $x=t^2, y=e^t$

   (ii) $u=e^x \sin y$, where $x=\log t, y=t^2$

6. If $u=\sqrt{(x^2+y^2)}$ and $x^3+y^3+3axy=5a^2$. Find the value of $\frac{du}{dx}$ at $x=a, y=a$.

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from the following implicit relations.

   (i) $x^2+y^2=a^2$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$

8. If $f(x,y,z)=0$, show that $\left(\frac{\partial y}{\partial z}\right)_{x \text{ const.}}\left(\frac{\partial z}{\partial x}\right)_{y \text{ const.}}\left(\frac{\partial x}{\partial y}\right)_{z \text{ const.}}=-1$.

## ANSWERS

1. $-\frac{y(\tan x)^{y-1}\sec^2 x - y^{\cot x}.\log y.\operatorname{cosec}^2 x}{(\tan x)^y \log\tan x + \cot x\, y^{\cot x-1}}$

2. $2x[\cos(x^2+y^2)]\left(1-\frac{a^2}{b^2}\right)$

5. (i) $\frac{du}{dt}=\frac{1}{x^2}(xe^t-2yt)\sinh\frac{y}{x}$ (ii) $\frac{du}{dt}=\frac{e^x}{t}(\sin y+2t^2\cos y)$, where $x=\log t, y=e^t$

6. 0

7. (i) $-\frac{x}{y}, \frac{-a^2}{y^3}$ (ii) $\frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}, \frac{d^2y}{dx^2}=\frac{a^{1/3}}{3x^{4/3}.y^{1/3}}$

## Summary

- If $u = f(x, y)$ be a continuous function of two independent variables $x$ and $y$, then the differential coefficient of $u$ w.r.t. $x$ (regarding $y$ as constant) is called the partial derivative or partial differential co-efficient of $u$ w.r.t. $x$ and is denoted by various symbols such as $\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y), f_x$.
- If $u$ is a function of $x, y$ and we are to differentiate partially w.r.t. $x$ then, $y$ is treated as constant.
- Similarly, if we are to differentiate $u$ partially w.r.t. $y$ then $x$ is treated as constant.
- If $u$ is a function of $x, y, z$ and we are to differentiate partially w.r.t. $x$, then $y$ and $z$ are treated as constant.
- If $z = u \pm v$, where $u$ and $v$ are functions of $x$ and $y$, then $\frac{\partial z}{\partial x}=\frac{\partial u}{\partial x}\pm\frac{\partial v}{\partial x}$ and $\frac{\partial z}{\partial y}=\frac{\partial u}{\partial y}\pm\frac{\partial v}{\partial y}$.
- If $z = uv$, where $u$ and $v$ are functions of $x$ and $y$, then $\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(uv)=u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}$ and

  $\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(uv)=u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}$.
- If $z=\frac{u}{v}$, where $u$, $v$ are functions of $x$ and $y$, then $\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}}{v^2}$ and

  $\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(\frac{u}{v}\right)=\frac{v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}}{v^2}$.
- If $z = f(u)$, where $u$ is a function of $x$ and $y$, then $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}$.
- A function $u = u(x, y)$ is said to be symmetric if, on interchanging $x$ and $y$, $u$ remains unchanged.

Notes

- A function $f(x, y)$ is said to be homogeneous function of degree $n$, if the degree of each of its terms in $x$ and $y$ is equal to $n$. Thus $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n$ is homogeneous function in $x$ and $y$ of order $n$.
- If $u$ be a homogeneous function of $x$ and $y$ of degree $n$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

## Objective Evaluation

### FILL IN THE BLANKS

1. $\cos^{-1}\frac{y}{x}$ is a homogeneous functionof degree ________.
2. If $\phi = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}$ is ________.
3. If $u = e^{my}\cos mx$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$ ________.
4. A function is said to be homogeneous if every term is of ________.
5. An expression in which every term is of the same degree is called ________ function.
6. If $z = f(y/x)$ then $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$ is ________
7. If $z = xy\, f(y/x)$ then $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$ is ________ .
8. If $u = e^{xyz}$, then $\frac{\partial^2 z}{\partial y \partial z}$ is ________ .
9. If $u(x, y)$ is a homogeneous function of $x$ and $y$ of degree $n$, then $x\frac{\partial}{\partial x}(u_x) + y\frac{\partial}{\partial y}(u_x) =$ ________ where $u_x = \frac{\partial u}{\partial x}$.
10. If $u = f(x, y)$, and its partial derivatives are continuous, then order of differentiation is ________ .

### TRUE/FALSE

**Write 'T' for True and 'F' for False statement.**

1. An expression in which every term is of same degree is called homogeneous function. **(T/F)**
2. In homogeneous function every term is not necessarily of same degree. **(T/F)**
3. If $u$ is a homogeneous function of $x$ and $y$ of degree $n$, then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also homogeneous function of degree $n$. **(T/F)**
4. If $x$ and $y$ are connected by an equation of the form $f(x, y) = 0$, then $\frac{dy}{dx}$ is $-\frac{\partial f/\partial x}{\partial f/\partial y}$. **(T/F)**
5. If $u$ is a homogeneous function of degree $n$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to $n$. **(T/F)**
6. If $f(x, y)$ be an implicit function of $x$ and $y$ and $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$, then $\frac{d^2 y}{dx^2} = -\frac{(q^2 r - 2pqs + p^2 t)}{q^3}$. **(T/F)**
7. If $u = \sqrt{(x^2 + y^2 + z^2)}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ is equal to $u$. **(T/F)**
8. The Euler's theorem for homogeneous function is not true for a function of more than two variables. **(T/F)**
9. If $u = f(x, y)$, where $x = g(t)$ and $y = \phi(t)$, then $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$. **(T/F)**
10. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(T/F)**

Notes

### MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. $\sin^{-1}(y/x)$ is a homogeneous function of degree :
   (a) 1 (b) 2 (c) 3 (d) 0
2. If $z = xy\, f\left(\frac{y}{x}\right)$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to :
   (a) $z$ (b) $2z$ (c) $xy$ (d) $yz$
3. If $f = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is :
   (a) $f$ (b) $2f$ (c) $\tan f$ (d) $\sin f$
4. A function $f(x, y)$ is said to be homogeneous of degree $n$ if :
   (a) $f(x, ty) = t^{2n} f(xy)$ (b) it is of the form $x^n f(x/y)$
   (c) it is of the form $x^n f(y/x)$ (d) $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}$
5. If $z = e^{ax} \sin by$, then $\frac{\partial^2 z}{\partial y \partial x}$ is :
   (a) $ae^{ax} \cos by$ (b) $be^{ax} \sin by$ (c) $abe^{ax} \cos by$ (d) $abe^{ax} \sin by$
6. If $z = f(y/x)$ then $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$ is :
   (a) 1 (b) 2 (c) −2 (d) 0
7. If $z = f(x + ay) + \phi(x - ay)$, then $\frac{\partial^2 z}{\partial y^2}$ is :
   (a) $\frac{\partial^2 z}{\partial x^2}$ (b) $a^2\frac{\partial^2 z}{\partial y^2}$ (c) $a^2\frac{\partial^2 z}{\partial x^2}$ (d) $a^2\frac{\partial^2 z}{\partial x \partial y}$
8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is :
   (a) $\frac{-9}{(x+y+z)^2}$ (b) $\frac{3}{x+y+z}$ (c) $\frac{9}{(x+y+z)^2}$ (d) $\frac{-3}{x+y+z}$
9. If $x = r\cos\phi$, $y = r\sin\theta$ then $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$ is :
   (a) $r$ (b) $-r$ (c) 1 (d) −1
10. If $u = \tan^{-1} y/x$ then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is :
   (a) 0 (b) 1 (c) $\sin 2u$ (d) $\cos 2u$

## ANSWERS

### Fill in the Blanks

| | | | | |
|---|---|---|---|---|
| 1. 0 | 2. $\tan\phi$ | 3. 0 | 4. same degree | 5. homogeneous |
| 6. 0 | 7. $2z$ | 8. $4x + 4x^2yz$ | 9. $(n-1)4x$ | 10. immaterial |

### True/False

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. T | 2. F | 3. F | 4. T | 5. F | 6. T | 7. F |
| 8. F | 9. T | 10. T | | | | |

### Multiple Choice Questions

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. (d) | 2. (b) | 3. (c) | 4. (b) | 5. (c) | 6. (d) | 7. (c) |
| 8. (a) | 9. (c) | 10. (a) | | | | |

*Chapter*

# 4 Tangent and Normal

## STRUCTURE

- Introduction
- Polar co-ordinates
- Angle between radius vector and tangent
- Angle of intersection of two curves
- Length of subtangent and subnormal
- Length of the perpendicular from pole to the tangent
- The pedal equation
- Differential coefficient of arc length (Cartesian form)
- Differential coefficient of arc length (Polar form)
  - Summary
  - Objective Evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:

- Some fundamental concepts of tangents
- The euqation of tangent and normal
- The angle between radius vector and tangents
- The concepts of subtangent and subnormal

## 4.1 INTRODUCTION

Let $P$ be a given point and $Q$ be any other point on it. Let $Q$ travel towards $P$ along the curve.

Let $Q$ travel towards $P$ along the curve. Then, the limiting position $PT$ of the secant $PQ$ is known as the tangent to the curve.

The line $PS$ through $P$ which is perpendicular to the tangent $PT$ is called the normal of the curve.

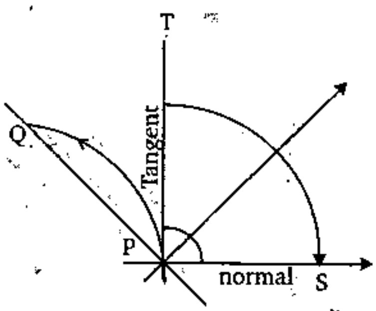


**Fig. 1**

### 4.1.1 SOME FUNDAMENTAL CONCEPTS

(i) Slope of a line, $m = \tan\theta$, where $\theta$ is the angle which the line makes with the positive direction of $x$-axis.

(ii) Slope of the line $ax + by + c = 0$ is given by $m = -\frac{a}{b}$

(iii) Slope of the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is $= \frac{y_2 - y_1}{x_2 - x_1}$

(iv) Slope of $x$-axis $= 0$, Slope of $y$-axis $= \infty$

(v) Two lines are parallel iff $m_1 = m_2$.

(vi) Two lines are perpendicular iff $m_1 m_2 = -1$.

(vii) Angle between two lines having slopes $m_1$ and $m_2$ is given by $\theta = \tan\left(\frac{m_1 \quad m_2}{m_1 m_2}\right)$

(viii) Equation of the line (one point form)

$$y - y_1 = m(x - x_1)$$

passing through the point $(x_1, y_1)$.

(ix) Perpendicular distance formula $= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Notes

## 4.1.2 EQUATION OF THE TANGENT

Let $y = f(x)$ be the equation of the curve, and $P(x_1, y_1)$ be any given point on this curve.

Let $Q = Q(x + \delta x, y + \delta y)$ be any neighbouring point of $P$. Let $PT$ be the tangent at the point $(x_1, y_1)$.

The slope of the tangent at $(x_1, y_1) = \frac{dy_1}{dx_1}$.

Now, tangent is a line through the point $P(x_1, y_1)$ and its slope $m = \frac{dy_1}{dx_1}$.

Hence, by Co-ordinate Geometry, the equation of the tangent is $y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$.

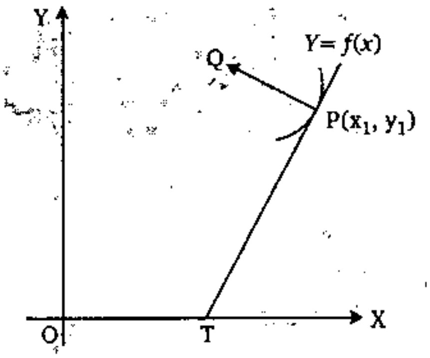


Fig. 2.

### REMARKS

- It should be clearly understood that by $\frac{dy_1}{dx_1}$ we mean the value of $\frac{dy}{dx}$ at $(x_1, y_1)$ and not as derivative of $y_1$ with respect to $x_1$.
- The equation of the tangent at a point $t_1$ to the curve $x = f(t), y = g(t)$ is given by

$$y - g(t_1) = \frac{g'(t_1)}{f'(t_1)}[x - f(t_1)].$$

## 4.1.3 GEOMETRICAL MEANING OF $\frac{dy}{dx}$

Let $y = f(x)$ be the given function and let it be represented by the curve $AB$. Take two neighbouring points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ on the curve $AB$. Join $PQ$ and let $PQ$ be produced to meet $OX$ at the point $R$.

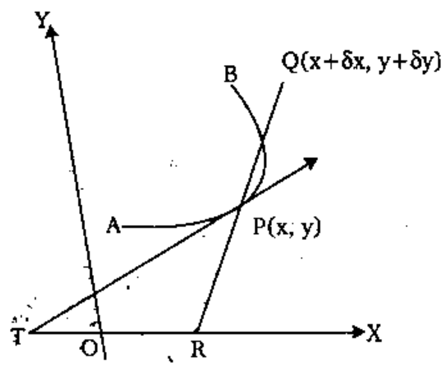


Fig. 3.

Slope of the secant $PQ$

$$= \frac{y + \delta y - y}{x + \delta x - x} = \frac{\delta y}{\delta x}. \qquad \ldots(1)$$

Now, let the point $Q$ move along the curve and approach the point $P$ in the limiting position. $\delta x \to 0$, $\delta y \to 0$ and the secant $PQ$ becomes the tangent $PT$ at $P$.

Therefore, from (1)

Slope of the tangent $PT$ at $(x, y) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

*i.e.*, the value of the derivative at a point $P$ of the curve is equal to the slope of tangent at that point to the curve.

### REMARKS

- If the tangent at a point on the curve $y = f(x)$ is parallel to $x$-axis, its slope is zero *i.e.*, $\frac{dy}{dx}$ at the point = 0.
- If the tangent at a point on the curve is perpendicular to $x$-axis, *i.e.*, parallel to $y$-axis. Its slope is $\infty$, *i.e.*, $\frac{dy}{dx}$ at the point = $\infty$.

## 4.13 EQUATION OF THE NORMAL

The normal to a curve at a given point is a line perpendicular to the tangent at that point and passes through the point. The slope of the normal at point $P(x_1, y_1)$ will be negative reciprocal of the slope of the tangent.

Hence, the slope of the normal at $(x_1, y_1) = -\frac{1}{dy_1/dx_1}$

$\therefore$ The equation of the normal at $P(x_1, y_1)$ is $y - y_1 = -\frac{1}{dy_1/dx_1}(x - x_1)$

## Solved Examples

**Example 1.** *Find the point on the curve $y = x^2 - x - 8$ at which the tangent is parallel to x-axis.*

**Solution.** Let the required point be $(x_1, y_1)$, then

$$y_1 = x_1^2 - x_1 - 8 \qquad \text{...(i)}$$

Given curve $y = x^2 - x - 8$

$$\therefore \quad \frac{dy}{dx} = 2x - 1$$

$\therefore$ The slope of the tangent at point

$$(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2x_1 - 1 \qquad \text{...(ii)}$$

Since the tangent is parallel to $x$-axis, therefore

$$m = \frac{dy}{dx} = 0$$

$\therefore$ From eqn. (ii),

$$2x_1 - 1 = 0 \Rightarrow x_1 = \frac{1}{2}$$

Putting $x_1 = \frac{1}{2}$ in eqn. (i), we get

$$y_1 = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 8 = \frac{1}{4} - \frac{1}{2} - 8$$

$$\therefore \quad y_1 = -\frac{33}{4}$$

Hence, required point is $\left(\frac{1}{2}, -\frac{33}{4}\right)$.

**Example 2.** *Prove that the straight line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve cut y-axis.*

**Solution.** Equation of the tangent

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{...(i)}$$

Equation of the curve

$$y = be^{-x/a} \qquad \text{...(ii)}$$

Since, curve cut $y$-axis. So, at the point where curve cut $y$-axis, $x = 0$. Putting in eqn. (ii), we get $y = b$

$\therefore$ Required point = $(0, b)$

We have to prove that the tangent at point $(0, b)$ on the curve is eqn. (i). From eqn. (ii);

$$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$$

Notes

$$\therefore \left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b}{a}$$

Equation of the tangent at point (0, b) is

$$y - b = -\frac{b}{a}(x - 0)$$

$$\Rightarrow \quad \frac{y-b}{b} = -\frac{x}{a}$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

**Example 3.** *Find the equation of the normal to the parabola $y^2 = 4ax$ at $(x_1, y_1)$.*

**Solution.** The given curve $y^2 = 4ax$

Differentiating w.r.t. $x$, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{2a}{y_1}$$

$$\therefore \text{The slope of the normal of the parabola} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = -\frac{y_1}{2a}$$

$\therefore$ The equation of the normal of the parabola at the point $(x_1, y_1)$ is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

$$\Rightarrow \quad \frac{y-y_1}{-y_1} = \frac{(x-x_1)}{2a}$$

**Example 4.** *Find the point on the curve $9x^2+4y^2=36$ at which the equation of the normal is (i) parallel to x-axis (ii) parallel to y-axis.*

**Solution.** The given curve $9x^2 + 4y^2 = 36$ ...(i)

Let $(x_1, y_1)$ be the required point on the curve, therefore

$$9x_1^2 + 4y_1^2 = 36 \quad \text{...(ii)}$$

Differentiating eqn. (i) w.r.t. $x$, we get

$$18x + 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \frac{-9x}{4y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{9x_1}{4y_1}$$

$$\therefore \text{Slope of the normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = \frac{4y_1}{9x_1}$$

(i) Since normal is parallel to $x$-axis, therefore

$$\frac{4y_1}{9x_1} = 0 \quad \Rightarrow \quad y_1 = 0$$

From eqn. (ii) $\quad 9x_1^2 = 36 \Rightarrow x_1 = \pm 2$

$\therefore$ Required point is $(\pm 2, 0)$

(ii) Since normal is parallel to $y$-axis

$$\therefore \quad \frac{4y_1}{9x_1} = \infty \quad \Rightarrow \quad \frac{9x_1}{4y_1} = 0$$

$\therefore \quad x_1 = 0$

From eqn. (ii)

$$4y_1^2 = 36 \quad \Rightarrow \quad y_1^2 = 9$$

$\therefore \quad y_1 = \pm 3$

$\therefore$ Required point is $(0, \pm 3)$.

## 4.2 POLAR CO-ORDINATES

Let $OX$ be a fixed straight line through fixed point $O$. The fixed point $O$ is called the pole, or the origin and the fixed straight line $OX$ is called initial line or the polar axis.

Let $P$ be any point in the plane through the line $OX$. Join $OP$, then

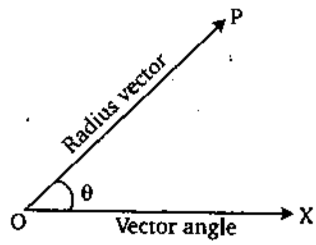


Fig. 4.

(i) The length $OP$ is called the radius vector of the point $P$ and is denoted by $r$.

(ii) The angle $XOP$ is called the vectorial angle of the point $P$ and denoted by $\theta$.

(iii) The number $r$ and $\theta$ taken together in this order and called $p$, the polar-co-ordinates of the point $P$ and we write it as $P(r, \theta)$.

(iv) If $(x, y)$ are the co-ordinates of $P$ referred to cartesian system, then it can be easily found that $x = r\cos\theta$, $y = r\sin\theta$.

## 4.3 ANGLE BETWEEN RADIUS VECTOR AND TANGENT

Let $(r, \theta)$ be the co-ordinate of any point $P'$ on the curve $r = f(\theta)$. Let the tangent at $P$ makes an angle $\psi$ with $OX$.

Let $\phi$ be the angle between the radius vector and the tangent at $P$, *i.e.*, $\angle MPN = \phi$ is the angle between the radius vector $OP$ and the tangent at $P$ to the curve $r = f(\theta)$.

To show that for any point $(r, \theta)$ of the curve $r = f(\theta)$, the angle $\phi$ between the radius vector and tangent is given by $\tan\phi = r\dfrac{d\theta}{dr}$.

Let $P(r, \theta)$ be any point on the given curve

$$r = f(\theta) \text{ or } f(r, \theta) = 0.$$

Let us suppose $Q(r + \delta r, \theta + \delta\theta)$ be the point in the neighbourhood of $P$ on the curve.

Join $OP$, $OQ$, $PQ$, then

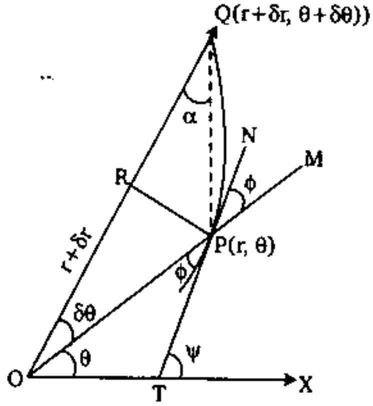


Fig. 5.

$$OP = r,\ OQ = r + \delta r$$

$\angle XOP = \theta$, $\angle XOQ = \theta + \delta\theta$ and $\angle POQ = \delta\theta$.

Draw $PR \perp OQ$ and $\angle PQR = \alpha$.

Now, let the angle between the radius vector $OP$ and the tangent $PT$ is $\phi$ *i.e.*,

$$\angle OPT = \phi$$

Also, we have

$$\frac{PR}{OP} = \sin\delta\theta \quad \Rightarrow PR = r\sin\delta\theta$$

$$RQ = OQ - OR = (r + \delta r) - OP\cos\delta\theta$$

$$= r + \delta r - r\cos\delta\theta$$

$$= \delta r + r(1 - \cos\delta\theta) = \delta r + 2r\sin^2\frac{\delta\theta}{2}.$$

Notes

$$\tan\alpha = \frac{PR}{QR} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$

Dividing the numerator and denominator by $\delta\theta$, we get

$$\tan\alpha = \frac{r.\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r.\dfrac{\sin\delta\theta/2}{\delta\theta/2}\sin\dfrac{\delta\theta}{2}}$$

when $Q \to P$ along the curve $\alpha \to \phi$ ($\because$ $PQ$ becomes the tangent $PT$ and $OQ$ coincides with OP).

$$\tan\phi = \lim_{Q\to P}\tan\alpha = \lim_{\delta\theta\to 0}\frac{r.\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r.\dfrac{\sin\delta\theta/2}{\delta\theta/2}\sin\dfrac{\delta\theta}{2}} = \frac{r.1}{dr/d\theta + r.1.0} = \frac{r}{dr/d\theta}$$

Hence, $\tan\phi = r\dfrac{d\theta}{dr}$.

**REMARKS**

- $\phi$ is the angle between the radius vector and tangent and taken to be positive when measured in the anticlockwise direction.
- Relation between $\theta$, $\phi$ and $\psi$ is $\psi = \theta + \phi$.

## 4.4 ANGLE OF INTERSECTION OF TWO CURVES

If the tangent to the two curves make angle $\phi_1$ and $\phi_2$ with the common radius vector to their point of intersection, then angle between the curves.

$$= \text{angle between tangents} = |\phi_1 - \phi_2|.$$

**REMARKS**

- The two curves intersect orthogonally if $\tan\phi_1 \tan\phi_2 = -1$.
- If $\dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1.\tan\phi_2}$ is positive, we shall get acute angle of intersection at $P$ and if $\dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1.\tan\phi_2}$ is negative, we get the obtuse angle of intersection at $P$.

## 4.5 LENGTH OF SUBTANGENT AND SUBNORMAL

Let $P$ be any point $(r, \theta)$ on a curve $f(r, \theta) = 0$. Let the tangent and normal at $P$ meet the straight line through the pole $O$ perpendicular to the radius vector $OP$ in $T$ and $N$ respectively. Then $OT$ and $ON$ are called polar subtangent and polar subnormal at $P$.

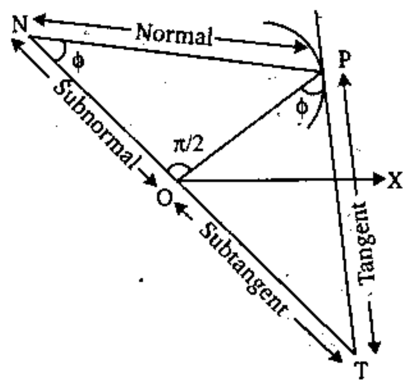


Fig. 6

Hence,

Polar subtangent $= r^2\dfrac{d\theta}{dr}$

Polar subnormal $= \dfrac{dr}{d\theta}$

## 4.6 LENGTH OF THE PERPENDICULAR FROM POLE TO THE TANGENT

Let $p$ be the length of the perpendicular from the pole to the tangent at any point $(r, \theta)$ of a curve $r = f(\theta)$, then

(i) $p = r\sin\phi$

(ii) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}.\left(\dfrac{dr}{d\theta}\right)^2$

(iii) $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$ where $u = \dfrac{1}{r}$

**Proof.** (i) Let $PT$ be the tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$ making an angle $\psi$ with the initial line $OX$.

From the pole $O$, draw $OR \perp$ to the tangent $PT$.

$\therefore \qquad OR = p.$

Joint $OP$, also, $\quad \angle OPT = \phi.$

Now from figure, we have

$$\frac{OR}{OP} = \sin\phi \quad \Rightarrow \quad \frac{p}{r} = \sin\phi$$

$$\Rightarrow \qquad p = r\sin\phi$$

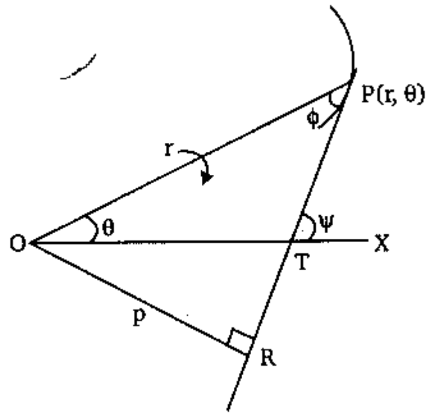


**Fig. 7.**

(ii) From (i), we have

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}\operatorname{cosec}^2\phi \qquad \text{...(1)}$$

Also, $\qquad \tan\phi = r\frac{d\theta}{dr}.$

$$\therefore \operatorname{cosec}^2\phi = 1 + \cot^2\phi = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

Substitute it in (1), we get

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] \quad \Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

(iii) Put $r = \frac{1}{u}$ in (ii),

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \Rightarrow u^2 + u^4\cdot\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2 \qquad \left(\because r = \frac{1}{u} \Rightarrow \frac{dr}{d\theta} = -\frac{1}{u^2}\cdot\frac{du}{d\theta}\right)$$

$$\Rightarrow \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

## 4.7 THE PEDAL EQUATION

Let $r$ be the distance of any point on the curve from the origin (or pole), and $p$, is the length prependicular from the origin to the tangent at that point, then

*The relation between p and r, where r is the distance of any point on the curve from the origin (or pole) and p is perpendicular from origin (or pole) to the tangent at that point is called the Pedal equation of the curve.*

### 4.7.1 PEDAL EQUATION OF A CURVE WHOSE CARTESIAN EQUATION IS GIVEN

Let the equation of the curve is

$$f(x, y) = 0 \qquad \text{...(1)}$$

Then, the equation of the tangent at any point $(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) = y_1(X - x) \text{ where } y_1 = \frac{dy}{dx}$$

$\Rightarrow \quad Xy_1 - Y + y - xy_1 = 0.$

If $p$ be the length prependicular from the origin to this tangent, then

$$p = \frac{y - xy_1}{\sqrt{1 + y_1^2}} \qquad \text{...(2)}$$

Also, $\qquad r^2 = x^2 + y^2 \qquad \text{...(3)}$

Eliminating $x, y$ from the equation (1), (2) and (3), we get the required pedal equation of the curve (1).

## 4.7.2 PEDAL EQUATION OF A CURVE WHOSE POLAR EQUATION IS GIVEN

Let $r = f(\theta)$ ...(1) be the polar curve. Find $\phi$ in terms of $\theta$.

Eliminating $\theta$ and $\phi$ from both the above equations and $p = r \sin \phi$, we get the required pedal equation of curve (1).

**REMARK**

- The pedal equation is sometimes more conveniently obtained by eliminating $\theta$ between (1) and the equation $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

## 4.8 DIFFERENTIAL COEFFICIENT OF ARC LENGTH (CARTESIAN FORM)

Let $y = f(x)$ be the given curve and $s$ denote the length of the arc, then

$$\frac{ds}{dx} = \pm\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

**REMARKS**

- If the equation of the curve is $x = f(y)$, then $\frac{ds}{dy} = \pm\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}$
- If the given equation is in parametric form *i.e.*, $x = f_1(t), y = f_2(t)$, then $\frac{ds}{dt} = \pm\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}$

## 4.9 DIFFERENTIAL COEFFICIENT OF ARC LENGTH (POLAR FORM)

*To prove that* $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$ *where* $r = f(\theta)$ *is the polar form of curve* :

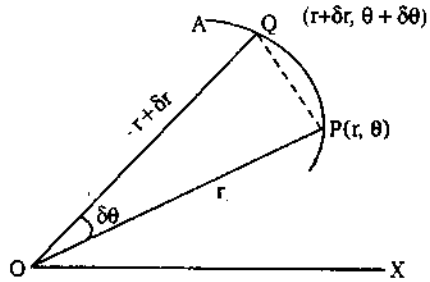


Fig. 8

Let $r = f(\theta)$ be the equation of the curve and $s$ denote the length of arc $AP$. Obviously $s$ is a function of $\theta$. Let $Q$ be the neighbouring point of $P$ such that

$AQ = s + \delta s \qquad \Rightarrow PQ = \delta s.$

As $Q \to P$, $\delta\theta \to \theta$ and $\delta r \to 0$

From $\Delta OPQ$, we have

$$(\text{chord } PQ)^2 = OP^2 + OQ^2 - 2OP.OQ\cos(\angle QOP)$$
$$= r^2 + (r+\delta r)^2 - 2r(r+\delta r)\cos\delta\theta$$
$$= (\delta r)^2 + 2r\delta r(1-\cos\delta\theta) + 2r^2(1-\cos\delta\theta)$$

Dividing by $(\delta\theta)^2$, we get

$$\left(\frac{\text{chord } PQ}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 .\delta r + r^2\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2$$

and $$\left(\frac{\text{chord } PQ}{\delta s}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 .\delta r + r^2\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2$$

Taking limit as $Q \to P$, we have

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r.1.0 + r^2.1 \qquad \left[\because \lim_{Q\to P} \frac{\text{chord } PQ}{PQ(=\delta s)} = 1 \text{ and } \lim_{\delta\theta\to\theta} \frac{\delta r}{\delta\theta} = \frac{dr}{d\theta}\right]$$

$$\Rightarrow \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \quad \Rightarrow \quad \frac{ds}{d\theta} = \pm\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

**REMARKS**

- Here + or − sign is to be taken according as $s$ increases or decreases as $\theta$ increases, we have

$$\frac{ds}{d\theta} = \pm\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

- If $\theta = f(r)$ is the given equation of the curve, then

$$\frac{ds}{dr} = \pm\sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}}$$

- The result $\cos\phi = \frac{dr}{ds}$ and $\sin\phi = r\frac{d\theta}{ds}$ can be remember with the help of adjoining figure(9).

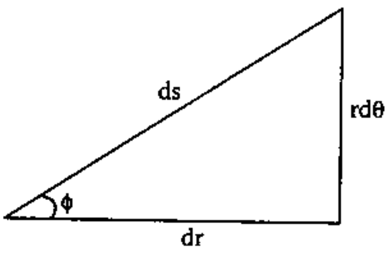


**Fig. 9.**

## Solved Examples

**Example 1.** *Find the equation on the tangent at the point t to the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$.*

**Solution.** We have $\quad x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t)$

and $\quad y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a\sin t$

Therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/2 \cdot \cos t/2}{2\cos^2 t/2} = \tan\frac{t}{2}$

Now, the equation of the tangent at '$t$' is $y - a(1-\cos t) = \tan\frac{t}{2}[x - a(t + \sin t)]$

$$\Rightarrow y - 2a\sin^2\frac{t}{2} = (x - at)\tan\frac{t}{2} - a\sin t.\tan\frac{t}{2}$$

$$\Rightarrow y - 2a\sin^2 \frac{«}{«} = (x - at)\tan - - - 2a\sin^2 -$$

$$\Rightarrow \qquad y = (x - at)\tan t/2.$$

**Example 2.** *Show that the parabolas $r = \frac{a}{(1+\cos\theta)}$ and $r = \frac{b}{(1-\cos\theta)}$ intersect orthogonally.*

**Solution.** Here we have $r = \frac{a}{(1+\cos\theta)}$ ...(1)

and $\quad r = \frac{b}{(1-\cos\theta)}$ ...(2)

Taking log of both sides of (1), we get

$$\log r = \log a - \log(1 + \cos\theta)$$

Differentiating with respect to $\theta$, we get

$$\frac{1}{r}.\frac{dr}{d\theta} = \frac{-(-\sin\theta)}{(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\Rightarrow \cot\phi = \tan\frac{\theta}{2} = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

Notes

$\Rightarrow \quad \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$

Now, from (2), we get

$$\log r = \log b - \log(1 - \cos\theta)$$

Differentiating with respect to θ, we get

$$\frac{1}{r}.\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\therefore \cot\phi = -\cot\frac{1}{2}\theta = \cot\left(\pi - \frac{1}{2}\theta\right)$$

$$\Rightarrow \quad \phi = \pi - \frac{1}{2}\theta \quad \Rightarrow \quad \phi_2 = \pi - \frac{1}{2}\theta$$

Now, the angle of intersection $= \phi_1 \sim \phi_2$

$$= \left(\pi - \frac{1}{2}\theta\right) - \left(\frac{1}{2}\pi - \frac{1}{2}\theta\right) = \frac{\pi}{2}$$

Both curves intersect orthogonally.

**Example 3.** *Show that the pedal equation of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$.

**Solution.** Here, the equation of the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let $x = a\cos t, y = b\sin t$.

$$\therefore \quad \frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$$

Therefore, the equation of the tangent at '*t*' is

$$Y - b\sin t = -\frac{b\cos t}{a\sin t}(X - a\cos t)$$

$$\Rightarrow ab - b\cos t.X - a\sin t.Y = 0 \quad \ldots(1)$$

Since *p* denote the length prependicular from (0, 0) to (1), therefore

$$p = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

$$\frac{1}{p^2} = \frac{a^2\sin^2 t + b^2\cos^2 t}{a^2b^2} \quad \ldots(2)$$

Now, $r^2 = x^2 + y^2 = a^2\cos^2 t + b^2\sin^2 t$

$$= a^2 + b^2 - a^2\sin^2 t - b^2\cos^2 t \quad \ldots(3)$$

From (3) $a^2\sin^2 t + b^2\cos^2 t = (a^2+b^2) - r^2$.

Therefore, from (3), we get

$$\frac{1}{p^2} = \frac{(a^2+b^2)-r^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}.$$

**Example 4.** *Find the pedal equation of* $r^n = a^n \sin n\theta$.

**Solution.** Here, the given curve is

$$r^n = a^n \sin n\theta \quad \ldots(1)$$

Taking logarithm of both the sides of (1), we get

$$n \log r = n \log a + \log \sin n\theta. \qquad \ldots(2)$$

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\cdot\frac{dr}{d\theta} = n\frac{\cos n\theta}{\sin n\theta} = n \cot n\theta$$

$$\Rightarrow \quad \cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta} = \cot n\theta$$

$$\therefore \quad \phi = n\theta$$

Also, $p = r \sin\phi \Rightarrow p = r \sin n\theta$ ...(3)

Now from (1) and (3), we have

$$\sin n\theta = \frac{p}{r}$$

Putting the value in (1), we get

$$pa^n = r^{n+1}.$$

**Example 5.** *Find the angle at which the radius vector cuts the curves* $\frac{l}{r} = 1 + e\cos\theta$.

**Solution .** Here, the given equation of the curve is

$$\frac{l}{r} = 1 + e\cos\theta$$

$$\Rightarrow \quad \log l - \log r = \log(1 + e\cos\theta).$$

Diff. w.r.t. θ, we get

$$-\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{(1+e\cos\theta)}(-e\sin\theta)$$

$$\therefore \quad \cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow \quad \tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

$$\Rightarrow \quad \phi = \tan^{-1}\left[\frac{1+e\cos\theta}{e\sin\theta}\right].$$

**Example 6.** *For the cardiod* $r = a(1 - \cos\theta)$, *prove that*

(*i*) $\phi = \frac{1}{2}\theta$ (*ii*) $2ap^2 = r^3$

**Solution.** Here the given curve is

$$r = a(1 - \cos\theta) \qquad \ldots(1)$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

(i) Since, we have

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2a\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\Rightarrow \phi = \frac{\theta}{2}$$

(ii) Since, we have $p = r\sin\phi = r\sin\theta/2$

$$\Rightarrow \quad r = 2a\sin^2\frac{\theta}{2} = 2a\frac{p^2}{r^2}$$

$$\therefore 2ap^2 = r^3$$

Notes

**Example 7.** *Find the pedal equation of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$.

**Solution.** Here, the given curve is

$$x^{2/3} + y^{2/3} = a^{2/3} \qquad \text{...(1)}$$

Let $x = a\cos^3 t, y = a\sin^3 t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t}.$$

Hence, the equation of tangent of (1) is

$$y - a\sin^3 t = -\frac{\sin t}{\cos t}(x - a\cos^3 t)$$

$$\Rightarrow x\sin t \Re y\cos t \quad a\sin t\cos t(\cos^2 t \quad \sin^2 t) = a\sin t\cos t \qquad \text{...(2)}$$

$p$ = the length of the prependicular from (0, 0) to (2)

$$= \frac{a\sin t\cos t}{\sqrt{\sin^2 t + \cos^2 t}} = a\sin t\cos t.$$

Now,

$$r^2 = x^2 + y^2 = a^2\cos^6 t + a^2\sin^6 t = a^2[(\cos^2 t)^3 + (\sin^2 t)^3]$$

$$= a^2[(\cos^2 t + \sin^2 t)^3 - 3\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)]$$

$$= a^2[1 - 3(p^2/a^2).1] = a^2 - 3p^2.$$

**Example 8.** *Show that for any curve* $\sin^2\phi\left(\frac{d\phi}{d\theta}\right) + r\left(\frac{d^2r}{ds^2}\right) = 0.$

**Solution.** We have $\frac{dr}{ds} = \cos\phi$

$$\Rightarrow \quad \frac{d^2r}{ds^2} = -\sin\phi\left(\frac{d\phi}{ds}\right) = -\sin\phi\left(\frac{d\phi}{d\theta}\right)\left(\frac{d\theta}{ds}\right)$$

$$\Rightarrow \quad r\left(\frac{d^2r}{ds^2}\right) = -\sin\phi\left(\frac{d\phi}{d\theta}\right).r\left(\frac{d\theta}{ds}\right)$$

$$\Rightarrow \quad r\left(\frac{d^2r}{ds^2}\right) = -\sin\phi\left(\frac{d\phi}{d\theta}\right).\sin\phi \qquad \left(\because r\frac{d\theta}{ds} = \sin\phi\right)$$

$$\therefore r\left(\frac{d^2r}{ds^2}\right) + \sin^2\phi\left(\frac{d\phi}{d\theta}\right) = 0.$$

## STUDENT ACTIVITY

1. For the curve $r^n = a^n\cos n\theta$, show that $a^{2n}\frac{d^2r}{ds^2} + nr^{2n-1} = 0$.

2. For the cycloid $x = a(1 - \cos t), y = a(t + \sin t)$, show that

(i) $\frac{ds}{dt} = 2a\cos\frac{t}{2}$ (ii) $\frac{ds}{dx} = \text{cosec}\frac{t}{2}$ (iii) $\frac{ds}{dy} = \sec\frac{t}{2}$

3. Show that for the curve $r^m = a^m \cos m\theta, \frac{ds}{d\theta} = \frac{a^m}{r^{m-1}}$.

4. Show that the pedal equation of the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.

## TEST YOURSELF

1. Find the angle of intersection of the curve $r^2 = 16 \sin 2\theta$ and $r^2 \sin 2\theta = 4$.
2. Show that in the curve $r = a\theta$, the polar subnormal is constant and in the curve $r\theta = a$, the polar subtangent is constant.
3. Show that the curves $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ intersect at right angles.
4. Show that the spiral $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.
5. Find the angle $\phi$ for the curve $a\theta = (r^2 - a^2)^{1/2} - a\cos^{-1} a/r$.
6. Show that the curves $r = (1 + \sin\theta)$ and $r = a(1 - \sin\theta)$ cut orthogonally.
7. Show that the curves $r = 2\sin\theta$ and $r = 2\cos\theta$ intersect at right angles.
8. Find the angle of intersection between the pair of curves $r = 6\cos\theta$ and $r = 2(1 + \cos\theta)$.
9. Show that the pedal equation of the
   (i) conic $\frac{l}{r} = 1 + e\cos\theta$ is $\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)$  (ii) curve $r = a\theta$ is $p^2 = \frac{r^4}{r^2 + a^2}$
   (iii) cardiod $r = a(1 + \cos\theta)$ is $r^3 = 2ap^2$.  (iv) spiral $r = a \operatorname{sech} n\theta$ is $\frac{1}{p^2} = \frac{A}{r^2} + B$.
   (v) hyperbola $r^2\cos 2\theta = a^2$ is $pr = a^2$.  (vi) lemniscate $r^2 = a^2\cos 2\theta$ is $r^3 = a^2 p$.
10. Show that the normal at any point $(r, \theta)$ to the curve $r^n = a^n \cos n\theta$ makes an angle $(n + 1)\theta$ with the initial line.
11. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle $\alpha$ to the radius vector.

### ANSWERS

1. $\frac{2\pi}{3}$  5. $\cos^{-1}\frac{a}{r}$  8. $\frac{\pi}{6}$

Notes

## Summary

- Slope of a line, $m = \tan\theta$, where $\theta$ is the angle which the line makes with the positive direction of $x$-axis.
- Slope of the line $ax + by + c = 0$ is given by $m = -\frac{a}{b}$
- Slope of the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is $= \frac{y_2 - y_1}{x_2 - x_1}$
- Slope of $x$-axis $= 0$, Slope of $y$-axis $= \infty$
- Two lines are parallel iff $m_1 = m_2$.
- Two lines are perpendicular iff $m_1 m_2 = -1$.
- Angle between two lines having slopes $m_1$ and $m_2$ is given by $\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$
- Equation of the line (one point form) $y - y_1 = m(x - x_1)$ passing through the point $(x_1, y_1)$.
- Perpendicular distance formula $= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
- The equation of the tangent is $y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$.
- The equation of the normal at $P(x_1, y_1)$ is $y - y_1 = -\frac{1}{dy_1 / dx_1}(x - x_1)$.
- If the tangent to the two curves make angle $\phi_1$ and $\phi_2$ with the common radius vector to their point of intersection, then angle between the curves = angle between tangents $= |\phi_1 - \phi_2|$.
- Let $p$ be the length of the perpendicular from the pole to the tangent at any point $(r, \theta)$ of a curve $r = f(\theta)$, then
  (i) $p = r\sin\phi$
  (ii) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\cdot\left(\frac{dr}{d\theta}\right)^2$
  (iii) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ where $u = \frac{1}{r}$
- The relation between $p$ and $r$, where $r$ is the distance of any point on the curve from the origin (or pole) and $p$ is perpendicular from origin (or pole) to the tangent at that point is called the Pedal equation of the curve.
- If the equation of the curve is $x = f(y)$, then $\frac{ds}{dy} = \pm\sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}$
- If the given equation is in parametric form *i.e.*, $x = f_1(t), y = f_2(t)$, then $\frac{ds}{dt} = \pm\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$

## Objective Evaluation

**FILL IN THE BLANKS**

1. The pedal equation of the curve $y^2 = 4a(x + a)$ is ________
2. If $\phi$ is the angle between the radius vector and the tangent of a curve, then $\tan\phi$ = ________
3. Polar subtangent for the curve $r = a\theta$ is ________
4. For the curve $r = f(\theta)$, the value of $\frac{ds}{d\theta}$ = ________
5. Polar subnormal for the curve $r = a\theta$, ________
6. For the cycloid $x = a(1 - \cos t), y = a(1 + \sin t)$, we have $\frac{ds}{dt}$ = ________
7. In the equiangular spiral $r = ae^{\theta\cot a}$, the tangent is inclined to the radius vector with angle ________

Notes

8. For the curve $r^2 = a^2 \cos 2\theta, \frac{ds}{d\theta}$ ________

## TRUE/FALSE

**Write 'T' for True and 'F' for False statement.**

1. The relation between $p$ and $r$ is called pedal equation. **(T/F)**
2. The relation between p and r is called polar equation. **(T/F)**
3. The pedal equation of the curve $r = a/\theta$ is $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2}$. **(T/F)**
4. The pedal equation of the curve $r^m = a^m \cos m\theta$ is $r^{m+1} = a^m.p$. **(T/F)**
5. For the curve $r = f(\theta)$, we have $\left(\frac{dr}{ds}\right)^2 + \left(r\frac{d\theta}{ds}\right)^2 = 1$. **(T/F)**
6. For any curve $r = f(\theta)$, the value of $\frac{ds}{d\theta}$ is $\frac{r^2}{p}$. **(T/F)**
7. If $p$ be the length of perpendicular drawn from the pole $O$ to the tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$ then $\frac{1}{p^2} \neq \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$. **(T/F)**
8. The pedal equation of the cardiod $r = a(1 - \cos\theta)$ is $r^3 = 2ap$. **(T/F)**

## MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. Two curves cut orthogonally if $\tan\phi_1.\tan\phi_2$ is equal to:
   (a) 1 (b) −1 (c) 0 (d) none of these
2. For the curve $r = f(\theta)$, the value of $\cos\phi$ is :
   (a) $r\frac{d\theta}{ds}$ (b) $r\frac{ds}{d\theta}$ (c) $\frac{ds}{dr}$ (d) $\frac{dr}{ds}$
3. The pedal equation of the curve $y^2 = 4a(x + a)$ is :
   (a) $p = a^2r^2$ (b) $p^2 = ar$ (c) $p^2 = r$ (d) $2ap^2 = r^3$
4. The angle at which the radius vector cuts the curve $r = a(1 - \cos\theta)$ is :
   (a) $\theta$ (b) $\theta/2$ (c) $\theta/3$ (d) $\theta/4$
5. In the equiangular spiral $r = ae^{\theta\cot a}$ the tangent isinclined to which angle to the radius vector:
   (a) $\alpha/2$ (b) $\alpha/3$ (c) $\alpha$ (d) $2\alpha$
6. Polar subtangent for the curve $r = a\theta$ is :
   (a) $r^2a$ (b) $r^2$ (c) $r^2/a$ (d) $(r/a)^2$
7. Polar subtangent for the curve $\frac{2a}{r} = 1 - \cos\theta$ is :
   (a) $2a\sin\theta$ (b) $-2a\cos\theta$ (c) $2a\tan\theta$ (d) $-2a\operatorname{cosec}\theta$
8. The angle of intersection of the curve $r = a\cos\theta, 2r = a$ is :
   (a) $\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi$
9. Polar subnormal for the curve $r = a\theta$ is :
   (a) $r^2a$ (b) $a$ (c) $r^2/a$ (d) $r^2$
10. For the cardiod $r = a(1 - \cos\theta)$, the value of $\phi$ is :
   (a) $\theta$ (b) $\frac{\theta}{2}$ (c) $-\frac{\theta}{2}$ (d) none of these

## ANSWERS

### FILL IN THE BLANKS

1. $p^2 = ar$ 2. $r\frac{d\theta}{dr}$ 3. $\frac{r_2^2}{a}$ 4. $\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$ 5. $a$ 6. $2a\cos\frac{t}{2}$

7. $a$ 8. $\frac{a^2}{r}$

Notes

**TRUE/FALSE**

| | | | | | | |
|---|---|---|---|---|---|---|
| **1.** T | **2.** F | **3.** T | **4.** T | **5.** T | **6.** T | **7.** T |
| **8.** F | | | | | | |

**MULTIPLE CHOICE QUESTIONS**

| | | | | | | |
|---|---|---|---|---|---|---|
| **1.** (b) | **2.** (d) | **3.** (b) | **4.** (b) | **5.** (c) | **6.** (c) | **7.** (d) |
| **8.** (c) | **9.** (b) | **10.** (b) | | | | |

□□□□

Chapter

# 5

# Curvature



## STRUCTURE



## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:

- Concepts of curvature and related formulae
- The formulas of radius of curvature in different form
- The concept of centre of curvature

## 5.1 INTRODUCTION

The measure of the sharpness of the bending of a curve at a particular point is called curvature of the curve at the point. In figure (1), curve $PQ$ bends more sharply than the curve $AB$. In this chapter, we shall find mathematical expressions for the curvature of a curve at a given point.

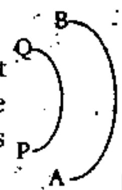


**Fig. 1.**

## 5.2 CURVATURE

Let $P$, $Q$ be two neighbouring points on a curve $AB$.

Also, let $AP = s$, arc $AQ = s + \delta s$ and arc $PQ = \delta s$.

Let the tangent to the curve at points $P$ and $Q$ makes angle $\psi$ and $\psi + \delta\psi$ respectively with a fixed line say $X$-axis, then

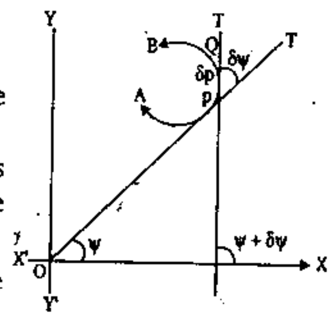


**Fig. 2.**

(i) The angle $\delta\psi$ through which the tangent turns as its points of contact travels along the arc $PQ$ is called the total bending or total curvature of arc $PQ$.

(ii) The ratio $\frac{\delta\psi}{\delta s}$ is called the mean or average curvature of arc $PQ$.

(iii) The limiting value of the mean curvature when $Q$ tends to $P$ is called the curvature of the curve at the point $P$. Therefore, the curvature $K$ at point $P$ is

$$\lim_{Q \to P} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

(iv) The reciprocal of the curvature of the given curve at $P$. (provided this curvature is not equal to zero), is called the radius of curvature of the curve at $P$. This is denoted by $\rho$.

$$\rho = \frac{1}{K} = \frac{ds}{d\psi}$$

Notes

## 5.3 FORMULA FOR RADIUS OF CURVATURE (CARTESIAN FORM)

Let $y = f(x)$ be the equation of curve. Then the slope of the tangent at any point $= \tan\psi = \frac{dy}{dx}$

Differentiating both sides, w.r.t. $s$, we get

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) \Rightarrow \sec^2\psi . \frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$$

$$\Rightarrow \quad \sec^2\psi . \frac{1}{\rho} = \frac{d^2y}{dx^2}.\cos\psi \qquad \left(\because \frac{dx}{ds} = \cos\psi\right)$$

Therefore
$$\rho = \frac{\sec^2\psi}{\cos\psi \frac{d^2y}{dx^2}} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} \Rightarrow \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

### REMARKS

- The positive root is taken in numerator of above formula, therefore, radius of curvature r, will be positive when $\frac{d^2y}{dx^2}$ is positive (*i.e.*, when the curve is concave upward) and negative when $\frac{d^2y}{dx^2}$ is negative (*i.e.*, when the curve is concave downward).
- At a point of inflexion, the curvature of a curve is not defined. $\left(\because \text{at the point of inflexion} \frac{d^2y}{dx^2} = 0\right)$
- When the equation of the curve is given in the form $x = f(y)$ then by interchanging $x$ and $y$ (It is justify because curvature is a length, and its value is independent of the choice of axis), we get
$$\rho = \frac{\left[1+(dx/dy)^2\right]^{3/2}}{d^2x/dy^2}$$
- When the equation of curve is given in parametric form, *i.e.*, $x = f(t)$ and $y = g(t)$, then radius of curvature is given by $\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$, where dash (′) denote the derivative w.r.t., '$t$'.
$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

## 5.4 RADIUS OF CURVATURE AT THE ORIGIN

Let the curve $y = f(x)$ passes through the origin. Then, we may use the following methods, to find the radius of curvature.

**(i) Method of direct substitution.** Since $y = f(x)$ be given. Calculate the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at origin and then use the following formula

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

**(ii) Method of Expansion.** Let $y = f(x)$ be the equation of curve. Since, it passes through the origin, therefore $f(0) = 0$.

Therefore, by Maclaurin's series expansion, we have

$$y = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\Rightarrow \quad y = xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \qquad [\because f(0) = 0]$$

$$\Rightarrow y = p_1x + \frac{1}{2!}p_2x^2 + \frac{1}{3!}p_3x^3 + \ldots (1)$$

where $p_1 = f'(0) = y_1(0), p_2 = f''(0) = y_2(0)$, etc.

Now, differentiating (1) with respect to $x$, we get

$$y_1 = p_1 + \frac{2p_2x}{2!} + \frac{3p_3x^2}{3!} + \ldots$$

Again differentiating w.r.t. $x$. we get

$$y_2 = \frac{2p_2}{2!} + \frac{6p_3 x}{3!} + \ldots$$

At the origin (*i.e.*, $x = 0$), we have

$$y_1 = p_1 \text{ and } y_2 = \frac{2p_2}{2!} = p_2$$

Now putting these values of $y_1$ and $y_2$ in the formula $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, We get

$$\rho = \frac{(1+p_1^2)^{3/2}}{p_2}$$

**REMARK**

- We can find the values of $p$ and $q$ in the following manner:
  Put the value of $y = p_1 x + \frac{p_2 x^2}{2!} + \frac{p_3 x^3}{3!} + \ldots$ in the given equation of the curve and equating the coefficients of the powers of $x$.

**(iii) Newton's Method.** If a curve passes through the origin, and axis of $x$ is the tangent at the origin, then radius of curvature $\rho$ at origin $= \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$

Since the axis of $x$ is the tangent at the origin, therefore, we have

$$y_1(0) = \left(\frac{dy}{dx}\right)_{(0,0)} = 0$$

Here, we observed that $\frac{x^2}{2y}$ is of the indeterminate form $\left(\frac{0}{0}\right)$ as $x \to 0, y \to 0$.

Using L Hospital rule, we have

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{2x}{2y_1} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{y_1} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{1}{y_2} = \frac{1}{y_2(0)} \qquad \ldots(1)$$

Now, $\rho$ at origin $\Re \frac{[1+y_1(0)]^{/}}{\ll\cdot(\,)} \quad \frac{(\,+\,)^{3/2}}{(\,)} \quad \frac{}{(\,)} \qquad \ldots(2)$

From (1) and (2), we have

$$\rho_{(\text{at origin})} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

**REMARK**

- If a curve passes through the origin and axis of $y$ is the tangent, then radius of curvature at the origin is given by $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$.

## Solved Examples

**Example 1.** *For $x = a(t + \sin t)$, $y = a(1 - \cos t)$, prove that* $\rho = 4a\cos\frac{t}{2}$.

**Solution.** We have $x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t)$

and $y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/2 \cos t/2}{2\cos^2 t/2} = \tan\frac{t}{2}$$

Also $\frac{d^2y}{dx^2} \Re \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \frac{d}{dx}\left(\tan\frac{t}{-}\right) \quad \frac{1}{-}\sec^2\frac{t}{-}.\frac{dt}{dx}$

Notes

$$= \frac{1}{2}\sec^2\frac{t}{2}\cdot\frac{1}{a(1+\cos t)} = \frac{1}{4a}\sec^4\frac{t}{2}$$

Now, putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

We get $\rho = \frac{[1+\tan^2 t/2]^{3/2}}{\frac{1}{4a}\sec^4 t/2} = \frac{4a\sec^3 t/2}{\sec^4 t/2} = 4a\cos t/2$

**Example 2.** *Find the curvature of the curve* $x^3 + y^3 = 3axy$ *at the point* $(3a/2, 3a/2)$.

**Solution .** The equation of the curve is

$$x^3 + y^3 = 3axy \qquad ...(1)$$

Differentiating w.r.t. $x$, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$\Rightarrow \quad x^2 + y^2\frac{dy}{dx} = ay + ax\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2} \qquad ...(2)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{at\left(\frac{3}{2}a,\frac{3}{2}a\right)} = -1$$

From (2), we have

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

$$\Rightarrow (ax - y^2)\frac{d^2y}{dx^2} = 2x + 2y\left(\frac{dy}{dx}\right)^2 - 2a\frac{dy}{dx} \qquad ...(3)$$

Putting $x = \frac{3a}{2}, y = \frac{3a}{2}$

and $\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1,$

We get $\left[\frac{d^2y}{dx^2}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3}\cdot\frac{1}{a}$

Hence, the radius of curvature $\rho$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$, we get

$$\rho = \left[\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right]_{at\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{(1+1)^{3/2}}{-\frac{32}{3}\cdot\frac{1}{a}} = -\frac{3a}{8\sqrt{2}}$$

Therefore, the curvature $\frac{1}{\rho} = +\frac{8\sqrt{2}}{3a}$. (By ignoring the negative sign)

**Example 3.** *Show that the radii of curvature of the curve* $y^2 = x^2\left(\frac{a+x}{a-x}\right)$ *at the origin are* $a\sqrt{2}$.

**Solution.** The equation of the curve is

$$y^2 = x^2\left(\frac{a+x}{a-x}\right)$$

$$\Rightarrow \quad y = \pm\frac{x(a+x)^{1/2}}{(a-x)^{1/2}} = \pm x\frac{a^{1/2}\left(1+\frac{x}{a}\right)^{1/2}}{a^{1/2}\left(1-\frac{x}{a}\right)^{1/2}}$$

$$\Rightarrow y = \pm x\left(1+\frac{x}{a}\right)^{1/2}\left(1-\frac{x}{a}\right)^{-1/2}$$

$$\Rightarrow y = \pm x\left(1+\frac{x}{2a}+\ldots\right)\left(1+\frac{x}{2a}+\ldots\right) \quad \text{(Expanding by Binomial Expansions)}$$

$$\text{or } y = \pm x\left(1+\frac{x}{2a}+\frac{x}{2a}+\frac{x^2}{4a^2}+\ldots\right)$$

$$\Rightarrow y = \pm\left(x+\frac{x^2}{a}+\frac{x^3}{4a^2}+\ldots\right)$$

Therefore,

$$\frac{dy}{dx} = y_1 = \pm\left(1+\frac{2x}{a}+\frac{3x^2}{4a^2}+\ldots\right)$$

$$\text{and } \frac{d^2y}{dx^2} = y_2 = \pm\left(\frac{2}{a}+\frac{6x}{4a^2}+\ldots\right)$$

At(0, 0) $y_1 = \pm 1$ and $y_2 = \pm\frac{2}{a}$

$$\therefore\ \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{\pm 2/a} = \pm 2\sqrt{2}.\frac{a}{2}$$

$$\Rightarrow \rho = \pm\sqrt{2}\cdot a = \sqrt{2}\cdot a. \quad \text{(Numerically)}$$

**Example 4.** *Apply Netwon's formula, find the radius of curvature at the origin for the curve*

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

**Solution.** Since, the curve passes through the origin. Equating to zero, the lowest degree terms, we may find $y = 0$

$\Rightarrow x$ axis is the tangent at the origin.

Therefore, by Newton's formula, $\rho$ at (0, 0)

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Dividing the equation of the curve by $2y$, we get

$$x.\frac{x^2}{2y} - x^2 + \frac{3}{2}xy - 2y^2 + 5.\frac{x^2}{2y} - 3x + \frac{7}{2}y - 4 = 0$$

Taking $\lim x \to 0$ and $y \to 0$, we get

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y} - 4 = 0 \Rightarrow 5\rho - 4 = 0 \Rightarrow \rho = \frac{4}{5}.$$

**Example 5.** *For the curve* $y = \frac{ax}{a+x}$, *if* $\rho$ *is the radius of curvature at any point* $(x, y)$, *show that*

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2.$$

Notes

**Solution.** Let $y = \dfrac{ax}{a+x}$ ...(1)

Therefore, $\dfrac{dy}{dx} = a\dfrac{a+x-x}{(a+x)^2} = a^2(a+x)^{-2}$

Now, again $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = -2a^2(a+x)^{-3} = \dfrac{-2a^2}{\left(\dfrac{ax}{y}\right)^3}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2y^3}{ax^3}$$

$$\therefore\ 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{a^4}{(a+x)^4} = 1+\frac{a^4}{\left(\frac{ax}{y}\right)^4} = 1+\frac{y^4}{x^4}$$

$$\therefore\ \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{[(x^4+y^4)/x^4]^{3/2}}{(-2y^3/ax^3)}$$

$$= -\frac{a(x^4+y^4)^{3/2}}{2x^6(y^3/x^3)} = -\frac{a}{2}\frac{(x^4+y^4)^{3/2}}{x^3y^3}$$

Hence, $\left(\dfrac{2\rho}{a}\right)^{2/3} = \dfrac{x^4+y^4}{x^2y^2} = \dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}$

$$\Rightarrow \left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2.$$

**Example 6.** *Find the radius of curvature at origin for the curve* $x^3+y^3-2x^2+6y=0$.

**Solution.** The curve passes through origin. Equating to zero the lowest degree terms we get $y=0$ i.e., $x$ axis as tangent to the curve at origin.

$\therefore$ By Newtons method, $\rho$ (at origin) $= \lim\limits_{x\to 0}\dfrac{x^2}{2y}$

Dividing by $2y$, the equation of the curve can be written as

$$x.\frac{x^2}{2y}+\frac{1}{2}y^2-2.\frac{x^2}{2y}+3=0$$

Taking limit as $x\to 0, y\to 0$ and $\lim\limits_{x\to 0}\dfrac{y^2}{2x} = \rho$, we get

$$0.\rho+0-2\rho+3=0 \quad i.e., \quad \rho = 3/2.$$

**Example 7.** *If $\rho_1$ and $\rho_2$ be the radii of curvature of the extremities of two conjugate diameters of an ellipse prove that* $(\rho_1^{2/3}+\rho_2^{2/3})(ab)^{2/3} = a^2+b^2$.

**Solution.** Let the equation of an ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1. \quad ...(1)$$

Let $P(a\cos\theta, b\sin\theta)$ and $Q(-a\sin\theta, b\cos\theta)$

be the extremities of two conjugate diameters of (1).

Differentiating both sides of (1) w.r.t $x$ we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}.\frac{dy}{dx} = 0$$

Notes

or $$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$ ..(2)

Again differentiating, we get

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right] = -\frac{b^2}{a^2}\left[\frac{y - x\left(-\frac{b^2x}{a^2y}\right)}{y^2}\right] = -\frac{b^2}{a^2}\left[\frac{\left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right)}{y^3}\right]b^2$$

$$= -\frac{b^4}{a^2y^3}$$ [Using (1)]

We know that

$$\Re \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\alpha}}{\frac{d^2y}{dx^2}} \quad \frac{\left[1+\left(-\frac{b^2x}{a^2y}\right)^2\right]^{\alpha}}{-b^4/a^2y^3}$$

$$\rho = \frac{(a^4y^2 + b^4x^2)^{3/2}}{-a^4b^4}$$

At $P(a\cos\theta, b\sin\theta)$, $\rho = \rho_1$

$$\therefore \rho_1 = \frac{(a^4.b^2\sin^2\theta + b^4a^2\cos^2\theta)^{3/2}}{-a^4b^4}$$

or $$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{-ab}$$

or $\rho_1(-ab) = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}$

or $\rho_1^{2/3}(ab)^{2/3} = a^2\sin^2\theta + b^2\cos^2\theta$ ...(3)

At $Q(-a\sin\theta, b\cos\theta), \rho = \rho_2$

$\therefore \rho_2^{2/3}(ab)^{2/3} = a^2\cos^2\theta + b^2\sin^2\theta$ ...(4)

Adding (3) and (4), we get

$(\rho_1^{2/3} + \rho_2^{2/3})(ab)^{2/3} = a^2 + b^2$

**Example 8.** *Prove that for the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2b^2}{p^3}$; *p being the perpendicular from centre upon the tangent at (x, y).*

**Solution.** We have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

and $$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right] = -\frac{b^4}{a^2y^3}$$

Let $(a\cos\theta, b\sin\theta)$ be any point on the ellipse. The equation of the tangent at this point is

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

or $bx\cos\theta + ay\sin\theta - ab = 0$ ...(2)

We are given that

$p =$ Perpendicular from (0, 0) to the tangent (2)

Notes

or $\quad p = \dfrac{-ab}{\sqrt{b^2 \cos^2 \Re \;\; a^2 \sin^2}}$ ...(3)

Now the radius of curvature ρ is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{a^2y^3\left(1+\frac{b^4x^2}{a^4y^2}\right)^{3/2}}{-b^4} = \frac{(a^4y^2+b^4x^2)^{3/2}}{-a^4b^4}$$

The ρ at $(a\cos\theta, b\sin\theta)$ is given by

$$\rho = -\frac{(a^4b^2\sin^2\theta + b^4a^2\cos^2\theta)^{3/2}}{a^4b^4} = -\frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

$$= -\frac{(-ab/p)^3}{ab} \qquad \text{[Using (3)]}$$

$$\rho = \frac{a^2b^2}{p^3}.$$

**Example 9.** *If $\rho_1$ and $\rho_2$ be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{\Re 2/3} + \rho_2^{2/3} = «\; a^{\;\;2/3}$.*

**Solution.** We have $\quad y^2 = 4ax$ ...(1)

Parametric form of (1) is given by

$x = at^2, y = 2at$

∴ $\quad x' = 2at, y' = 2a$

and $\quad x'' = 2a, y'' = 0$

Therefore, radius of curvature ρ at $(at^2, 2at)$ is given by

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'} = \frac{(4a^2t^2+4a^2)^{3/2}}{0-4a^2} = 2a(1+t^2)^{3/2} \quad \text{(Ignore –ve sign)}$$

If $P(t_1)$ and $Q(t_2)$ be the extremities of the focal chord of the parabola, then

$$t_1t_2 = -1 \Rightarrow t_2 = -\frac{1}{t_1}$$

So, $\quad \rho_1$ at $P(t_1) = 2a(1+t_1^2)^{3/2}$

$\rho_2$ at $Q(t_2) = 2a(1+t_2^2)^{3/2}$

∴ $\quad \rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}.[(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$

$$= (2a)^{-2/3}.\left[\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}\right] = (2a)^{-2/3}.$$

## STUDENT ACTIVITY

1. Show that the curvature at a point of the curve $y = f(x)$ is given by $\dfrac{d^2y}{dx^2}\cos^3\psi$, where ψ is the inclination of the tangent at the point to the axis of $x$.

2. Show that for the curve $s = ae^{x/a}$, $a\rho = s(s^2 - a^2)^{1/2}$.

Notes

3. Show that if $\rho$ be the radius of curvature at any point $P$ on the parabola $y^2 = 4ax$ and $S$ be its focus, then $\rho$ varies as $(SP)^2$.

4. Show that for any curve $\frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

## TEST YOURSELF

1. Find the radius of curvature of the following curves:
   (i) $x^{1/2} + y^{1/2} = a^{1/2}$ (ii) $a^2y = x^3 - a^3$ (iii) $x^{2/3} + y^{2/3} = a^{2/3}$
   (iv) $x^m + y^m = 1$ (v) $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$
   (vi) $s = 4a \sin\psi$ at $(s, \psi)$ (vii) $ay^2 = x^3$
   (viii) $y = e^x$ at the point where it cuts the $y$-axis.
   (ix) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a\cos^3\theta, a\sin^3\theta)$
   (x) $y = 4\sin x - \sin 2x$ at $x = \frac{\pi}{2}$ (xi) $y = x^3(x-a)$ at $(a, 0)$
2. Find the radius of curvature at the origin of the following curves :
   (i) $x^3 + y^3 = 3axy$ (ii) $y = x^3 + 5x^2 + 6x$
   (iii) $5x^3+7y^3+4x^2y+xy^2+2x^2+3xy+y^2+4x=0$
   (iv) $a(y^2 - x^2) = x^3$ (v) $y - x = x^2 + 2xy + y^2$
   (vi) $2x^4+4x^3+xy^2+6y^3-3x^2-2xy+y^2-4x=0$
   (vii) $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

### ANSWERS

1. (i) $\frac{2(x+y)^{3/2}}{\sqrt{y}}$ (ii) $\frac{(a^4+9x^4)^{3/2}}{6a^4x}$ (iii) $3a^{1/3}x^{1/3}y^{1/3}$ (iv) $\frac{(x^{2m-2}+y^{2m-2})^{3/2}}{(1-m)x^{m-2}y^{m-2}}$ (v) $\frac{1}{\sqrt{2}}$
   (vi) $4a\cos\psi$ (vii) $\frac{1}{6a}(4a+9x)^{3/2}x^{1/2}$ (viii) $\sqrt{8}$ (ix) $3a\sin\theta\cos\theta$ (x) $\frac{5\sqrt{5}}{4}$
   (xi) $(1+a^3)^{3b}/6a^2$

2. (i) $\frac{3a}{2}$ (ii) $\frac{37\sqrt{37}}{10}$ (iii) $-2$ (iv) $2a\sqrt{2}$ (v) $\frac{1}{2\sqrt{2}}$ (vi) 2 (vii) $\frac{a}{\sqrt{2}}$

## 5.5 RADIUS OF CURVATURE FOR PEDAL EQUATIONS

*To prove that* $\rho = r\dfrac{dr}{dp}$

**Proof.** Let the pedal equation of the curve be

$$p = f(r).$$

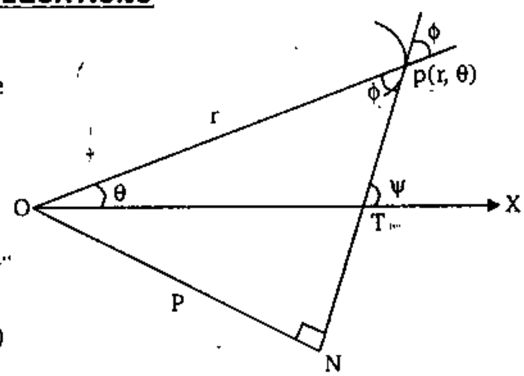


Fig. 3

Form the adjoining figure, we have

$$\psi = \theta + \phi$$

$$\Rightarrow \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \Rightarrow \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad \ldots(1)$$

Since, we know that $p = r\sin\phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r.\frac{d\theta}{ds} + r\frac{dr}{ds}.\frac{d\phi}{dr} \qquad \left[\because \sin\phi = r.\frac{d\theta}{ds} \text{ and } \cos\phi = \frac{dr}{ds}\right]$$

$$= r\left[\frac{d\theta}{ds} + \frac{d\phi}{ds}\right] = r\frac{1}{\rho}$$

$$\text{or } \frac{dp}{dr} = r\frac{1}{\rho} \qquad \therefore \rho = \frac{r}{dp/dr} = r.\frac{dr}{dp} \quad \Rightarrow \quad \rho = r\frac{dr}{dp}.$$

## 5.6 RADIUS OF CURVATURE FOR TANGENTIAL POLAR EQUATIONS $p = f(\psi)$

*To prove that* $\rho = p + \dfrac{d^2p}{d\psi^2}$

**Proof.** Let $p$ be the length of the perpendicular drawn from the origin on the tangent to curve at the point $P(x, y)$. Also, let $\psi$ be the angle which the tangent makes with $X$-axis.

Here we observe that $OL$ makes an angle $\psi - \frac{\pi}{2}$ with the positive direction of $X$-axis.

$\therefore$ Equation of the tangent $PT$ is

$$« = \cos\left(\psi - \frac{\pi}{2}\right) + \sin\left(\psi - \frac{\pi}{2}\right)$$

[Normal form: $x\cos\alpha + y\sin\alpha = p$]

$$\Rightarrow \quad p = X\sin\psi - Y\cos\psi$$

where $X$ and $Y$ are cartesian co-ordinates of any point on the tangent $PT$.

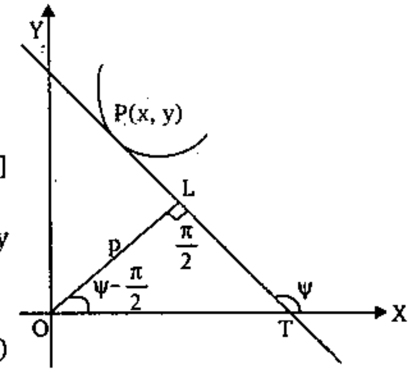


Fig. 4.

Since, $P(x, y)$ lies on $PT$, therefore

$$p = x\sin\psi - y\cos\psi \qquad \ldots(1)$$

$$\Rightarrow \frac{dp}{d\psi} = x\cos\psi + \sin\psi\frac{dx}{d\psi} + y\sin\psi - \cos\psi.\frac{dy}{d\psi}$$

$$= x\cos\psi + y\sin\psi + \sin\psi\frac{dx}{ds}.\frac{ds}{d\psi} - \cos\psi.\frac{dy}{ds}.\frac{ds}{d\psi}$$

$$= x\cos\psi + y\sin\psi + \sin\psi.\rho.\cos\psi - \cos\psi.\rho.\sin\psi$$

$$\left(\because \frac{dx}{ds} = \cos\psi \text{ and } \frac{dy}{ds} = \sin\psi\right)$$

$$= x\cos\psi + y\sin\psi$$

Differentiating again w.r.t. $\psi$, we get

$$\frac{d^2p}{d\psi^2} = -x\sin\psi + \cos\psi.\frac{dx}{d\psi} + y\cos\psi + \sin\psi.\frac{dy}{d\psi}$$

Notes

$$= -x\sin\psi + y\cos\psi + \cos\psi.\frac{dx}{ds}.\frac{ds}{d\psi} + \sin\psi.\frac{dy}{ds}.\frac{ds}{d\psi}$$

$$= (-x\sin\psi + y\cos\psi) + \cos\psi.\cos\psi.\rho + \sin\psi.\sin\psi.\rho$$

$$= -p + \rho[\cos^2\psi + \sin^2\psi] \qquad \text{(Using (1))}$$

$$\Rightarrow \qquad \rho = p + \frac{d^2p}{d\psi^2}.$$

**WORKING PROCEDURE**

To transform polar equation to pedal equation, proceed as follows :

**STEP 1.** Find $\phi$, using formula $\tan\phi = \dfrac{r}{dr/d\theta}$.

**STEP 2.** Substitute the value of $\phi$ in $p = r\sin\phi$.

**STEP 3.** Eliminate $\theta$.

## 5.7 RADIUS OF CURVATURE IN POLAR FORM

*To prove that* $\rho = \dfrac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$

**Proof.** We know that $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. ...(1)

Differentiating (1) w.r.t. $r$, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left\{\frac{d}{dr}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left[\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)^2\right].\frac{d\theta}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2r}{d\theta^4}$$

$$\frac{1}{p^3}.\frac{dp}{dr} = \frac{1}{r^5}\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]$$

Now $$\rho = r\frac{dr}{dp} = \frac{r.\dfrac{1}{p^3}}{\dfrac{1}{r^5}\left[r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right]}$$

Form (1), we have

$$\frac{1}{p^3} = \left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]^{3/2} = \frac{1}{r^6}\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}$$

Hence, $$\rho = \frac{r^6.\dfrac{1}{r^6}\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}} \Rightarrow \rho = \frac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

### Solved Examples

**Example 1.** *Find the radius of curvature for the curve* $r^n = a^n\cos n\theta$.

**Solution.** We have $r^n = a^n\cos n\theta$

$$\Rightarrow \quad n\log r = n\log a + \log\cos n\theta.$$

Notes

Now differentiating w.r.t. θ, we get

$$\frac{n}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n\sin n\theta) = -n\tan n\theta \qquad \ldots(1)$$

$\Rightarrow \; r_1 = -r\tan n\theta$

Again diiferentiating, we get

$$r_2 = -r.n.\sec^2 n\theta - r_1.\tan n\theta = -rn\sec^2 n\theta + r\tan^2 n\theta. \qquad \ldots(2)$$

Putting all these values in

$$\rho = \frac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(r^2+r^2\tan^2 n\theta)^{3/2}}{r^2+2r^2\tan^2 n\theta + r^2.n\sec^2 n\theta - r^2\tan^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{(n+1)r^2\sec^2 n\theta} = \frac{r\sec n\theta}{(n+1)} = \frac{r}{n+1}\cdot\frac{1}{\cos n\theta} = \frac{r}{(n+1)\dfrac{r^n}{a^n}} = \frac{a^n}{(n+1)r^{n-1}}$$

**Example 2.** *Show that in the rectangular hyperbola $r^2\cos 2\theta = a^2$, the radius of curvature $\rho = \dfrac{r^3}{a^2}$.*

**Solution.** The given curve is

$$r^2\cos 2\theta = a^2 \qquad \ldots(1)$$

$\Rightarrow 2\log r + \log\cos 2\theta = 2\log a$

Differentiating w.r.t. θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}(-2\sin 2\theta) = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \cot\phi = \tan 2\theta = \cot\left(\frac{\pi}{2} - 2\theta\right) \quad \Rightarrow \quad \phi = \frac{\pi}{2} - 2\theta$$

Now $p = r\sin\phi = r\sin\left(\dfrac{\pi}{2} - 2\theta\right) = r\cos 2\theta = r.\dfrac{a^2}{r^2} = \dfrac{a^2}{r}$

$$\Rightarrow \quad \frac{dp}{dr} = -\frac{a^2}{r^2}$$

Hence, $\rho = r\dfrac{dr}{dp} = -\dfrac{r^3}{a^2} = \dfrac{r^3}{a^2}$. (By neglecting the negative sign)

**Example 3.** *Show that at any point on the equiangular spiral $r = ae^{\theta\cot\alpha}$, $\rho = r\,\mathrm{cosec}\,\alpha$ and that it subtends a right angle at the pole.*

**Solution.** The given equation is $r = ae^{\theta\cot\alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we have

$$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}.\cot\alpha = r\cot\alpha.$$

$$\therefore \quad (1/r)\frac{dr}{d\theta} = \cot\alpha$$

or $\quad \cot\phi = \cot\alpha \Rightarrow \phi = \alpha.$

Now, $p = r\sin\phi$, thus the pedal equation of (1) is $p = r\sin\alpha$.

Therefore, $\dfrac{dp}{dr} = \sin\alpha.$

Now $\rho = r\dfrac{dr}{dp} = \dfrac{r}{\sin\alpha} = r\,\mathrm{cosec}\,\alpha.$

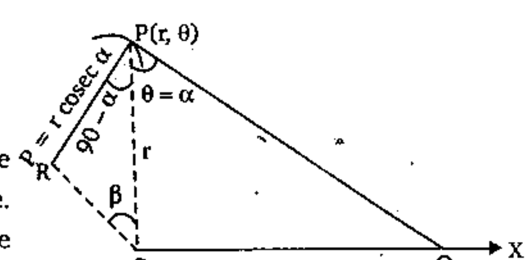

Fig. 5.

**Second part.** Let $P(r, \theta)$ be any point on the given curve. $PQ$ is the tangent and $PR$ is the normal to the curve at $P$. Let $R$ be center of curvatrure of the point

$P$ of the curve. Then $PR$ = the radius of curvature of the curve at $P = r \operatorname{cosec} \alpha$.

Intersect $OP$ and $OR$, where $O$ is the pole.

Let $\angle POR = \beta$. Then to show that $\beta = 90°$.

We have $\angle OPQ = \phi = \alpha$

$\angle OPR = 90° - \alpha$, (since $PR$ is normal at $P$)

*i.e.*, perpendicular to the tangent $PQ$.

Now in $\Delta OPR$, we have $\angle ORP = 180° - (90° - \alpha + \beta) = 90° + \alpha - \beta$.

Therefore, applying the sine theorem for $\Delta OPR$, we get

$$\frac{OP}{\sin \angle ORP} = \frac{PR}{\sin\beta} \text{ or } \frac{r}{\sin(90+\alpha-\beta)} = \frac{\rho}{\sin\beta} \text{ or } \frac{r}{\cos(\alpha-\beta)} = \frac{r \operatorname{cosec}\alpha}{\sin\beta}$$

$(\because \rho = r \operatorname{cosec} \alpha)$

$\sin\alpha \sin\beta = \cos(\alpha - \beta)$

or $\sin\alpha\sin\beta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ or $\cos\alpha\cos\beta = 0$ or $\cos\beta = 0$.

Hence, $\beta = 90°$.

## STUDENT ACTIVITY

1. Show that for the hypercycloid $P = A \sin B\psi$, $\rho$ varies as $P$.

2. Find the radius of curvature at the point $(p, r)$ on the spiral $p^2 = r^4/(r^2 + a^2)$.

3. Prove that for any curve $\dfrac{r}{\rho} = \sin\phi\left(1 + \dfrac{d\phi}{d\theta}\right)$, where $\rho$ is the radius of curvature and $\tan\phi = r\dfrac{d\theta}{dr}$.

## TEST YOURSELF

1. Find the radius of curvature in polar form on each of the following curves :
   (i) $r = a(1 - \cos\theta)$ (ii) $r(1 + \cos\theta) = 2a$ (iii) $r^2 = a^2 \cos 2\theta$
2. Find the radius of curvature at any point $(p,r)$ on the following curves :
   (i) $r^2 = ar$ (ii) $r^2 = a^2 - b^2 + \dfrac{a^2b^2}{p^2}$

(iii) $2ap^2 = r^3$ (iv) $pa^2 = r^3$

**3.** Show that the radius of curvature of the cardoid $r = a(1 + \cos\theta)$ at the origin is 0.

**4.** Show that the radius of curvature at any point on the curve $r = a(1 \pm \cos\theta)$ varies as square root of the radius vector.

**5.** If $\rho_1$, $\rho_2$ be the radii of curvature at the extrimities of any chord of the cardoid $r = a(1 + \cos\theta)$, which passes through the pole, then $\rho_1^2 + \rho_2^2 = 16a^2/9$.

**6.** Show that the radius of curvature at the point $(p, r)$ of the ellipse $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}$ is $\frac{a^2b^2}{p^3}$.

**7.** Show that the radius of curvature for the hyperbola $p^2 = a^2\cos^2\psi + b^2\sin^2\psi$ is $\frac{a^2b^2}{p^3}$.

**8.** Show that the curvature of the curves $r = a\theta$ and $r\theta = a$ at their common point are in the ratio 3 : 1.

**9.** By Newton's method, show that the radius of curvature of the curve $r = a\sin n\theta$ at the origin is $\frac{na}{2}$.

**10.** Show that the radius of curvature at each point of the curve $x \Re a\left(\cos t \quad \log\tan\frac{t}{2}\right), y \quad a\sin t$ is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis.

## ANSWERS

**1.**(i) $\frac{2}{3}\sqrt{2ar}$ (ii) $2\sqrt{(r^3/a)}$ (iii) $\frac{a^2}{3r}$ **2.** (i) $\frac{2r^{3/2}}{\sqrt{a}}$ (ii) $\frac{a^2b^2}{p^3}$ (iii) $\frac{2}{3}\sqrt{2ar}$ (iv) $\frac{a^2}{3r}$

## 5.8 CENTRE OF CURVATURE

For any point $P$ of a curve, the centre of curvature is the point on the positive direction of the normal at $P$, at a distance $\rho$ from it.

Let $PD$ be the normal curve at $P$ and $C$ be a point on it such that $PC = \rho$, then $C$ is said to be the center of curvature at $P$.

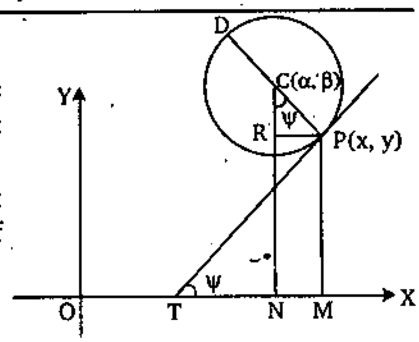


**Fig. 6.**

### 5.8.1 EVOLUTE OF A CURVE

The locus of the center of curvature of the given curve is called the evolute of the curve.

### 5.8.2 CIRCLE OF CURVATURE

The circle with its center at the center of curvature $C$ and radius equal to $\rho$ is called the circle of curvature.

**REMARK**

- The circle of curvature touches the curve at $P$ and both the curve and the circle of curvature have the same curvature at this point.

## 5.9 CO-ORDINATES OF THE CENTRE OF CURVATURE

Let $y = f(x)$ be the given curve and $P(x, y)$ be any given point.

Let $C(\alpha, \beta)$ be the center of curvature corresponding to any point $P(x, y)$ on the given curve, then from above fig. (7), we have $PC = \rho$.

Suppose, the tangent $TP$ makes an angle $\psi$ with positive direction of x-axis. Draw $PM$ and $CN$ perpendicular to x-axis

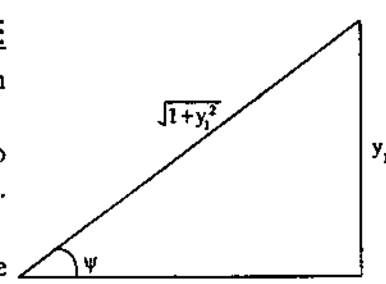


**Fig. 7.**

and draw perpendicular to $CN$. Then

$$\angle PCN = 90° - \angle CPR = 90° - (90° - \angle RPT) = \angle RPT = \angle PTX = \psi$$

$$\therefore \quad \alpha = ON = OM - NM = OM - RP = x - CP\sin\psi = x - \rho\sin\psi \quad \ldots(1)$$

Also, $$\beta = NC = NR + RC = MP + RC = y + CP\cos\psi = y + \rho\cos\psi \quad \ldots(2)$$

Since, we know that $y_1 = \tan\psi$

$$\Rightarrow \quad \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}} \text{ and } \cos\psi = \frac{1}{\sqrt{1+y_1^2}}.$$

Also, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Putting all these values in (1) and (2), we get

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \text{ and } \beta = y + \frac{(1+y_1^2)}{y_2}.$$

**REMARKS**

- From (1) and (2) we have $\alpha = x - \rho\sin\psi$ and $\beta = y + \rho\cos\psi$. Since $x, y, \rho, \psi$ depends upon $s$, therefore the above equations may be treated as parametric equations of the evolute.
- The equation of the circle of curvature at the given point is $(x-\alpha)^2 + (y-\beta)^2 = \rho^2$.

## 5.10 CHORD OF CURVATURE

The length intercepted by the circle of curvature of the curve at $P$, on a straight line drawn through $P$ in any given direction is called chord of curvature through $P$ in that direction.

Let the chord of curvature $PQ$ makes an angle $\alpha$, with the normal $PD$, then its length $PQ$ is given by

$$PQ = PD\cos\alpha$$

($\because \angle DQP$, being a semicircle is a right angle.)

$= 2\rho\cos\alpha$, which is the chord of curvature perpendicular to radius vector

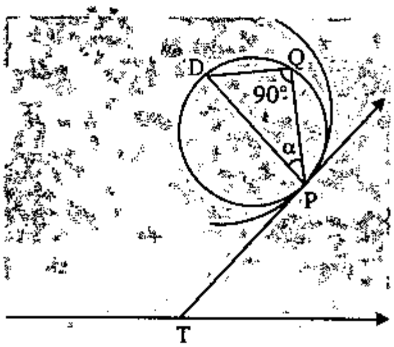


Fig. 8.

**REMARK**

- The chord of curvature through pole is given by $2\rho\sin\alpha$.

## 5.11 LENGTH OF THE CHORD OF CURVATURE

**(1) Cartesian form.** Since, the tangent at $P$ makes an angle $\psi$ with the $x$-axis therefore, the chord of curvature $PA$ is parallel to $x$-axis, which makes an angle $90 - \psi$ with the normal $PCD$ and chord of curvature $PB$ parallel to $y$-axis makes angle $\psi$ with the normal $PCD$.

$\therefore \quad C_x$ = length of the chord of curvature $PA$, parallel to $x$-axis.

$$= PD\cos(90-\psi) = 2\rho\sin\psi$$

$$= \frac{2(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{1+y_1^2}} = \frac{2y_1(1+y_1^2)}{y_2}$$

Similarly, $$C_y = \frac{2(1+y_1^2)^{3/2}}{y_2}.$$

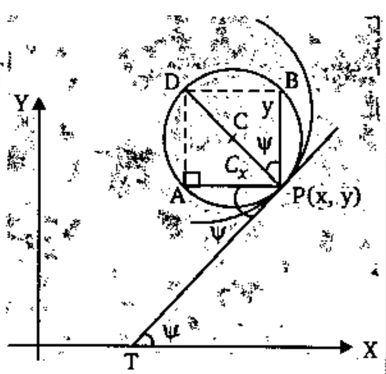


Fig. 9.

Notes

**(2) Polar form.** Let the chord of curvature $PL$ makes an angle $90 - \phi$ with $PCD$, the normal of the curve at $P$, and $PM$, the chord of curvature perpendicular to the radius vector $OP$, makes an angle $\phi$ with the normal $PCD$.

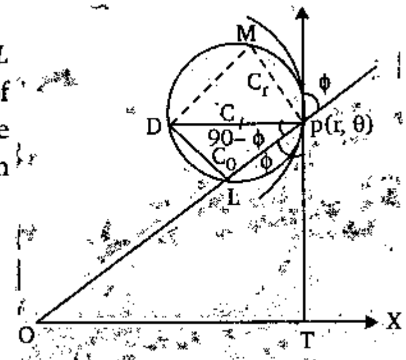


**Fig. 10.**

$\therefore C_o$ = Length of the chord of curvature $PL$ through origin (or pole)

$$= PD(\cos 90 - \phi)$$

$$= 2\rho \sin\phi = \frac{2(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \cdot \frac{r}{\sqrt{r^2 + r_1^2}}$$

$$= \frac{2r(r^2 + r_1^2)}{r^2 + 2r_1^2 - rr_2}$$

and $C_p$ = length of the chord of curvature $PM$ perpendicular to radius vector.

$$= PD \cos\phi = 2\rho\cos\phi = \frac{2(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \cdot \frac{r}{\sqrt{r^2 + r_1^2}} = \frac{2r(r^2 + r_1^2)}{r^2 + 2r_1^2 - rr_2}$$

**(3) Pedal form.** Let $p = f(r)$ be the given equation of the curve.

Let $C_o$ = length of the chord of curvature through pole along radius vector

$$= PD\cos(90 - \phi) = 2\rho\sin\phi \quad \ldots(1)$$

Now using $\rho = r\frac{dr}{dp}$ and $\sin\phi = \frac{p}{r}$ in (1), we get $C_o = 2r\frac{dr}{dp}.\frac{p}{r} = 2p.\frac{dr}{dp}$ ...(2)

Now $p = f(r) \Rightarrow \frac{dp}{dr} = f'(r)$ and $\sin\phi = \frac{p}{r} = \frac{f(r)}{r}$

$\therefore$ From (1), $C_o = 2\rho\sin\phi = 2.r.\frac{dr}{dp}.\sin\phi = 2r.\frac{1}{f'(r)}.\frac{f(r)}{r} = \frac{2f(r)}{f'(r)}$

Also $C_p$ = length of the chord perpendicular to the radius vector

$$= DP\cos\phi = 2\rho\cos\phi$$

$$= 2.r.\frac{dr}{dp}\frac{\sqrt{r^2 - p^2}}{r} \qquad \left[\because \sin\phi = \frac{p}{r} \text{ and } \cos\phi = \frac{\sqrt{r^2 - p^2}}{r}\right]$$

$$= 2.\sqrt{r^2 - p^2}.\frac{dr}{dp}.$$

## Solved Examples

**Example 1.** *Find the chord of curvature through the pole of the cardioid* $r = a(1 + \cos\theta)$.

**Solution.** We have $r = a(1 + \cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\frac{1}{2}\theta = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

Now $p = r\sin\phi = r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$

$$\therefore 2p^2 = r^2\left(2\cos^2\frac{\theta}{2}\right) = r^2(1 + \cos\theta) = r^2 - \frac{r}{a} = \frac{r^3}{a}$$

$\Rightarrow 2p^2a = r^3$ is the pedal equation of the curve. On differentiating w.r.t. $r$ we get

$$4ap\frac{dp}{dr} = 3r^2$$

$$\therefore \rho = r\frac{dr}{dp} = r.\frac{4ap}{3r^2} = \frac{4ap}{3r}$$

Therefore, the chord of curvature through the pole

$$= 2\rho \sin\phi = 2.\frac{4ap}{3r}.\frac{p}{r} \qquad [\because p = r\sin\phi]$$

$$= \frac{8ap^2}{3r^2} = \frac{8}{3r^2}.\frac{r^3}{2} = \frac{4r}{3} \qquad [\because 2ap^2 = r^3]$$

**Example 2.** *Show that the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$ is $\frac{2r}{n+1}$.*

**Solution.** The given curve is $r^n = a^n \cos n\theta$

$\Rightarrow n \log r = n \log a + \log \cos n\theta$

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n}{\cos n\theta}.\sin n\theta$$

$$\Rightarrow \quad \cot\phi = -\tan n\theta = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2} + n\theta$$

Now $p = r\sin\phi = r\sin\left(\frac{\pi}{2} + n\theta\right) = r\cos n\theta$

$\because$ Pedal equation of the curve is $p = \frac{r^{n+1}}{a^n}$.

$$\therefore \quad \frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$$

Also, $\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$

Therefore, the chord of curvature through pole is

$$= 2\rho\sin\phi = 2\rho\sin\left(\frac{\pi}{2} + n\theta\right) = 2\rho\cos n\theta$$

$$= 2\frac{a^n}{(n+1)r^{n-1}}.\frac{r^n}{a^n} = \frac{2r}{(n+1)}.$$

**Example 3.** *Find the co-ordinate of the centre of curvature at any point of the parabola $y^2 = 4ax$. Hence, show that its evolute is $27ay^2 = 4(x-2a)^3$.*

**Solution.** We have $y^2 = 4ax$

$$\Rightarrow \quad 2yy_1 = 4a \text{ i.e., } y_1 = \frac{2a}{y} \text{ and } y_2 = -\frac{2a}{y^2}.y_1 = -\frac{4a^2}{y^3}$$

If $(\overline{x}, \overline{y})$ be the centre of curvature, then

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{-4a^2/y^3}$$

$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = 3x + 2a \qquad \ldots(1)$$

and $$\overline{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^3}$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad \ldots(2)$$

Therefore, the required centre of curvature is $\left\{(3x+2a), -2x\sqrt{\frac{x}{a}}\right\}$. To find the required evolute, eliminate $x$ from (1) and (2), we have

$$(\overline{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\overline{x}-2a}{3}\right)^3$$

Notes

Notes

$$\Rightarrow \quad 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3 \quad ...(3)$$

Now, locus of $(\bar{x}, \bar{y})$ is $27ay^2 = 4(x - 2a)^3$ which is the required equation of evolute.

**Example 4.** *Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.*

**Solution.** We have $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\Rightarrow \quad y_1 = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\frac{\theta}{2}$$

$$\text{Now } y_2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx} = -\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the center of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(4a\sin^4\frac{\theta}{2}\right)\left(1 + \cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2}.4a\sin^4\frac{\theta}{2}.\operatorname{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\frac{\theta}{2}.\cos\frac{\theta}{2} = a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a(\theta + \sin\theta)$$

$$\text{and} \quad \bar{y} = y + \frac{1+y_1^2}{y_2} = a(1-\cos\theta) + \left(1 + \cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2.\operatorname{cosec}^2\theta/2 = a(1-\cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

Hence, the required evolute is given by $x = a(\theta + \sin\theta), y = -a(1-\cos\theta)$ which is another equal cycloid.

## TEST YOURSELF

1. In the curve $y = a\log\sec\left(\frac{x}{a}\right)$, show that the chord of curvature parallel to the axis of $y$ is of constant length.
2. Prove that the centre of curvature $(\alpha, \beta)$ for the curve $x = 3t, y = t^2 - 6$ is $\alpha = -\frac{4}{3}t^3, \beta = 3t^2 - \frac{3}{2}$.
3. If $C_x$ and $C_y$ be the chords of curvature parallel to the axis at any point of the curve $y = ae^{x/a}$, show that $\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}$.
4. Show that the centre of curvature $(\alpha, \beta)$ at the point determined by $t$ on the ellipse $x = a\cos t, y = b\sin t$, is given by $\alpha = \frac{a^2-b^2}{a}\cos^3 t, \beta = -\left(\frac{a^2-b^2}{b}\right)\sin^3 t$.
5. Show that in any curve the chord of curvature perpendicular to the radius vector is $2\rho\sqrt{(r^2-p^2)}/r$.
6. Show that the chord of curvature through the pole of the equiangular spiral $r = ae^{m\theta}$ is $2r$.
7. Find the coordinates of the centre of curvature of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $x = a\cos\theta, y = b\sin\theta$. Hence, show that the equation of its evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.
8. Find the chord of curvature through the pole of the curve $a\theta = \sqrt{r^2 - a^2} - a\cos^{-1}(a/r)$.
9. If $C_r$ and $C_\theta$ be the chords of curvature of the curve $r = a(1+\cos\theta)$ through the pole and perpendicular to the radius vector, then prove that $3(C_r^2 + C_\theta^2) = 8rC_r$.

Notes

## Answers

7. $\left(\dfrac{a^2-b^2}{a}\cos^3\theta, -\dfrac{a^2-b^2}{b}\sin^3\theta\right)$ 8. $\dfrac{2(r^2-a^2)}{r}$

## Summary

- The measure of the sharpness of the bending of a curve at a particular point is called curvature of the curve at the point.
- At a point of inflexion, the curvature of a curve is not defined.
- When the equation of the curve is given in the form $x = f(y)$ then by interchanging $x$ and $y$ (It is justify because curvature is a length, and its value is independent of the choice of axis), we get $\rho = \dfrac{\left[1+(dx/dy)^2\right]^{3/2}}{d^2x/dy^2}$
- $\rho = r\dfrac{dr}{dp}$.
- $\rho = p + \dfrac{d^2p}{d\psi^2}$.
- $\rho = \dfrac{\left[r^2+(dr/d\theta)^2\right]^{3/2}}{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}$.

## Objective Evaluation

FILL IN THE BLANKS

1. $\rho = \dfrac{ds}{d\psi}$ is intrinsic formula for ________ of curvature.
2. The relation between ________ is called the intrinsic equation of a curve.
3. The relation between $s$ and $\psi$ for any curve is called ________ equation.
4. The curvature of the curve at any point $P$ is defined as the ________ of the radius of curvature at $P$.
5. For a curve $y = f(x)$, the radius of curvature $\rho$ = ________ .
6. If the curve is in pedal form *i.e.*, $p = f(r)$, then $\rho$ = ________.
7. Locus of centre of curvature is known as ________ of that curve.
8. Chord of curvature through origin is ________ .
9. When curve is in tangential polar form $\rho$ = ________ .
10. The curvature of the curve at any point $P$ is equal to ______.

TRUE/FALSE

**Write 'T' for True and 'F' for False statement.**

1. The curvature of the curve at any point $P$ is defined as the reciprocal of the radius of curvature of $P$. **(T/F)**
2. If the given curve is in parametric form $x = f(t)$ and $y = \phi(t)$ then

   $\rho = \left[\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2\right]^{3/2} \Bigg/ \left[\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}\right]$ **(T/F)**
3. If the curve is $r = f(\theta)$, then $\rho = \dfrac{\left[r^2+(dr/d\theta)^2\right]^{3/2}}{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}$. **(T/F)**
4. The chord of curvature parallel to $x$-axis is $2\rho\cos\psi$. **(T/F)**
5. The chord of curvature parallel to $y$-axis is $2\rho\cos\psi$. **(T/F)**

Notes

6. If $y$-axis is the tangent to the given curve at the origin, then radius of curvature at the origin is equal to $\lim_{x\to 0} \frac{y^2}{2x}$. (T/F)
7. The curvature of the circle and circle of curvature, both are the same. (T/F)
8. The tangential polar formula for radius of curvature is $\rho = p + \frac{d^2p}{d\psi^2}$. (T/F)
9. The chord of curvature through the origin is $2\rho \sin \phi$. (T/F)
10. The chord of curvature at the origin of the curve $3x^2 + 4x - 12y = 0$ is zero. (T/F)

MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. The radius of curvature of the curve $y = e^x$ at the point where it crosses the $y$-axis is :
   (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 1
2. For the curve $xy = a2$ the radius of curvature at (2, 2) is :
   (a) 4 (b) 16 (c) 10 (d) none of these
3. Radius of curvature at any point $(s, \psi)$ of the curve $s = c \log \sec \psi$ is :
   (a) $c \sec \psi$ (b) $c \cot \psi$ (c) $c \operatorname{cosec} \psi$ (d) $c \tan \psi$
4. Radius of curvature at any point $(s, \psi)$ of the curve $S = a \log \cot(\pi/2 - \psi/2) + a\frac{\sin\psi}{\cos^2\psi}$ is :
   (a) $2a \cos^2\psi$ (b) $a \tan^2\psi$ (c) $2a \sec^2\psi$ (d) $a \cot^2\psi$
5. Radius of curvature at $(x, y)$ of the curve $y = 1/2c[e^{x/c} + e^{-x/c}] = c \cosh x/c$ is :
   (a) $y^2/c$ (b) $x^2/c$ (c) $y/c$ (d) $x/c$
6. Radius of curvature at point $(p, r)$ on curve $p^2 = ar$ is :
   (a) $2p^3/a^2$ (b) $2p^2/a^2$ (c) $p^3/a^2$ (d) $p^2a^2$
7. Radius of curvature at point $(p, r)$ on curve $r^3 = a^2p$ is :
   (a) $\frac{2}{5}\sqrt{ar}$ (b) $\frac{a^2}{3r}$ (c) $\frac{2}{3}\sqrt{2ar}$ (d) $\sqrt{2ar}$
8. Radius of curvature at $(r, \theta)$ of curve $r = a(1 - \cos\theta)$ is :
   (a) $\frac{2}{3}\sqrt{2ar}$ (b) $ar$ (c) $\sqrt{2ar}$ (d) $\frac{2}{3}\sqrt{ar}$
9. The radius of curvature is :
   (a) square of the curvature (b) reciprocal of curvature
   (c) equal to curvature (d) none of these
10. The radii of curvature at the origin for the curve $x^3 + y^3 = 3axy$ are each equal to :
    (a) $2a/3$ (b) $a/3$ (c) $3a/2$ (d) none of these

## ANSWERS

FILL IN THE BLANKS

1. Radius 2. $s$ and $\psi$ 3. intrinsic 4. reciprocal 5. $\frac{(1+y_1^2)^{3/2}}{y_2}$
6. $r\frac{dr}{dp}$ 7. Evolute 8. $2\rho \sin\phi$ 9. $p + \frac{d^2p}{d\psi^2}$ 10. $\frac{d\psi}{ds}$

TRUE/FALSE

1. T 2. T 3. T 4. F 5. T 6. T 7. F.
8. T 9. T 10. F

MULTIPLE CHOICE QUESTIONS

1. (c) 2. (d) 3. (d) 4. (c) 5. (a) 6. (a) 7. (b)
8. (a) 9. (b) 10. (c)

□□□□

# Chapter 6
# Asymptotes and Singular Points

## STRUCTURE

- Introduction
- Asymptotes of general equation
- Existence of asymptotes
- Number of asymptotes of a curve
- Asymptotes parallel to co-ordinates axes
- Intersection of a curve with its asymptotes
- Asymptotes of non-algebraic curves
- Asymptotes of polar curves
- Point of inflexion
- Multiple and singular points
- Types of double point
- Nature of a cusp
  - Summary
  - Objective evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:
- The concepts of asymptotes
- How to find the asymptotes of different curves
- The concepts of singular points

## 6.1 INTRODUCTION

In calculus, there are some curves whose branches seem to go to infinity. It is not necessary that there always exists a definite straight line for all such curves which seems to touch the branch of the curves at infinite but more or less there are some certain curves for which this type of definite straight line exists, this straight line is therefore known as asymptote.

**Definition.** *A definite straight line whose distance from branch of the curve continuously decreases as we move away from the origin along the branch of the curve and seems to touch the branch at infinity, provided the distance of this line from origin should be finite initially, is called an asymptote of the curve.*

Suppose in the equalion of a curve, two or more than two values of $y$ exists for every value of $x$, then we obtain different branches of the curve corresponding to these distinct values of $y$. If each branch have its own separate asymptote, then we can say that a curve may have more than one asymptote.

## 6.2 DETERMINATION OF ASYMPTOTES

Consider a curve $f(x, y) = 0$ ...(1)

and also consider that there are no asymptotes parallel to $y$-axis. Thus we shall take the equation which is not parallel to $y$-axis. in the form of

$$y = mx + c \quad \text{...(2)}$$

Let us take a point $P(x, y)$ on the curve (1), therefore this point as tends to infinity along the straight line (2), $x$ must tend to infinity. Now find the tangent to the curve $f(x, y) = 0$ at the point $P(x, y)$.

Notes

$\therefore$ The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right). \quad \ldots(3)$$

The equation (3) is of the form $y = mx + c$, so in order to exist the asymptote of the curve there must both $\frac{dy}{dx}$ and $\left(y - x\frac{dy}{dx}\right)$ tend to finite limits as $x$ tends to infinity. Therefore, if the equation (3) tends to the straight line given in (2) as $x$ tends to infinity, then the line (2) will be an asymptote of the curve $f(x, y) = 0$ and also we have

$$m = \lim_{x\to\infty} \frac{dy}{dx} \quad \text{and} \quad c = \lim_{x\to\infty}\left(y - x\frac{dy}{dx}\right)$$

Since $c$ is finite, then we have

$$\lim_{x\to\infty}\left(\frac{y - x\frac{dy}{dx}}{x}\right) = \lim_{x\to\infty}\frac{c}{x} = 0 \quad \text{or} \quad \lim_{x\to\infty}\left(\frac{y}{x} - \frac{dy}{dx}\right) = 0$$

or $$\lim_{x\to\infty}\left(\frac{y}{x}\right) = \lim_{x\to\infty}\frac{dy}{dx} \quad \text{or} \quad \lim_{x\to\infty}\frac{y}{x} = m.$$

Also $$c = \lim_{x\to\infty}\left(y - x\frac{dy}{xx}\right) \quad \text{or} \quad c = \lim_{x\to\infty}(y - mx).$$

Hence, if $y = mx + c$ is an asymptote to the curve $f(x, y) = 0$, then we obtain

$$m = \lim_{x\to\infty}\frac{dy}{dx} = \lim_{x\to\infty}\frac{y}{x} \quad \text{and} \quad c = \lim_{x\to\infty}(y - mx).$$

## 6.3 ASYMPTOTES OF GENERAL EQUATION

Let the general rational algebraic equation of a curve be

$$\{a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \ldots + a_{n-1}yx^{n-1} + a_nx^n\}$$
$$+ \{b_1y^{n-1} + b_2y^{n-2}x + \ldots + b_{n-1}yx^{n-2} + b_nx^{n-1}\}$$
$$+ \{c_2y^{n-2} + c_3y^{n-3} + \ldots + c_{n-1}yx^{n-3} + c_nx^{n-2}\} + \ldots = 0 \quad \ldots(1)$$

or $$x^n\left\{a_0\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + \ldots + a_{n-1}\left(\frac{y}{x}\right) + a_n\right\}$$
$$+ x^{n-1}\left\{b_1\left(\frac{y}{x}\right)^{n-1} + b_2\left(\frac{y}{x}\right)^{n-2} + \ldots + b_n\right\}$$
$$+ x^{n-2}\left\{c_2\left(\frac{y}{x}\right)^{n-2} + c\left(\frac{y}{x}\right)^{n-} + \ldots + c_n\right\} + \ldots \quad 0$$

or $$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0 \quad \ldots(2)$$

where $\phi_k\left(\frac{y}{x}\right)$ is a polynomial of degree $k$ in $\left(\frac{y}{x}\right)$.

Divide (2) by $x^n$, we get

$$\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + \frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right) + \frac{1}{x^n}\phi_0\left(\frac{y}{x}\right) = 0$$

Now taking limit as $x \to \infty$, and assuming there is no asymptote parallel to $y$-axis then $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$, we get $\phi_n(m) = 0$. ...(3)

This equation (3) is of degree $n$ in $m$ so it has at most $n$ roots, real as well as imaginary. Out of these $n$ roots some roots may be identical. Thus we get $n$ values of $m$ corresponding to the $n$ branches of the curve (1). Since, we will have only real values of $m$ so ignore all imaginary roots of (3) if they exists. Further if $y = mx + c$ is an asymptote of (1), then we have

$$c = \lim_{x\to\infty}(y - mx), \text{ for each specified value of } m.$$

**Determination of c.** For the determination of $c$ corresponding to each distinct value of $m$, we put $y = mx + p$ in the equation of curve (2), where $p \to c$ as $x \to \infty$.

Now putting $y = mx + p$ i.e., $\frac{y}{x} = m + \frac{p}{x}$, in the (2), we get

$$x^n\phi_n\left(m+\frac{p}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{p}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{p}{x}\right)+...+x\phi_1\left(m+\frac{p}{x}\right)+\phi_0\left(m+\frac{p}{x}\right)=0.$$

Expand each term by Taylor's expansion, we get

$$x^n\left[\phi_n(m)+\frac{p}{x}\phi_n'(m)+\frac{p^2}{2!x^2}\phi_n''(m)+...\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{p}{x}\phi_{n-1}'(m)+...\right]$$
$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{p}{x}\phi_{n-2}'(m)+...\right]+...=0$$

or $x^n\phi_n(m)+x^{n-1}[p\phi_n'(m)+\phi_{n-1}(m)]+x^{n-2}\left[\frac{p^2}{2!}\phi_n''(m)+\frac{p}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]+...=0$

Since we know that $\phi_n(m) = 0$, then

$$x^{n-1}[p\phi_n'(m)+\phi_{n-1}(m)]+x^{n-2}\left[\frac{p^2}{2!}\phi_n''(m)+\frac{p}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]+...=0$$

Dividing by $x^{n-1}$ and taking limit as $x \to \infty$, we get

$$\lim_{x\to\infty}[p\phi_n'(m)+\phi_{n-1}(m)]=0 \qquad \text{or} \qquad \left(\lim_{x\to\infty}p\right)\phi_n'(m)+\phi_{n-1}(m)=0$$

or $\quad c\phi_n'(m)+\phi_{n-1}(m)=0 \qquad \left(\because \lim_{x\to\infty} p = c\right)$

Hence, from above relation we can determine the value of $c$ for each distinct value of $m$.

**REMARK**

- To find the polynomial $\phi_n(m)$. We should put $y = m$ and $x = 1$ in the $n^{th}$ degree terms of the curve. Similarly to get $\phi_{n-1}(m)$ we put $y = m$ and $x = 1$ in the $(n-1)^{th}$ degree terms of the curve. Therefore in general, to get $\phi_k(m)$ we should put $y = m$ and $x = 1$ in the $k^{th}$ degree terms of the curves.

## 6.4 EXISTENCE OF ASYMPTOTES

From the equation $\phi_n(m) = 0$, if we obtain one or more than one values of $m$ such that $\phi_n'(m) = 0$ and $\phi_{n-1}(m) \neq 0$, then from the equation for the determining of $c$. we obtain $0.c + \phi_{n-1}(m) = 0$.

Thus we get $c$ is either, $+\infty$ or $-\infty$. Hence, we can say that corresponding to such values of $m$ no asymptotes will exists.

## 6.5 DETERMINATION OF c CORRESPONDING TO SOME IDENTICAL VALUES OF m

Let us suppose some of the roots of the equation $\phi_n(m) = 0$ are identical and let these identical values be $r$ in number which will make $\phi_n'(m),\phi_n''(m),...\phi_m^{r-1}(m)$ equal to zero. Now for the existence of the asymptotes $\phi_{n-1}(m)$ must be zero corresponding to the identical values of $m$. Also, if it will make $\phi_{n-1}'(m),\phi_{n-1}''(m),...\phi_{n-1}^{r-2}(m);\phi_{n-2}(m),\phi_{n-2}'(m),...\phi_{n-2}^{r-3}(m);\phi_{n-3}'(m),\phi_{n-3}'(m),...\phi_{n-3}^{r-4}(m)...;$ $\phi_{n-r+2}(m),\phi_{n-r+2}'(m)$ and $\phi_{n-r+1}(m)$ equal to zero, then the equation to determine $c$ will become

$$0.c^{r-1}+0.c^{r-2}+...+0.c+0=0$$

and thus we cannot find the value of $c$ in this way.

So to determine $c$ let us put $\phi_n(m),\phi'(m),...,\phi_n^{r-1}(m);\phi_{n-1}(m),\phi_{n-1}'(m),...\phi_{n-1}^{r-2}(m);$ $\phi_{n-r+1},\phi_{n-2}'(m),...\phi_{n-2}^{r-3}(m);\phi_{n-3}(m),(\phi_{n-3}'(m),\ ...\ \phi_{n-3}^{r-4}(m)\ ...\ \phi_{n-r+2}(m),\ \phi_{n-r+2}'(m)$ and $\phi_{n-r+1}(m)$ equal to zero in the following equation

Notes

$$x^n\phi_n(m)+x^{n-1}[p\phi_n'(m)+\phi_{n-1}(m)]+x^{n-2}\left[\frac{p^2}{2!}\phi_n''(m)+\frac{p}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]+\ldots$$

$$+x^{n-r+1}\left[\frac{p^{r-1}}{r-1!}\phi_n^{r-1}(m)+\frac{p^{r-2}}{r-2!}\phi_{n-1}^{r-2}(m)+\ldots+\frac{p}{1!}\phi_{n-r+2}'(m)+\phi_{n+r+1}'(m)\right]$$

$$+x^{n-r}\left[\frac{p^r}{r!}\phi_n^r(m)+\frac{p^{r-1}}{r-1!}\phi_{n-1}^{r-1}(m)+\frac{p^{r-2}}{r-2!}\phi_{n-2}^{r-2}(m)+\ldots+\frac{p}{1!}\phi_{n-r+1}'(m)+\phi_{n-r}(m)\right]$$

Now dividing above equation by $x^{n-r}$ and taking the limit as $x \to \infty$, we get

$$\frac{c^r}{r!}\phi_n^r(m)+\frac{c^{r-1}}{r-1\,!}\phi_{n-1}^{r-1}(m)+\ldots+\frac{c}{1!}\phi_{n-r+1}'(m)+\phi_{n-r}(m)=0 \text{ where } c=\lim_{x\to\infty} p.$$

Therefore this equation gives $r$ values of $c$ corresponding to the identical values of $m$. Hence, we obtain $r$ parallel asymptotes.

## 6.6 NUMBER OF ASYMPTOTES OF A CURVE

Suppose the degree of an algebraic curve is $n$, then we find a polynomial $\phi_n(m)$ by putting $y = m$ and $x = 1$ in the $n^{\text{th}}$ degree terms of the curve. Thus the equation $\phi_n(m) = 0$ is of degree $n$ in $m$ and which gives atmost $n$ values of $m$ real as well as imaginary. These $n$ values of $m$ are nothing but the slopes of the asymptotes, which are not parallel to $y$ axis. If there are some asymptotes, parallel to $y$-axis, then the degree of $\phi_n(m)$ will be smaller than $n$ by the same number of parallel asymptotes. Suppose all the roots of $\phi_n(m) = 0$ are distinct and real, then to each value of $m$ we obtain one value of $c$. Hence, we obtain $n$ asymptotes. In case, there some roots say $r$ (out of $n$) of $\phi_n(m) = 0$ are same, then we can find the values of $c$ for these same roots the following equation

$$\frac{c^r}{r!}\phi_n^r(m)+\frac{c^{r-1}}{r-1!}\phi_{n-1}^{r-1}(m)+\ldots+\phi_{n-r}(m)=0$$

This equation in $c$ is of degree $r$ so we get $r$ distinct values of $c$ for the same roots, hence, again we obtain $n$ asymptotes. Therefore we can say that the total number of asymptotes of a curve are equal to the degree of the curve. These asymptotes are real as well as imaginary but we have required only real asymptotes so we ignore all the imaginary asymptotes.

## 6.7 ASYMPTOTES PARALLEL TO CO-ORDINATES AXES

**(a) Asymptotes parallel to x-axis.** Let the general equation of an algebraic curve in decreasing powers of $x$ be

$$x^n\phi(y)+x^{n-1}\phi_1(y)+x^{n-2}\phi_2(y)+\ldots=0 \qquad \ldots(1)$$

where $\phi(y),\phi_1(y),\phi_2(y),\ldots$ are the function of $y$ only.

Now divide (1) by $x^n$, we get

$$\phi(y)+\frac{1}{x}\phi_1(y)+\frac{1}{x^2}\phi_2(y)+\ldots=0. \qquad \ldots(2)$$

If $y = k$ is an asymptote parallel to $x$-axis, then we can say that $x$ alone tends to infinity as a point $P(x, y)$ on the curve tends to infinity along the line $y = k$ and also we have $k = \lim_{x\to\infty} y$.

Now taking the limit of both sides of (2) as $x \to \infty$ and $y \to k$, we get $\phi(k) = 0$.

Thus $k$ is a root of the equation $\phi(y) = 0$. If $k_1$, $k_2$, etc. are the roots of $\phi(y) = 0$, then the asymptotes parallel to $x$-axis are given by $y = k_1$, $y = k_2$, etc. Since $k$ is a root of the equation $\phi(y) = 0$, then $(y - k)$ is a factor of the equation $\phi(y) = 0$. Also $\phi(y)$ is the coefficient of the highest power of $x$ i.e., $x^n$ in the equation of the curve. Hence, we obtain the asymptotes parallel to $x$-axis by taking the coefficient of highest power of $x$ in the equation of the curve equal to zero.

**(b) Asymptotes parallel to y-axis.** Similarly, we may obtain the asymptotes parallel to $y$-axis by taking the coefficient of highest power of $y$ in the equation of the curve equal to zero.

**REMARK**

- If the coefficient of highest power of $x$ or $y$ or both are constant, then no asymptotes parallel to either $x$ or $y$ or both axes exists respectively.

## Solved Examples

**Example 1.** *Find the asymptotes of the curve* $x^3+y^3-3axy=0$.

**Solution.** Obviously, the degree of the curve is 3, so it will have 3 asymptotes real as well as imaginary. Here the coefficient of highest degree in $x$ and $y$ are constant so no asymptote parallel to co-ordinate axis exist. Let

$$y = mx + c \quad \ldots(1)$$

be the asymptote of the curve.

So putting $y = m$ and $x = 1$ in the highest degree terms of the curve, we get

$$\phi_3(m) = 1 + m^3.$$

Solving the equation

$$\phi_3(m) = 0$$

i.e., $1 + m^3 = 0$

or $(1 + m)(m^2 - m + 1) = 0$ or $m = -1$

is only real root and other two roots are imaginary so ignore them.

Next, putting $y = m$ and $x = 1$ is second degree terms in the equation of the curve (1), we get

$$\phi_2(m) = -3am.$$

Now we find value of $c$ by the following equation

$$c\phi_n'(m)+\phi_{n-1}(m) = 0 \quad \text{or} \quad c\phi_3'(m)+\phi_2(m) = 0$$

or $c(3m^2) + (-3am) = 0$ $[\because \phi_3(m) = 1 + m^3 \Rightarrow \phi_3'(m) = 3m^2]$

If $m = -1$, then

$$c[3(-1)^2]+[-3a(-1)] = 0$$

$$3c + 3a = 0 \quad \text{or} \quad c = -a.$$

Hence, the asymptote is $y = -x - a$

or $x + y + a = 0$.

**Example 2.** *Find all the asymptotes of the curve* $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$.

**Solution.** The degree of the curve is 3 so it has 3 asymptotes which are real as well as imaginary. Since the coefficients of highest degree *i.e.*, 3rd degree of $x$ and $y$ are constant so there are no asymptotes parallel to co-ordinate axes. Thus there are oblique asymptotes of the form $y = mx + c$.

Now putting $y = m$ and $x = 1$ in the third degree terms of the curve, we get

$$\phi_3(m) = 1 + m - m^2 - m^3.$$

Solving the equation

$$\phi_3(m) = 0 \text{ i.e, } 1 + m - m^2 - m^3 = 0,$$

we get $(1 + m)(1 - m^2) = 0$ or $m = -1, -1, 1$.

**Determination of c.** For $m = 1$, we use the following equation

$$c\phi_n'(m)+\phi_{n-1}(m) = 0$$

or $c\phi_3'(m)+\phi_2(m) = 0 \quad \ldots(1)$

Putting $y = m$ and $x = 1$ in the second degree terms of the equation we get

$$\phi_2(m) = 0.$$

From (1), we get

$$c(1-2m-3m^2)+0=0$$

at $m = 1$

$$c(1-2-3)+0=0$$

or $$-4c=0$$

or $$c=0$$

Thus one of the asymptote is $y = x$

**Determination of c for m = –1, –1.** Since two out of three roots of the equation $\phi_3(m) = 0$ are same, then we use the following formula to determine $c$

$$\frac{c^2}{2!}\phi_3''(m)+\frac{c}{1!}\phi_2'(m)+\phi_1(m)=0. \quad ...(2)$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation we obtain $\phi_1(m) = -3-m$.

From (2), we have

$$\frac{c^2}{2!}(-2-6m)+\frac{c}{1!}.0+(-3-m)=0$$

at $m = -1$

$$\frac{c^2}{2}(-2+6)-3+1=0$$

or $$2c^2-2=0$$ or $$c=\pm1$$

Thus other two asymptotes are $y = -x + 1,\ y = -x - 1$.

Hence, all the asymptotes of the given curve are $y = x, x+y-1=0, x+y+1=0$.

**Example 3.** *Find all the asymptotes of the curve* $(x-2y)^2(x-y)-4y(x-2y)-(8x+7y)=0$.

**Solution.** Simplifying the equation of curve

$$(x^2+4y^2-4xy)(x-y)-4xy+8y^2-8x-7y=0$$

or $x^3+8xy^2-5x^2y-4y^3-4xy+8y^2-8x-7y=0. \quad ...(1)$

The degree of the curve (1) is 3 so it has 3 asymptotes which are real as well as imaginary. Obviously there are no asymptotes parallel to co-ordinate axis. Thus there are only oblique asymptotes of the form $y = mx + c$.

Putting $y = m$ and $x = 1$ in the highest degree *i.e.*, third degree terms of the curve (1), we obtain

$$\phi_3(m) = 1-5m+8m^2-4m^3.$$

Solving the equation $\phi_3(m) = 0$

*i.e.*, $1-5m+8m^2-4m^3=0$

or $(1-m)(1-2m)^2=0$

or $m=\frac{1}{2},\frac{1}{2},1.$

**Determination of c for m = 1 :**

Putting $y = m$ and $x = 1$ in the second degree terms of the curve (1), we obtain

$$\phi_2(m) = -4m+8m^2.$$

Applying the formula

$$c.\phi_3'(m)+\phi_2(m)=0$$

or $c(-5+16m-12m^2)-4m+8m^2=0.$

Substitute $m = 1$, we get

$$c(-5+16-12)-4+8=0$$

or $$-c+4=0$$

or $$c=4.$$

Thus the asymptote is $y = x + 4$

or $$x-y+4=0.$$

**Determination of c for $m=\frac{1}{2},\frac{1}{2}$:**

Putting $y = m$ and $x = 1$ in the first degree terms of the curve (1) we obtain

$$\phi_1(m) = -8-7m.$$

Since $m=\frac{1}{2},\frac{1}{2}$ are two repeated roots of $\phi_3(m) = 0$, then apply the following formula to determine $c$,

$$\frac{c^2}{2!}[\phi_3''(m)]+\frac{c}{1!}\phi_2'(m)+\phi_1(m)=0$$

or $\frac{c^2}{2!}(16-24m)+c(-4+16m)-8-7m=0$

At $m=\frac{1}{2}$

$$\frac{c^2}{2}(16-12)+c(-4+8)-8-\frac{7}{2}=0$$

or $2c^2+4c-\frac{23}{2}=0$

or $4c^2+8c-23=0 \Rightarrow c=\frac{-2\pm 3\sqrt{3}}{2}.$

Thus the other asymptotes are

$$y=\frac{1}{2}x+\frac{-2\pm 3\sqrt{3}}{2}$$

or $2y=x-2\pm 3\sqrt{3}.$

Hence, all the three asymptotes of the curve are

$x-y+4=0,\ 2y=x-2\pm 3\sqrt{3}.$

**Example 4.** *Find asymptotes of the curve* $x^2y^2-x^2y-xy^2+x+y+1=0$

**Solution.** Degree of the given curve is 4, so it has at most 4 asymptotes (Real and imaginary).

**Asymptote parallel to x-axis :**

Equating the coefficient of highest degree term of $x$ (*i.e.*, $x^2$) to zero, we get

$$y^2-y=0 \quad \Rightarrow \quad y(y-1)=0 \quad \Rightarrow \quad y=0 \text{ and } y=1$$

Thus, $y = 0$ and $y = 1$ are two asymptotes parallel to $x$-axis.

**Asymptote parallel to y-axis :**

Equating the coefficient of highest degree term of $y$ (*i.e.*, $y^2$) to zero, we get

$$x^2-x=0 \quad \Rightarrow \quad x(x-1)=0$$

$$\Rightarrow \quad x=0 \quad \text{and} \quad x=1$$

Thus, $y= 0$ and $x = 1$ are two asymptotes parallel to $x$-axis.

Hence, $x=0, y=0, x=1$ and $y=1$ are the required asymptotes.

**Example 5.** *Find asymptotes parallel to axes for the curve* $y^2(x^2-a^2)=x.$

**Solution.** The given curve is a degree 4, so it cannot have more than four asymptotes. Now, equating to zero the coefficient of the highest power of $y$ (*i.e.*, of $y^2$), the asymptotes parallel to $y$-axis are given by

$$x^2-a^2=0 \quad \Rightarrow \quad x=\pm a.$$

Again equating to zero the coefficient of the highest power of $x$ (*i.e.*, of $x^2$), the asymptotes parallel to $x$-axis are given by

$$y^2=0 \quad \Rightarrow \quad y=0, y=0.$$

Hence, all the four asymptotes are given by $x=\pm a, y=0, y=0.$

Notes

## STUDENT ACTIVITY

1. Find all the asymptotes of the curve $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$.

2. Find all the asymptotes of the curve $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$.

3. Find all the asymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.

4. Find all the asymptotes of the curve $(x + y)^2(x + 2y + 2) = x + 9y + 2$.

5. Find all the asymptotes of the curve $x^2(x - y)^2 + a^2(x^2 - y^2) - a^2xy = 0$.

## TEST YOURSELF

*Find all the asymptotes of the following curves:*

1. $a^2/x^2 - b^2/y^2 = 1$
2. $a^2/x^2 + b^2/y^2 = 1$
3. $y^2(a^2 - x^2) = x^4$
4. $x^2y^2 = a^2(x^2 + y^2)$
5. $x^2y^2 - x^2y - xy^2 - y + 1 = 0$
6. $3x^3 + 2x^2y - 7xy^2 + 2y^3 + 14xy + 7y^2 + 4x + 5y = 0$
7. $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$
8. $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$
9. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0$

**10.** $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

**11.** $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$ **12.** $(x^2-y^2)(y^2-4x^2)-6x^3+5yx^2+3xy^2-2y^3-x^2+3xy-1=0$

**13.** $y^3 = x^3 + ax^2$ **14.** $x^2y^3 + x^3y^2 = x^3 + y^3$.

**15.** $(y-x)(y-2x)^2 + (y+3x)(y-2x) + 2x + 2y - 1 = 0$.

## ANSWERS

**1.** $x = \pm a$ **2.** $x = \pm a, y = \pm b$ **3.** $x = \pm a$ **4.** $x = \pm a, y = \pm a$ **5.** $y = 0; y = 1; x = 0; x = 1$

**6.** $x + 2y = 1, 2x - 2y = -7, 6x - 2y = 15$ **7.** $x + y - 2 = 0; x - y + 2 = 0; 2x - y - 4 = 0$

**8.** $x = 0; x + y = 0; x + y - 1 = 0$ **9.** $x - y = 0; 2x - y + 2 = 0; 2x - y + 1 = 0$

**10.** $x - y - 1 = 0; x + y + 2 = 0; 2x - y = 0$ **11.** $x + y = 0; x - y = 0; x - y + 1 = 0$

**12.** $x - y = 0; 2x - y = 0; x + y + 1 = 0; 2x + y + 1 = 0$ **13.** $3x - 3y + a = 0$

**14.** $y = \pm 1; x = \pm 1; x + y = 0$ **15.** $2x - y - 2 = 0; 2x - y - 3 = 0; x - y + 4 = 0$

## 6.8 OTHER METHODS FOR FINDING THE ASYMPTOTE OF AN ALGEBRAIC CURVE

**THEOREM 1.** *The asymptotes of an algebraic curve are parallel to the lines which obtained by equating to zero the linear factors of the highest degree terms of the equation of curve.*

**Proof.** Let us suppose the equation of the curve is of degree $n$ and let $y - mx$ be a linear factor of the $n^{th}$ degree term in the equation of the curve. Since $\phi_n(m)$ is a polynomial of degree $n$ in $m$ and obtained by putting $y = m$ and $x = 1$ in the $n^{th}$ degree terms of the curve, then $(m - m1)$ is a factor of $\phi n\ (m)$. Thus $m1$ is a root of the equation $\phi n\ (m) = 0$ which gives the slope of the asymptote. Hence, there is an asymptote parallel to the line $y = m1x = 0$.

Conversely, let $m_1$ be a root of the equation $\phi n(m) = 0$ so that there is an asymptote which is parallel to the line $y - m_1x = 0$, then $(m_1 - m)$ must be a factor of $\phi_n(m)$ and therefore, $(y/x - m_1)$ will be a linear factor of $\phi_n(y/x)$. Hence $(y - m_1x)$ is a linear factor of $x^n\phi_n(y/x)$ which is the highest degree terms in the equation of the curve.

Hence the theorem is proved.

Since we know that if $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then we have

$$\left.m = \lim_{x\to\infty} \frac{y}{x} \text{ and } c = \lim_{x\to\infty}(y - mx) = \lim_{x\to\infty,\frac{y}{x}\to\infty}(y - mx)\right\} \quad \ldots(1)$$

With the help of (1) and above theorem we may find the asymptotes of an algebraic curves.

**WORKING PROCEDURE**

**STEP 1.** First we collect all the highest degree terms in the equation of the curve and then resolve into linear factors.

**STEP 2.** After getting linear factors there may arise following cases.

**CASE I.** If the linear factor $(y - m_1x)$ of the highest degree *i.e.*, $n^{th}$ degree terms in the equation of the curve is simple (non-repeated). Then the given equation of the curve can be written as

$$(y - m_1x)F_{n-1} + P_{n-1} = 0. \quad \ldots(2)$$

where $F_{n-1}$ contains only terms of degree $n - 1$ and $P_{n-1}$ contains the terms of various degree not exceeding $n - 1$. Therefore $y - m_1x = c$ is an asymptote of the curve where $c$ is to be determined. Let us take a point $(x, y)$ on the curve (1), then we have

$$y - m_1x = -\frac{P_{n-1}}{F_{n-1}}.$$

Now taking the limit as $x \to \infty, y/x \to m_1$, then we have

Notes

$$\lim_{x\to\infty,\frac{y}{x}\to m_1}(y-m_1x)=\lim_{x\to\infty,\frac{y}{x}\to m_1}\left(-\frac{P_{n-1}}{F_{n-1}}\right) \text{ or } c=\lim_{x\to\infty,\frac{y}{x}\to m_1}\left(-\frac{P_{n-1}}{F_{n-1}}\right).$$

Now substitute this value of $c$ in the equation $y = m_1x + c$

We obtained the asymptote which is parallel to the line $y - m_1x = 0$ corresponding to the linear factor $(y - m_1x)$. Similarly we may obtain other asymptotes.

**Case II.** If $(y - m_1x)$ is a linear factor of the $n^{th}$ degree terms of order two but $(y - m_1x)$ is not a factor of the $(n - 1)^{th}$ degree terms of the curve, then we have $\phi'_n(m_1) = 0$ and $\phi'_{n-1}(m_1) \neq 0$. Therefore, no asymptotes corresponding to $(y - m_1x)^2$ will exist. On the other hand if there are no terms of $(n - 1)^{th}$ degree in the equation of the curve, then make them by adding with zero coefficient and thus we can say that ( $y - m_1x$) is now a factor of $(n - 1)^{th}$ degree terms, then we have the case III.

**Case III.** If $(y - m_1x)^2$ is a linear factor of $n^{th}$ degree terms and $(y - m_1x)$ is a factor of $(n - 1)^{th}$ degree terms, then the equation of the curve can be written as

$$(y-m_1x)^2F_{n-2}+(y-m_1x)G_{n-2}+P_{n-2}=0 \qquad \ldots(3)$$

where $F_{n-2}$ and $G_{n-2}$ contain only the terms of degree $n-2$, and $P_{n-2}$ contains various degree terms not exceeding $n - 2$. Now divide (2) by $F_{n-2}$ and taking the limit as $x \to \infty$ and $y/x \to m_1$, we get

$$\lim_{\substack{x\to\infty,\\(y/x)\to m_1}}(y-m_1x)^2+\lim_{\substack{x\to\infty,\\(y/x)\to m_1}}(y-m_1x)\left(\frac{G_{n-2}}{F_{n-2}}\right)+\lim_{\substack{x\to\infty,\\(y/x)\to m_1}}\left(\frac{P_{n-2}}{F_{n-2}}\right) \qquad \ldots(4)$$

Since we know that $c=\lim_{x\to\infty,(y/x)\to m_1}(y-m_1x)$

and $A=\lim_{x\to\infty,(y/x)\to m_1}\left(\frac{G_{n-2}}{F_{n-2}}\right)$ and $B=\lim_{x\to\infty,(y/x)\to m_1}\left(\frac{P_{n-2}}{F_{n-2}}\right)$

then (4) becomes $c^2 + Ac + B = 0$.

This is a quadratic equation in $c$ so it has two roots let $c_1$ and $c_2$ be these two roots. Then we obtain two asymptotes $y - m_1x = c_1$ and $y - m_1x = c_2$ corresponding to $m_1$.

## REMARK

- As a consequence we can say that the two asymptotes corresponding to the factor $(y - m_1x)^2$ may obtain by solving the quadratic equation $(y - m_1x)^2 + A(y - m_1x) + B = 0$.

  Similarly, we can also find the asymptotes corresponding to the factor $(y - m_1x)^3$ , etc. of the $n^{th}$ degree terms in the equation of the curve.

**Case IV.** Suppose the equation of the curve is of the form

$$(ax + by + c)P_{n-1} + Q_{n-1} = 0 \qquad \ldots(5)$$

where $P_{n-1}$ and $Q_{n-1}$ contain various degree term not exceeding the degree $(n-1)^{th}$, and $P_{n-1}$ contains atleast one term of degree $(n - 1)$ such that (5) becomes of degree $n$. Therefore, we can say that $(ax+ by)$ is a linear factor of $n^{th}$ degree terms in the equation (5). Thus (5) can also be written as

$$(ax + by)\, P_{n-1} + cP_{n-1} + Q_{n-1} = 0.$$

Divide this equation by $P_{n-1}$ and taking the limit as $x \to\infty$ and $y/x \to -a/b$, we obtain

$$(ax + by + c) + \lim_{x\to\infty,y/x\to(-a/b)}(Q_{n-1} / P_{n-1}) = 0$$

This the required equation of the asymptote.

**Case V.** Let the equation of the curve of $n^{th}$ degree be of the form

$$F_n + P = 0 \quad ...(1)$$

where $F_n$ is of degree $n$ and $P$ is of degree $n-2$ or lower and if $F_n = 0$ can be expressed as the product of $n$ linear factors which give $n$ straight lines such that no two of them are parallel or coincident, then all the asymptotes of the curve (1) are obtained by equating to zero the linear factors of $F_n$.

## Solved Examples

**Example 1.** *Find the asymptotes of* $(x-y)^2(x^2+y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$.

**Solution.** We have

$$(x-y)^2 - 10(x-y)\lim_{x\to\infty\ y/x\to 1}\frac{x^2}{x^2+y^2} + 12 + \lim_{x\to\infty\ y/x\to 1}\frac{y^2}{x^2+y^2} = 0$$

or $(x-y)^2 - 5(x-y) + 6 = 0$

which gives parallel asymptotes $x - y = 2$ and $x - y = 3$.

The other two asymptotes are imaginary. Since the remaining linear factors of the four degree terms in the equation to the curve are imaginary.

**Example 2.** *Find the asymptotes of* $(x-y-1)^2(x^2+y^2+2) + 6(x-y-1)(xy+7) - 8x^2 - 2x - 1 = 0$.

**Solution.** Dividing by the coefficient of $(x-y-1)^2$ and taking limits, we see that the asymptotes parallel to $x - y - 1 = 0$ are

$$(x-y-1)^2 + 6(x-y-1)\lim_{x\to\infty\ \frac{y}{x}\to 1}\frac{xy+7}{x^2+y^2+2} + \lim_{x\to\infty\ \frac{y}{x}\to 1}\frac{-8x^2-2x-1}{x^2+y^2+2} = 0$$

$$\Rightarrow (x-y-1)^2 + 3(x-y-1) - 4 = 0$$

$$\Rightarrow x - y - 1 = \frac{-3 \pm \sqrt{9+16}}{2} = 1, -4.$$

Hence, the two asymptotes are $x - y - 2 = 0$ and $x - y + 3 = 0$ the remaining two asymptotes are imaginary.

## 6.9 ASYMPTOTES BY EXPANSION

**THEOREM.** *Let the equation of the curve be of the form* $y = mx + c + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \ldots \quad ...(1)$ *then* $y = mx + c$ *is the asymptote of* (1).

**Proof.** Since the equation of the curve is

$$y = mx + c + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \ldots; \text{ where } \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \ldots \text{ is convergent}$$

for sufficiently large values of $x$.

Differentiating (1) w.r.t. '$x$', we get $\frac{dy}{dx} = m - \frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots$

Now the equation of the tangent to (1) at the point $P(x, y)$ is

$$Y - y = \left(m - \frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots\right)(X - x)$$

or $$Y = \left(m - \frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots\right)X + c + \frac{2A_1}{x} + \frac{3A_2}{x^2} + \ldots \quad \text{[Using(1)]}$$

Now taking the limit as $x \to \infty$, we get

$$Y = mX + c.$$

Hence $y = mx + c$ is an asymptote of the curve $y = mx + c + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \ldots$

## Solved Examples

**Example 1.** *Find the asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Notes

**Solution .** The equation of the curve can be written as

$$y^2 = b^2\left(-1+\frac{x^2}{a^2}\right)$$

$$\text{or } y=\pm b\sqrt{\left(-1+\frac{x^2}{a^2}\right)}=\pm\frac{b}{a}x\sqrt{\left(1-\frac{a^2}{x^2}\right)}$$

$$y=\pm\frac{b}{a}x\left[1-\frac{1}{2}\frac{a^2}{x^2}-\frac{1}{8}\frac{a^4}{x^4}+...\right]$$ [Using binomial expansion]

Since we know that $y = mx + c$ is an asymptote of the curve

$$y = mx+c+\frac{A_1}{x}+\frac{A_2}{x^2}+...$$

Hence, $y=\pm\frac{b}{a}x$ are the asymptotes of the given curve.

**Example 2.** *Find all the asymptotes of the curve* $(y^2-x^2)(y-2x)-7xy+3y^2+2x^2+2x+2y+1=0$.

**Solution .** The given equation can be written as

$$(y-x)(y+x)(y-2x)-7xy+3y^2+2x^2+2x+2y+1=0. \qquad ...(1)$$

The slope of the asymptote corresponding to the factor $y-x$ is 1. Thus the asymptote corresponding to this factor is

$$y-x=\lim_{\substack{x\to\infty,\\ \frac{y}{x}\to 1}}\frac{7xy-3y^2-2x^2-2x-2y-1}{(y+x)(y-2x)}=\lim_{\substack{x\to\infty,\\ \frac{y}{x}\to 1}}\frac{7\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2-2-\frac{2}{x}-2\frac{y}{x}\left(\frac{1}{x}\right)-\frac{1}{x^2}}{\left(\frac{y}{x}+1\right)\left(\frac{y}{x}-2\right)}$$

$$=\frac{7-3-2}{2(1-2)}=\frac{2}{-2}=-1.$$

$\therefore y-x+1=0$

Similarly the second asymptote corresponding to the factor $(y+x)$ is

$$x+y=\lim_{\substack{x\to\infty,\\ \frac{y}{x}\to -1}}\frac{7xy-3y^2-2x^2-2x-2y-1}{(y-x)(y-2x)}=\lim_{\substack{x\to\infty,\\ \frac{y}{x}\to -1}}\frac{7\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2-2-\frac{2}{x}-2\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)-\frac{1}{x^2}}{\left(\frac{y}{x}-1\right)\left(\frac{y}{x}-2\right)}$$

$$=\frac{7(-1)-3(-1)^2-2}{(-1-1)(-1-2)}=\frac{-7-3-2}{(-2)(-3)}=-2$$

$\therefore \quad x+y+2=0$

and the third asymptote corresponding to the factor $y-2x$ is

$$y-2x=\lim_{x\to\infty,\frac{y}{x}\to 2}\frac{7xy-3y^2-2x^2-2x-2y-1}{(y-x)(y+x)}$$

$$=\lim_{x\to\infty,\frac{y}{x}\to 2}\frac{7\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2-2-2\left(\frac{1}{x}\right)-2\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)-\frac{1}{x^2}}{\left(\frac{y}{x}-1\right)\left(\frac{y}{x}+1\right)}$$

$$=\frac{7(2)-3(2)^2-2}{(2-1)(2+1)}=\frac{14-12-2}{3}=0.$$

$\Rightarrow y-2x=0$

Hence, all the asymptotes are $y-x+1=0, x+y+2=0$ and $y-2x=0$.

**Example 3.** *Find all the asymptotes of the curve* $(y-x)(y-2x)^2+(y+3x)(y-2x)+2x+2y-1=0$.

**Solution .** The equation of the curve is

$$(y-x)(y-2x)^2+(y+3x)(y-2x)+2x+2y-1=0$$

The asymptotes corresponding to the factor $(y-2x)^2$ are

$$(y-2x)^2+(y-2x)\lim_{\substack{x\to\infty,\\ y/x\to 2}}\frac{y+3x}{y-x}+\lim_{x\to\infty,y/x\to 2}\frac{2x+2y-1}{(y-x)}=0$$

or $$(y-2x)^2+(y-2x)\lim_{\substack{x\to\infty,\\ y/x\to 2}}\left(\frac{\frac{y}{x}+3}{\frac{y}{x}-1}\right)+\lim_{\substack{x\to\infty,\\ y/x\to 2}}\frac{2+2(y/x)-1/x}{(y/x-1)}=0$$

or $(y-2x)^2+5(y-2x)+6=0$ or $(y-2x)=\dfrac{-5\pm\sqrt{(25-24)}}{2}=\dfrac{-5\pm 1}{2}$

or $y-2x=-2$, and $y-2x=-3$

or $y-2x+2=0$ and $y-2x+3=0$

And the asymptote corresponding to the factor $(y-x)$ is

$$(y-x)+\lim_{\substack{x\to\infty,\\ y/x\to 1}}\frac{(y+3x)(y-2x)}{(y-2x)^2}+\lim_{x\to\infty,y/x\to 1}\frac{2x+2y-1}{(y-2x)^2}=0$$

or $$(y-x)+\lim_{\substack{x\to\infty,\\ y/x\to 1}}\frac{(y/x+3)(y/x-2)}{(y/x-2)^2}+\lim_{\substack{x\to\infty,\\ y/x\to 1}}\frac{2+2(y/x)-1/x}{x(y/x-2)^2}=0$$

or $(y-x)+\dfrac{(1+3)(1-2)}{(1-2)^2}+0=0$

or $y-x-4=0$

Hence, all the asymptotes of the given curve are $y-2x+2=0$, $y-2x+3=0$ and $y-x-4=0$.

## STUDENT ACTIVITY

1. Find the asymptotes of the curve $(x-y+1)(x-y-2)(x+y)=8x-1$.

2. Find the asymptotes of the curve $(x^2-3x+2)(x+y-2)+1=0$.

3. Find the asymptotes of the curve $x^2(x+y)(x-y)^2+ax^3(x-y)-a^2y^3=0$.

4. Find the asymptotes of the curve $(y-a)^2(x^2-a^2)=x^4+a^4$.

Notes

## TEST YOURSELF

*Find all the asymptotes of the following curves:*

1. $(x^2-y^2)(x+2y+1)+x+y+1=0$
2. $x^5-y^5=a^3xy$
3. $(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0$
4. $x^2(x^2-y^2)(x-y)+2x^3(x-y)-4y^3=0$
5. $xy(x^2-y^2)(x^2-4y^2)+xy(x^2-y^2)+x^2+y^2-7=0$
6. $(x-2y)^2(x-y)-4y(x-2y)-(8x+7y)=0$
7. $(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0$
8. $(x-y-1)^2(x^2+y^2+2)+6(x-y-1)(xy+7)-8x^2-2x-1=0$
9. $(\alpha_1x+\beta_1y+\gamma_1)(\alpha_2x+\beta_2y+\gamma_2)+\gamma_3=0$
10. $(x-y+2)(2x-3y+4)(4x-5y+6)+5x-6y+7=0$

### ANSWERS

1. $x-y=0, x+y=0, x+2y+1=0$
2. $y-x=0$
3. $x-y=0, 2x-y=0, x+y+1=0, 2x+y+1=0$
4. $x-y+2=0, x-y-1=0, x+y+1=0, x+2=0$
5. $x=0, y=0, x-y=0, x+y=0, x-2y=0$ and $x+2y=0$
6. $x-y+4=0, x-2y=2\pm3\sqrt{3}$
7. $x-y-2=0, x-y-3=0$
8. $x-y-2=0, x-y+3=0$
9. $\alpha_1x+\beta_1x+\gamma_1=0, \alpha_2x+\beta_2y+\gamma_2=0$
10. $x-y+2=0, 2x-3y+4=0, 4x-5y+6=0$

## 6.10 INTERSECTION OF A CURVE WITH ITS ASYMPTOTES

Let the equation $\quad y=mx+c \quad$ ...(1)

be an asymptote of the curve

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0 \qquad \ldots(2)$$

Solving (1) and (2) to find the intersection points so eliminating $y$ between (1) and (2), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\ldots=0.$$

Now expand each term of above equation by Taylor's theorem, we have

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{x^2}\cdot\frac{1}{2!}\phi_n''(m)+\ldots\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\ldots\right]$$
$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{c}{x}\phi_{n-2}'(m)+\ldots\right]=0$$

or $$x^n\phi_n(m)+[c\phi_n'(m)+\phi_{n-1}(m)]x^{n-1}+\left[\frac{c^2}{2!}\phi_n''(m)+\frac{c}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\ldots=0. \qquad \ldots(3)$$

Since $y=mx+c$ is an asymptotes of the curve (2), then we have $\phi_n(m)=0$ and $c\phi_n'(m)+\phi_{n-1}(m)=0$.

Thus (3) becomes $\left[\frac{c^2}{2!}\phi_n''(m)+\frac{c}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\ldots=0.$ ...(4)

This is a equation of degree $n-2$ in $x$ so it will have atmost $n-2$ values of $x$ provided there is no asymptote parallel to $y = mx + c$ of the given curve.

Hence, in general we can say that any asymptote of a curve of the $n^{th}$ degree cuts the curve in $(n-2)$ points.

## REMARKS

- Since one asymptote of the curve of $n^{th}$ degree cuts the curve in $(n-2)$ points so $n$ asymptotes of that curve will cut in $n(n-2)$ points.
- If the equation of the curve of degree $n$ can be written as $F_n + P = 0$, where $F_n$ contains $n$ non-repeated linear factors and $P$ contains the terms almost of degree $n-2$, then $n(n-2)$ points of intersection of the curve will lie on the curve $P = 0$.

## Solved Examples

**Example 1.** *Show that the four asymptotes of the curve*

$$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0.$$

*Cut the curve in eight points which lie on the circle* $x^2+y^2=1$.

**Solution.** The given equation of the curve can be written as

$$(x-y)(x+y)(y-2x)(y+2x)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0 \quad \ldots(1)$$

The asymptote corresponding to the factor $x-y$ is

$$x-y+\lim_{\substack{x\to\infty,\\ y/x\to1}}\frac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x+y)(y-2x)(y+2x)}=0$$

$$\text{or } x-y+\lim_{\substack{x\to\infty,\\ \frac{y}{x}\to1}}\frac{6-5\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^3-\frac{1}{x}+3\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)-\frac{1}{x^3}}{\left(1+\frac{y}{x}\right)\left(\frac{y}{x}-2\right)\left(\frac{y}{x}+2\right)}=0$$

$$\text{or } x-y+\lim_{x\to\infty,\frac{y}{x}\to1}\frac{6-5-3+2}{(1+1)(1-2)(1+2)}=0$$

or $x-y=0.$

The asymptote corresponding to the factor $x+y$ is

$$x+y+\lim_{\substack{x\to\infty,\\ y/x\to-1}}\frac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x-y)(y-2x)(y+2x)}=0$$

$$\text{or } x+y+\lim_{\substack{x\to\infty,\\ y/x\to-1}}\frac{6-5(y/x)-3(y/x)^2+2(y/x)^3-(1/x)+3(y/x)(1/x)-(1/x^3)}{(1-y/x)(y/x-2)(y/x+2)}=0$$

$$\text{or } x+y+\frac{6-5(-1)-3(-1)^2+2(-1)^3}{(1+1)(-1-2)(-1+2)}=0$$

or $x+y-1=0.$

Now the asymptote corresponding to the factor $y-2x$ is

$$y-2x+\lim_{\substack{x\to\infty,\\ y/x\to2}}\frac{\{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1\}}{(x-y)(x+y)(y+2x)}=0$$

$$\text{or } y-2x+\lim_{\substack{x\to\infty,\\ y/x\to2}}\frac{\{6-5(y/x)-3(y/x)^2+2(y/x)^3-(1/x)+3(y/x)(1/x)-(1/x^3)\}}{(1-y/x)(1+y/x)(y/x+2)}=0$$

$$\text{or } y-2x+\frac{6-5(2)-3(2)^2+2(2)^3}{(1-2)(1+2)(2+2)}=0 \quad \text{or} \quad y-2x=0.$$

Notes

The asymptote corresponding to the factor $y + 2x$ is

$$y+2x+\lim_{\substack{x\to\infty,\\ y/x\to -2}} \frac{\{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1\}}{(x-y)(x+y)(y-2x)}=0$$

or

$$y+2x+\lim_{\substack{x\to\infty,\\ y/x\to -2}} \frac{\{6-5(y/x)-3(y/x)^2+2(y/x)^3-(1/x)+3(y/x)(1/x)-(1/x^3)\}}{(1-y/x)(1+y/x)(y/x-2)}=0$$

or $y+2x+\dfrac{6-5(-2)-3(-2)^2+2(-2)^3}{(1+2)(1-2)(-2-2)}=0,$

or $\quad y + 2x - 1 = 0.$

Hence, all the four asymptotes are $x - y = 0$, $x + y - 1 = 0$, $y - 2x = 0$ and $y + 2x - 1 = 0$.

Since one asymptote cuts the curve in $(4-2) = 2$ points so all the four asymptotes cut the curve in $4 \times 2 = 8$ points. Now combine all the asymptotes, we get

$(x-y)(x+y-1)(y-2x)(y+2x-1) = 0$

or $[x^2-y^2-(x-y)][y^2-4x^2-(y-2x)] = 0$

or $(x^2-y^2)(y^2-4x^2)-(x^2-y^2)(y-2x)-(x-y)(y^2-4x^2)+(x-y)(y-2x) = 0$

or $(x^2-y^2)(y^2-4x^2)-(x^2y-2x^3-y^3+2xy^2)-(xy^2-4x^3-y^3$

$\qquad +4x^2y)+xy-2x^2-y^2-2xy=0$

or $(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-2x^2-y^2+3xy=0. \qquad ...(2).$

Now subtract (2) from (1), we get $x^2 + y^2 = 1$.

Hence, all the eight points of intersection lie on the circle $x^2 + y^2 = 1$.

## STUDENT ACTIVITY

1. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3-6x^2y+11xy^2-6y^3+x+y+1=0$$

and which passes through the points (0, 0), (1, 0) and (0,1).

2. Show that the asymptotes of the curve $y^2(x^2-a^2) = x^2(x^2-4a^2)$ form two right angle triangles with the $x$-axis. $(y > 0)$.

## TEST YOURSELF

1. Show that the asymptotes of the curve $4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$ cut the curve in eight points which lie on the ellipse $x^2+4y^2=4$.
2. Find the asymptotes of the curve $x^2y-xy^2+xy+y^2+x-y=0$ and show that they cut the curve again in three points which lie on the straight line $x+y=0$.

3. Show that the eight points of intersection of the curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie on a rectangular hyperbola.

4. Show that the asymptotes of the cubic $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$ cut the curve in three points which lie on the straight line $x - y + 1 = 0$.

ANSWERS

2. $y = 0, x = 1, x - y + 2 = 0$ 5. $x^3 - 6x^2y + 11xy^2 - 6y^3 - x + 6y = 0$.

## 6.11 ASYMPTOTES OF NON-ALGEBRAIC CURVES

**Definition.** *A curve in which there are some terms involving cosine, sine, etc. is called non-algebraic curve.*

The method for finding the asymptotes of non-algebraic curves can be explained by following example.

**Example.** Let the equation of the curve be $y = \sec x$, then differentiating this w.r.t. '$x$', we get

$$\frac{dy}{dx} = \sec x \tan x.$$

Therefore, the tangent at $P(x, y)$ on the curve is

$$Y - \sec x = \frac{dy}{dx}(X - x)$$

or $$Y - \sec x = \sec x \tan x(X - x)$$

or $$Y \cos^2 x - \cos x = (X - x)\sin x. \qquad \ldots(1)$$

Now taking the distance of $P(x, y)$ from $(0, 0)$ infinity as $x \to \pi/2$ and $y \to \infty$, we get

$$Y.0 - 0 = (X - \pi/2).1 \text{ or } X = \pi/2.$$

This is one asymptote and the other asymptotes are $X = -\pi/2, \pm 3/2\pi, \ldots$

## 6.12 ASYMPTOTES OF POLAR CURVES

**(i) Equation of a line in polar form.** Let $O$ be the pole and $OX$ the initial line and let $P(r, \theta)$ be any point on the line whose equation is to be required as shown in Fig. 1.

Draw a perpendicular $OM$ from $O$ to the line such that $OM = p$ and $\angle MOX = \alpha$ (say).

$\therefore$ In $\Delta OPM$

$$\angle POM = \theta - \alpha$$

then, $$\frac{OM}{OP} = \cos \angle POM$$

or $$\frac{p}{r} = \cos(\theta - \alpha)$$

or $$p = r\cos(\theta - \alpha).$$

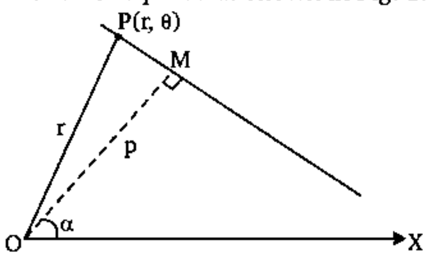


**Fig. 1.**

This is the equation of line in polar form, where $p$ is the perpendicular length from pole to this line and $\alpha$ is an angle which the perpendicular makes with initial line.

**(ii) Asymptotes of polar curves.**

**THEOREM 1.** *If $\theta = \alpha$ is a root of the equation $f(\theta) = 0$, then $r \sin(\theta - \alpha) = 1/f'(\alpha)$ is an asymptote of the curve $1/r = f(\theta)$.*

**Proof.** Since the equation of a curve in polar form is $\frac{1}{r} = f(\theta)$. ...(1)

Let $P(r, \theta)$ be any point on this curve and draw a line through $O$ perpendicular to $OP$, then radius vector which meets the tangent at $P$ in $T$ as show in Fig. 2. Then $OT$ is a polar subtangent of the curve at $P$.

Notes

$$OT = r^2 \frac{d\theta}{dr} \quad \text{(From calculus)}$$

Now differentiating (1) w.r.t. 'θ', we get

$$-\frac{1}{r^2}\frac{dr}{d\theta} = f'(\theta).$$

$$\therefore \quad OT = r^2 \frac{d\theta}{dr} = -\frac{1}{f'(\theta)}.$$

Since $\alpha$ is a root of $f(\theta) = 0$ as $\theta \to \alpha$, then $r \to \infty$ from (1) and the tangent $PT$ tends to the asymptote and

$$OT \to \left[-\frac{1}{f'(\theta)}\right]_{\theta=\alpha}, f'(\alpha) \neq 0.$$

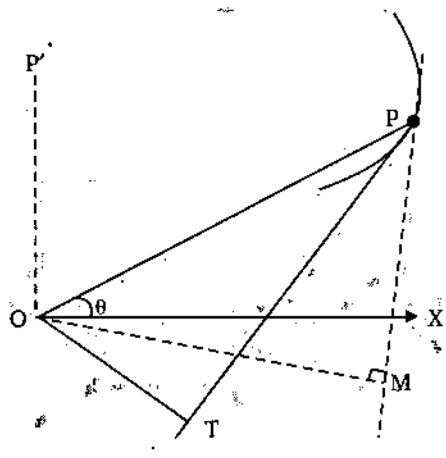


**Fig. 2.**

And $OP$, $PT$ will become parallel to lines shown dotted in the figure 2. Thus $\angle OTP \to \pi/2$ and $OT \to OM$, where $OM$ is a perpendicular distance from $O$ to the asymptote.

$$\therefore \quad OM = -\frac{1}{f'(\alpha)}$$

when $\theta \to \alpha$ *i.e.*, $OP \to OP'$. Then $\angle XOP' = \alpha$

$$\therefore \quad \angle MOX = -\left(\frac{\pi}{2} - \alpha\right) \quad \text{(In the clockwise direction)}$$

Therefore the equation of the asymptote is

$$r\cos\left[\theta - \left\{-\left(\frac{\pi}{2} - \alpha\right)\right\}\right] = -\frac{1}{f'(\alpha)} \quad [\text{using } p = r\cos(1-\alpha)]$$

$$\text{or} \quad r\cos\left(\frac{\pi}{2} + \theta - \alpha\right) = -\frac{1}{f'(\alpha)} \quad \text{or} \quad -r\sin(\theta - \alpha) = -\frac{1}{f'(\alpha)}$$

$$\text{or} \quad r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

## WORKING PROCEDURE

To find the asymptotes of polar curves, we use the follows steps :

**STEP 1.** Convert the equation of the given curve in the form $\frac{1}{r} = f(\theta)$.

**STEP 2.** Find the roots of the equation $f(\theta) = 0$ *i.e.*, values of θ. Suppose α, β, etc. are the roots of $f(\theta) = 0$.

**STEP 3.** Now the asymptote corresponding to $\theta = \alpha$ is

$$r\, sin\, (\theta - \alpha) = \frac{1}{f'(\alpha)}$$

where $f'(\alpha)$ = value of $f'(\theta)$ at $\theta = \alpha$.

## Solved Examples

**Example 1.** *Find the asymptotes of the curve $r \sin n\theta = a$.*

**Solution.** **Step I.** Convert the given curve into the form

$$\frac{1}{r} = f(\theta).$$

$$\therefore \quad \frac{1}{r} = \frac{\sin n\theta}{a} = f(\theta). \quad \ldots(1)$$

**Step II.** Solve the equation $f(\theta) = 0$.

$$i.e., \quad \frac{\sin n\theta}{a} = 0.$$

or $\sin n\theta = \sin r\pi,\ r = 0, 1, 2, ...,$

or $n\theta = r\pi$ or $\theta = \frac{r\pi}{n}, r = 0, 1, 2, 3, ....$

Let $\alpha = \frac{r\pi}{n}$.

Now differentiating (1) w.r.t. 'θ', we get $f'(\theta) = +\frac{n\cos n\theta}{a}$.

$\therefore\ f'(\alpha) = \frac{n\cos n\alpha}{a} = \frac{n}{a}\cos r\pi = \frac{n}{a}(-1)^r$.

**Step III.** Therefore, the asymptotes of the curve are $r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$

or $r\sin\left(\theta - \frac{r\pi}{n}\right) = \frac{a}{n(-1)^r}$, where $r$ is any integer.

**Example 2.** *Find the asymptotes of the curve r sin θ = a cos 2θ.*

**Solution.** First put the equation in the form of $\frac{1}{r} = f(\theta)$.

*i.e.,* $\frac{1}{r} = \frac{\sin\theta}{a\cos 2\theta}$.

$\therefore\ f(\theta) = \frac{\sin\theta}{a\cos 2\theta}$. ...(1)

Now solve the equation $f(\theta) = 0$. Then

$$\frac{\sin\theta}{a\cos 2\theta} = 0$$

or $\sin\theta = \sin n\pi$ or $\theta = n\pi$.

Let $\alpha = n\pi$ be the root of the equation $f(\theta) = 0$.

Now differentiating (1) w.r.t. 'θ', we get

$$f'(\theta) = \frac{1}{a}\left[\frac{\cos 2\theta.\cos\theta + 2\sin 2\theta\sin\theta}{\cos^2 2\theta}\right]$$

$\therefore\ f'(\alpha) = \frac{1}{a}\left[\frac{\cos 2\alpha.\cos\alpha + 2\sin 2\alpha\sin\alpha}{\cos^2 2\alpha}\right] = \frac{1}{2a}\left[\frac{\cos 2n\pi.\cos n\pi + 2\sin 2n\pi\sin n\pi}{\cos^2 2n\pi}\right]$ $(\because \alpha = n\pi)$

$= \frac{1}{a}\cos n\pi$.

The asymptote corresponding to $\alpha = n\pi$ is $r\sin(\theta - n\pi) = \frac{1}{f'(\alpha)} = \frac{a}{\cos n\pi}$

or $r(\sin\theta\cos n\pi - \cos\theta\sin n\pi) = \frac{a}{\cos n\pi}$

or $r\sin\theta\cos n\pi = \frac{a}{\cos n\pi}$ $(\because \sin n\pi = 0)$

or $r\sin\theta\cos^2 n\pi = a$

or $r\sin\theta = a$ $(\because \cos n\pi = 1)$

**Example 3.** *Find the asymptotes of the curve rθ = a.*

**Solution.** First putting the equation of curve in the form $\frac{1}{r} = f(\theta)$ so we have $\frac{1}{r} = \frac{\theta}{a}$.

$\therefore\ f(\theta) = \frac{\theta}{a}$. ...(1)

Putting $f(\theta) = 0$, we get $\theta = 0$.

Then $\alpha = 0$ is the root of $f(\theta) = 0$.

Now differentiating (1) w.r.t. 'θ', we get

$$f'(\theta) = \frac{1}{a}. \Rightarrow f'(\alpha) = \frac{1}{a}.$$

Thus the asymptote corresponding to $\theta = \alpha$ is

Notes

$$r\sin(\theta-\alpha)=\frac{1}{f'(\alpha)}.$$

$$\therefore\quad r\sin(\theta-0)=\frac{1}{(1/a)}\quad\text{or}\quad r\sin\theta=a.$$

**Example 4.** *Find the circular asymptotes of the curve* $r = a.\dfrac{\theta}{\theta-1}$.

**Solution.** The circular asymptote is given by. $r = a\lim_{\theta\to\infty}\dfrac{\theta}{\theta-1} = a$.

Thus $r = a$ is the circular asymptote.

## STUDENT ACTIVITY

1. Find the asymptotes of the curve $r\cos\theta = a\sin\theta$.

2. Find the asymptotes of the curve $r(1 + 2\sin\theta) = 2$.

3. Find the asymptotes of the curve $r\sin\theta = 2\theta$.

## TEST YOURSELF

*Find the asymptotes of the following curves:*

1. $y = \tan x$.
2. $r = a\,\text{cosec}\,\theta + b$
3. $r\sin 2\theta = a$
4. $r\sin\theta = 2\cos 2\theta$
5. $r\sin\theta = 2\cos\theta$
6. $r\theta\cos\theta = a\cos 2\theta$
7. $r(1 - 2\cos\theta) = 2a$
8. $r = 4(\sec\theta + \tan\theta)$
9. $r\cos\theta = 4\sin^2\theta$
10. $r(e^\theta - 1) = a(e^\theta + 1)$

## ANSWERS

1. $x = \pm\pi/2, \pm 3\pi/2...$
2. $r\sin\theta = a$
3. $r\sin\theta=\pm\frac{1}{2}a, r\cos\theta=\pm\frac{1}{2}a$
4. $r\sin\theta = 2$
5. $r\sin\theta = \pm 2$
6. $r\sin\theta = a, r\cos\theta = \dfrac{a}{\left(k+\frac{1}{2}\right)\pi}$, $k$ is any integer
7. $r\sin\left(\theta-\frac{\pi}{3}\right)=\frac{2a}{\sqrt{3}}, r\sin\left(\theta+\frac{\pi}{3}\right)=-\frac{2a}{\sqrt{3}}$
8. $r\cos\theta = 8$
9. $r\cos\theta = 4$
10. $r\sin\theta = 2a$

## 6.13 CONCAVE AND CONVEX CURVES

If $P$ is any point on a curve and $CD$ is any given line which does not passes through this point $P$. Then the curve is said to be concave at $P$ with respect to the line $CD$ if the small arc of the curve containing $P$ lies entirely within the acute angle between the tangent at $P$ to the curve and the line $CD$ and the curve is said to be convex at $P$ if the arc of the curve containing $P$ lies wholly outside the acute angle between that tangent at $P$ and the line $CD$ which are shown in figures below :

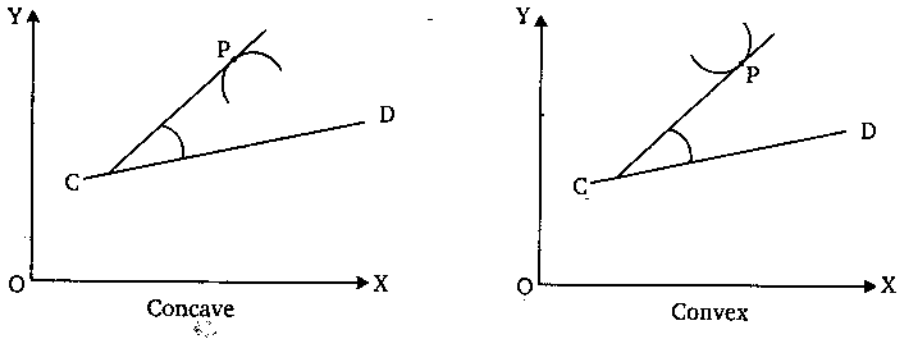


**Fig. 3**

## 6.14 POINT OF INFLEXION

A point $P$ on the curve is said to be the point of inflexion, if the curve in one side of $P$ is concave and other side of $P$ is convex with respect to the line $CD$ which does not passes through the point $P$ as shown in fig. 5.

**Inflexion tangent.** The tangent at the point of inflexion of a curve is said to be inflexion tangent. In the fig. 4 the line $PQ$ is the inflexion tangent.

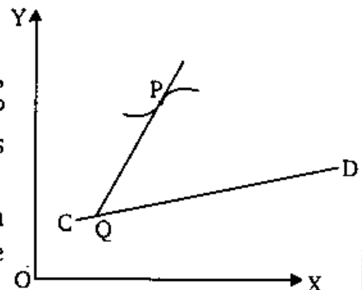


**Fig. 4**

## 6.15 DETERMINATION OF THE POINTS OF INFLEXION

Let $y = f(x)$ be the equation of a curve and let $P(x, y)$ be any point on the curve and assuming that the tangent at $P$ is not parallel to $y$-axis as shown in fig. 5.

Since the tangent is taken not to be parallel to $y$-axis, then $\frac{dy}{dx} = f'(x)$ must be finite. Let $Q(x + h, y + k)$ be any point on the curve in the neighbourhood of $P$. We may take this point $Q$ either side of $P$. Suppose the ordinate $OM$ of $Q$ intersects the tangent line at $Q'$.

$$Y - y = f'(x)(X - x) \qquad \ldots(1)$$

Since at point $Q(x + h, x + k)$ we have $X = x + h$ so putting $X = x + h$ in (1), we get

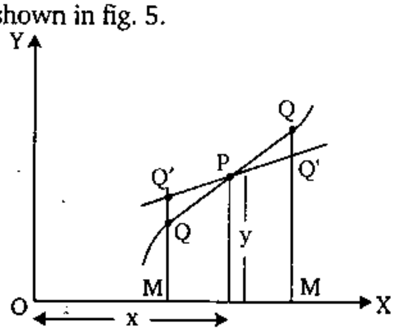


**Fig. 5**

$$Q'M - y = f'(x)(x + h - x) \qquad [\because Y = Q'M]$$

or $$Q'M = y + hf'(x)$$

or $$Q'M = f(x) + hf'(x). \qquad [\because y = f(x)]$$

But we know that $$QM = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

(Using Taylor's theorem)

$$\therefore\ QM - Q'M = \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x + \theta h) \text{ where } 0 < \theta < 1. \qquad \ldots(2)$$

Notes

Let us suppose $f''(x) \neq 0$ and taking $h$ sufficiently small, then $(QM - Q'M)$ will have the same sign as $\frac{h^2}{2!}f''(x)$. But $\frac{h^2}{2!}f''(x)$ will have invariable sign because $h^2$ will always be positive. This means that on both sides of $P$ the curve will be either concave or convex. Hence, we can say that the necessary condition for the existence of a point of inflexion at $P$ is given by

$$f''(x) = 0 \text{ or } \frac{d^2y}{dx^2} = 0.$$

Thus (2) now becomes

$$QM - Q'M = \frac{h^2}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + ... + \frac{h^n}{n!}f^{(n)}(x + \theta h) \qquad ...(3)$$

Further, if $f'''(x) \neq 0$ and taking $h$ to be very small, then $(QM - Q'M)$ will have the same sign as $\frac{h^3}{3!}f'''(x)$ and this changes sign when $h$ changes sign. Thus we can say that the curve with respect to the $x$-axis is concave on one side of $P$ and convex on other side of $P$. Hence, there will exist a point of inflexion at $P$.

Consequently, we can have a point of inflexion at $P$, if $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$.

**REMARKS**

- The position of a point of inflexion is independent of the choice of co-ordinate axes so we can say that a point of inflexion at $P$ exists if $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$.
- If $f''(x) = 0 = f'''(x) = ... = f^{(n-1)}(x)$ and $f^{(n)}(x) \neq 0$, then there will be a point of inflexion if $n$ is odd and if $n$ is even and greater than 2, then the point is called point of undulation.
- If the tangent at $P$ is parallel to $y$-axis, then $\frac{dy}{dx}$ will be infinite at $P$ so change the curve to the form $x = f(y)$ and then find the point of inflexion.

## Solved Examples

**Example 1.** *Find the points of inflexion of the curve $x = (\log y)^3$.*

**Solution.** The equation of the curve is

$$x = (\log y)^3. \qquad ...(1)$$

Differentiating (1) with respect to '$y$', we get

$$\frac{dx}{dy} = 3(\log y)^2 . \frac{1}{y}$$

Again differentiating w.r.t. $y$

$$\frac{d^2x}{dy^2} = 3\left[\frac{2\log y}{y^2} - \frac{(\log y)^2}{y^2}\right]. \qquad ...(2)$$

Again differentiating w.r.t. '$y$', we get

$$\frac{d^3x}{dy^3} = 3\left[\frac{2}{y^3} - \frac{4\log y}{y^3} - \frac{2\log y}{y^3} - \frac{2(\log y)^2}{y^2}\right]. \qquad ...(3)$$

For the point of inflexion, we have

$$\frac{d^2x}{dy^2} = 0.$$

$$\therefore \quad 3\left[\frac{2\log y - (\log y)^2}{y^2}\right] = 0$$

Notes

or $\quad 3(\log y)(2-\log y) = 0$

or $\quad \log y = 0, \log y = 2 \quad$ or $\quad y = 1, y = e^2$

From (3) it is obvious that at $y = 1, y = e^2$,

$$\frac{d^3x}{dy^3} \neq 0.$$

Hence, the points of inflexion are $(0, 1)(8, e^2)$.

**Example 2.** *Find the points of inflexion of the curve*

$$y^2 = x(x+1)^2.$$

**Solution.** The equation of the curve can be written as

$$y = (x+1)\sqrt{x}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{3}{2}.x^{1/2} + \frac{1}{2\sqrt{x}}.$$

Again differentiating w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{x}} - \frac{1}{4x^{3/2}}. \qquad \ldots(2)$$

and again differentiating w.r.t. 'x', we get

$$\frac{d^3y}{dx^3} = -\frac{3}{8x^{3/2}} + \frac{3}{8x^{5/2}}. \qquad \ldots(3)$$

For the point of inflexion, we have

$$\frac{d^2y}{dx^2} = 0.$$

$$\therefore \quad \frac{3}{4\sqrt{x}} - \frac{1}{4x\sqrt{x}} = 0$$

or $\quad \left(3 - \frac{1}{x}\right) = 0 \quad$ or $\quad x = 1/3.$

From (3) it is obvious that at $x = 1/3$, $\dfrac{d^3y}{dx^3} \neq 0$.

Thus, the point of inflexion are given by $(1/3, \pm 4/3\sqrt{3})$.

## STUDENT ACTIVITY

**1.** Show that the points of inflexion on the curve $y = be^{-(x/a)^2}$ are given by $x = \pm a/\sqrt{2}$.

**2.** Find the points of inflexion on the curve $r(\theta^2 - 1) = a\theta^2$.

**3.** Show that the points of inflexion of the curve $r = b\theta^n$ are given by $r = b\{-n(n+1)\}^{n/2}$.

Notes

## TEST YOURSELF

1. Find the points of inflexion of the curve $x = \log(y/x)$.
2. Find the points of inflexion of the curve $y(a^2+x^2) = x^3$.
3. Find the points of inflexion of the curve $y = (x-1)^4(x-2)^3$.
4. Find the points of inflexion of the curve $xy = a^2\log(y/a)$.
5. Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$.
6. Show that the origin is a point of inflexion of the curve $a^{m-1}. y = x^m$, if $m$ is odd and greater than 2.
7. Show that the points of inflexion of the curve $x^2y=a^2(x-y)$ are given by $x = 0, x = \pm a\sqrt{3}$.
8. Prove that the curve $y = (1-x)/(1+x^2)$ has three points of inflexion which lie on a straight line.
9. Show that the abscissae of the points of inflexion on the curve $y^2 = f(x)$ satisfy the equation $[f'(x)]^2 = 2f(x)f''(x)$.

### ANSWERS

**1.** $(-2, -2/e^2)$ **2.** $(0,0), \left(\sqrt{3}a, \frac{3\sqrt{3}}{4}a\right), \left(-\sqrt{3}a, \frac{-3\sqrt{3}}{4}a\right)$ **3.** Point of inflection at $x = 2$, $(11\pm\sqrt{2})/7$

**4.** $\left(\frac{3}{2}ae^{-3/2}, ae^{3/2}\right)$

## 6.16 MULTIPLE AND SINGULAR POINTS

**Definition 1.** *A point on the curve is said to be multiple points if through this point more than one branches of a curve passes.*

**Definition 2.** *A point on the curve is called a double point if through it two branches of the curve passes.*

**Definition 3.** *If three branches of the curve passes through a point, then this point is called triple point.*

**Definition 4.** *If n branches passes through a point on the curve, then this point is called a multiple point of $n^{th}$ order.*

**Definition 5.** *The point of inflexion and multiple points are also called the singular points. Or An unusual point on the curve is basically called a singular point.*

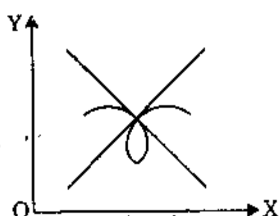

**Fig. 6**

### 6.15.1 TYPES OF DOUBLE POINT

**(i) Node.** A double point on a curve is said to be a node, if through this double point two branches of the curve passes which are real and having two different tangents at that point (Fig. 6).

**(ii) Cusp.** A double point on a curve is called a cusp if through this double point two real branches of the curve passes and have real coincident tangents at that point (Fig. 7).

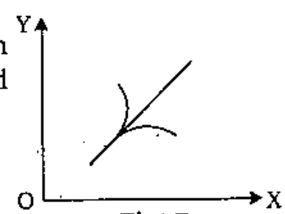

**Fig. 7**

Notes

(iii) **Conjugate point.** A point $P$ on the curve is said to be conjugate point if there are no real points on the curve in the neighbourhood of that point and having no real tangent at that point (Fig. 8).

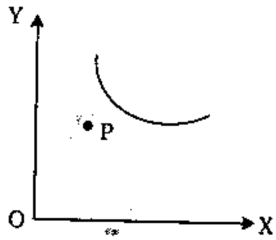


**Fig. 8**

## 6.17 SPECIES OF CUSP

**Definition.** *A cusp is said to be single if the curve lies entirely on one side of the common tangent* (Fig. 9(ii)).

**Definition.** *A cusp is said to be double if the curve lies on both sides of the common tangent* (Fig. 9(i)).

**Definition.** *A cusp is said to be of first species if the two branches of the curve lie on opposite sides of common tangent* (Fig. 9(iii)).

**Definition.** *A cusp is said to be of second species if the two branches of the curve lie on same side of the common tangent* (Fig. 9(ii)).

There are five different types of cusp :

(i) Single cusp of first species (ii) Single cusp of second species

(iii) Double cusp of first species (iv) Double cusp of second species

(v) Double cusp with change of species.

These all five types of cusp are shown below respectively :

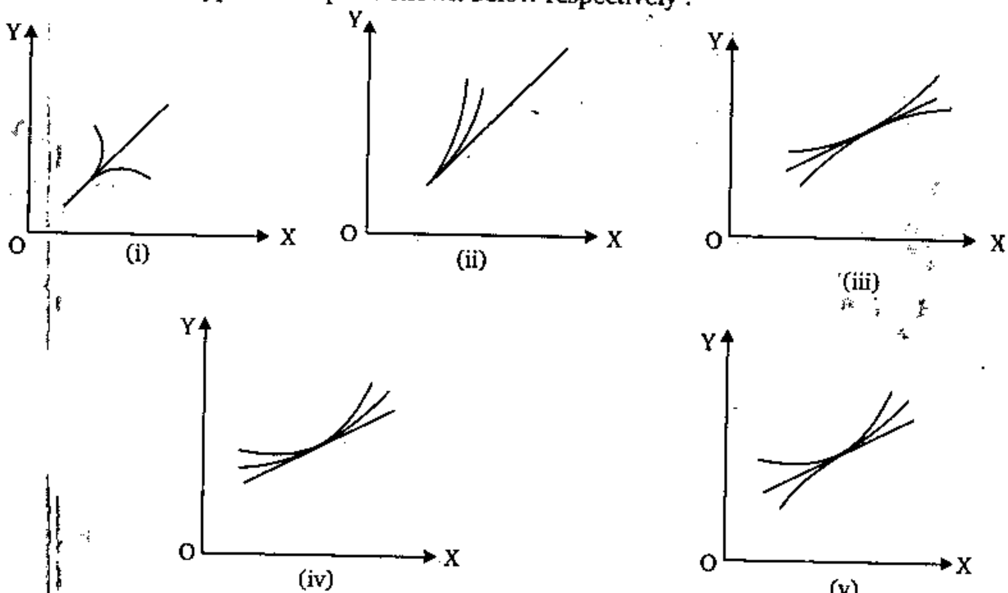


**Fig. 9**

## 6.18 POSITION AND NATURE OF DOUBLE POINTS

Let $P(x, y)$ be any point on the curve $f(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \qquad ...(1)$$

Therefore, the slope of the tangent at $P(x, y)$ is equal to $dy/dx$ which is given above.

Since by the definition of a multiple point we know that the curve has atleast two tangents so $\frac{dy}{dx}$ has atleast two values at a multiple point. But the equation (1) is of first degree in $dy/dx$. Therefore $dy/dx$ will have two values or more than one value, if and only if

Notes

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$$

Thus the necessary and sufficient condition for any point of the curve $f(x, y) = 0$ to be a multiple point are that

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}.$$

Hence, to find the multiple point of the curve $f(x, y) = 0$ we shall simultaneously solve the following equations

$$f(x,y)=0, \frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0.$$

Next, differentiating (1) w.r.t. 'x', we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right)+\frac{d}{dx}\left(\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x}\right)=0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dx}+\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\frac{\partial f}{\partial y}\cdot\frac{d^2y}{dx^2}=0$$

or $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dx}+\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dx}\right]\frac{dy}{dx}+\frac{\partial f}{\partial y}\cdot\frac{d^2y}{dx^2}=0.$$

Sinceatthemultiplepoint $\frac{\partial f}{\partial y}=0$. Therefore, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dy}{dx}+\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2=0$

or $$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2=0 \qquad \ldots(2)$$

$$\left(\because \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}\right)$$

This is a quadratic equation in $\frac{dy}{dx}$ and the multiple point will be double point if the equation (2) will remain quadratic in $\frac{dy}{dx}$, and for the quadratic in $\frac{dy}{dx}$ it is assumed that $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y^2}$ are not all zero. From the equation (2) it is obvious that the two values of $dy/dx$ will be real and distinct, coincident, or imaginary according as

$$\left[\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\right]>,= \text{or} < 0.$$

Therefore, the two tangents will be real and distinct, coincident or imaginary according as

$$\left[\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\right]>,= \text{or} < 0.$$

Hence we obtained that the double point will be node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 > \text{or} = \text{or} < \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}.$$

**REMARK**

- If $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y^2}$ are all zero, then the point $P(x, y)$ will be a multiple point of order greater that two.

## 6.19 NATURE OF A CUSP AT THE ORIGIN

Let (0, 0) be a cusp of the curve. Then there will be two coincident tangents at (0, 0). Therefore, the curve will be of the form

$$(ax + by)^2 + \text{terms of degree greater then two} = 0 \qquad ...(1)$$

Thus the common tangent to the curve (1) at the origin is

$$ax + by = 0. \qquad ...(2)$$

Let us suppose $p$ is perpendicular from any point $P(x, y)$ to the equation (2), then

$$p = \frac{ax + by}{\sqrt{a^2 + b^2}} \qquad ...(3)$$

where $P(x, y)$ is any point in the neighbourhood of (0,0).

From the equation (3) it is obvious that $p$ is proportional to $ax + by$ so let us take

$$p = ax + by. \qquad ...(4)$$

Now eliminating either $x$ or $y$ between (1) and (4), we get the equation involving $p$ and $x$. Since $p$ is small and there are two branches of the curve passes through the origin, therefore, neglecting all those terms having the degree of $p$ greater than two. Thus we obtain a quadratic in $p$ of the form

$$Ap^2 + Bp + C = 0 \qquad ...(5)$$

where $A, B, C$ are the functions of $x$ only.

Now solving (5), we get $\qquad p = -\frac{B \pm \sqrt{(B^2 - 4AC)}}{2A}$ also $p_1p_2 = C/A$

where $p_1$ and $p_2$ are the roots of (5).

Now there arises following cases :

**Case I.** If for all numerically small values of $x$ either negative or positive, the values of $p$ obtained from (5) are imaginary, then the origin will be a conjugate point.

**Case II.** If the values of $p$ are real for all numerically small values of $x$, then the origin will be a double cusp.

**Case III.** If the reality of $p$ depends on the sign of $x$, then origin will be a single cusp.

**Case IV.** If $p$ is real for numerically small values of $x$ and if $p_1p_2 > 0$, then $p_1$ and $p_2$ will have same sign. Therefore the origin will be a cusp of second species because the two perpendiculars $p_1$ and $p_2$ lie on the same side of the common tangent. On the other hand if $p_1p_2 < 0$,then $p_1$ and $p_2$ are of opposite signs. Then the origin will be a cusp of the first species because the two perpendicular line on the opposite sides of the common tangent.

## 6.20 NATURE OF A CUSP AT ANY POINT

In order to find the nature of the cusp at any point $(h, k)$. We first shift the origin at $(h, k)$ and then apply above process discussed in § 6.19.

### Solved Examples

**Example 1.** *Show that the origin is a node on the curve* $x^3 + y^3 - 3axy = 0$.

**Solution .** The tangent at the origin are obtained by equating to zero the lowest degree terms *i.e.*, second degree term in the given equation of the curve.

$$\therefore \qquad -3axy = 0 \text{ or } x = 0, y = 0.$$

Thus at the origin there are two real and distinct tangents. Hence (0, 0) is a node.

**Example 2.** *Find the double point of the curve* $(x - 2)^2 = y(y - 1)^2$.

**Solution .** Let $\qquad f(x, y) \equiv (x - 2)^2 - y(y - 1)^2 = 0 \qquad ...(1)$

Notes

Differentiating (1) partially w.r.t. $x$ and $y$, we get

$$\frac{\partial f}{\partial x} = 2(x-2) \quad \text{...(2)}$$

and $$\frac{\partial f}{\partial y} = -(y-1)^2 - 2y(y-1). \quad \text{...(3)}$$

Since the necessary and sufficient condition for a double points are

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \Rightarrow 2(x-2) = 0 \quad \text{...(4)}$$

$$-(y-1)^2 - 2y(y-1) = 0. \quad \text{...(5)}$$

Now solving $f(x, y) = 0, \frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

From (4), we get $x = 2$ and from (5), we get

$$-(y-1)(-y-1+2y) = 0$$

or $-(y-1)(2y-1) = 0$ or $y = 1$ and $y = 1/3$.

$\therefore$ Possible double points are (2, 1) and (2, 1/3)

But (2, 1/3) does not satisfy $f(x, y) = 0$. Hence only double point is (2, 1).

**Example 3.** *Examine the nature of the origin on the following curve :* $y^2 = a^2x^2 + bx^3 + cxy^2$.

**Solution .** The given curve is $f(x, y) \equiv y^2 - a^2x^2 - bx^3 - cxy^2 = 0$. ...(1)

Equating to zero the lowest degree terms in the equtaion of curve (1), we get

$$y^2 - a^2x^2 = 0 \text{ or } y = \pm\, ax.$$

Thus we have obtained two real and distinct tangents at (0, 0). Hence (0, 0) is a node.

**Example 4.** *Find the position and nature of the double points on the curve* $x^2y^2 = (a+y)^2(b^2-y^2)$ *if*

(*i*) $b > a$ (*ii*) $b = a$ (*iii*) $b < a$.

**Solution .** Let $f(x, y) \equiv x^2y^2 - (a+y)^2(b^2-y^2) = 0$. ...(1)

Differentiating (1) partially w.r.t. 'x' and 'y' respectively, we get

$$\frac{\partial f}{\partial x} = 2xy^2 \quad \text{...(2)}$$

and $$\frac{\partial f}{\partial y} = 2x^2y - 2(a+y)(b^2-y^2) + 2y(a+y)^2. \quad \text{...(3)}$$

Again differentiating, we get

$$\frac{\partial^2 f}{\partial x^2} = 2y^2 \; ; \qquad \frac{\partial^2 f}{\partial x \partial y} = 4xy$$

and $$\frac{\partial^2 f}{\partial y^2} = 2x^2 - 2(b^2-y^2) + 4(a+y)y + 2(a+y)^2 + 4y(a+y)$$

For double point, we have

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \therefore 2xy^2 = 0 \quad \text{...(4)}$$

$$2x^2y - 2(a+y)(b^2-y^2) + 2y(a+y)^2 = 0 \quad \text{...(5)}$$

From (4) we get $x = 0, y = 0$

From (5) and $x = 0$, we get

$$2(a+y)[-(b^2-y^2) + y(a+y)] = 0$$

or $$2(a+y)(2y^2 + ay - b^2) = 0$$

or $$y = -a \text{ and } y = \frac{-a \pm \sqrt{(a^2+8b^2)}}{4}$$

Notes

Thus we obtain $(0, -a)$ and $\left(0, \frac{-a \pm \sqrt{(a^2+8b^2)}}{4}\right)$ and from (5) and $y=0$, we get two points.

Hence, $(0, -a)$ and $\left(0, \frac{-a \pm \sqrt{(a^2+8b^2)}}{4}\right)$ are possible double points. But only $(0, -a)$ satisfies the equation $f(x, y) = 0$. Hence, $(0, -a)$ is only the double point.

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,-a)} = (2y^2)_{(0,-a)} = 2a^2$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,-a)} = (4xy)_{(0,-a)} = 0$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,-a)} = [2x^2 - 2(b^2 - y^2) + 4y(a+y) + 2a(a+y)^2 + 4y(a+y)]_{(0,-a)}$$
$$= 2(a^2 - b^2)$$

Then $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}.\frac{\partial^2 f}{\partial y^2} = 0 - 2a^2[2(a^2 - b^2)] = 4a^2(b^2 - a^2).$

(i) If $b > a$, then $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > \frac{\partial^2 f}{\partial x^2}.\frac{\partial^2 f}{\partial y^2}$ and thus $(0, -a)$ is a node.

(ii) If $b = a$, then $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2}.\frac{\partial^2 f}{\partial y^2}$ and thus $(0, -a)$ is a cusp.

(iii) If $b < a$, then $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < \frac{\partial^2 f}{\partial x^2}.\frac{\partial^2 f}{\partial y^2}$ and thus $(0, -a)$ is a conjugate point.

**Example 5.** *Find the nature of origin on the curve* $x^4 + y^3 + 2x^2 + 3y^2 = 0$.

**Solution.** Let $f(x, y) = x^4 + y^3 + 2x^2 + 3y^2 = 0$

Then $\frac{\partial f}{\partial x} = 4x^3 + 4x, \frac{\partial f}{\partial y} = 3y^2 + 6y$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 4, \frac{\partial^2 f}{\partial y^2} = 6y + 6$$

and $\frac{\partial^2 f}{\partial x \partial y} = 0.$

At (0, 0) $\frac{\partial^2 f}{\partial x^2} = 4, \frac{\partial^2 f}{\partial y^2} = 6, \frac{\partial^2 f}{\partial x \partial y} = 0.$

$$\therefore \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$

Hence, the origin is a conjugate point.

## STUDENT ACTIVITY

1. Examine the nature of the double points of the curve $2(x^3 + y^3) - 3(3x^2 + y^2) + 12x = 4$.

Notes

2. Find the position and nature of the double points of the curve $a^4y^2 = x^4(2x^2 - 3a^2)$.

3. Find the position and nature of the double points of the curve $x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0$.

## TEST YOURSELF

1. Find the equation of the tangents at the origin to the following curves :
   (a) $(x^2 + y^2)(2a - x) = b^2x$ (b) $a^4y^2 = x^4(x^2 - a^2)$
   (c) $x^4 + 3x^3y + 2xy - y^2 = 0$ (d) $x^3 + y^3 = 3axy$
2. Examine the nature of the origin on the curve $(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0$.
3. Show that the origin is a conjugate point on the curve $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$.
4. Show that the origin is a conjugate point on the curve $y^2 = 2x^2y + x^4y - 2x^4$.
5. Find the position and nature of double points of the curve $y^3 = x^3 + ax^2$.

### ANSWERS

1. (a) $x = 0$ (b) $y = 0, y = 0$ (c) $y = 0, 2x - y = 0$ (d) $x = 0, y = 0$
2. Origin is a single cusp of first species
5. A cusp at (0, 0)

## Summary

- A definite straight line whose distance from branch of the curve continuously decreases as we move away from the origin along the branch of the curve and seems to touch the branch at infinity, provided the distance of this line from origin should be finite initially, is called an asymptote of the curve.
- We obtain the asymptotes parallel to $x$-axis by taking the coefficient of highest power of $x$ in the equation of the curve equal to zero.
- We may obtain the asymptotes parallel to $y$-axis by taking the coefficient of highest power of $y$ in the equation of the curve equal to zero.
- If the coefficient of highest power of $x$ or $y$ or both are constant, then no asymptotes parallel to either $x$ or $y$ or both axes exists respectively.
- The asymptotes of an algebraic curve are parallel to the lines which obtained by equating to zero the linear factors of the highest degree terms of the equation of curve.
- A curve in which there are some terms involving cosine, sine, etc. is called non-algebraic curve.
- A point $P$ on the curve is said to be the point of inflexion, if the curve in one side of $P$ is concave and other side of $P$ is convex with respect to the line $CD$ which does not passes

Notes

through the point *P*.

- A point on the curve is said to be multiple points if through this point more than one branches of a curve passes.
- A point on the curve is called a double point if through it two branches of the curve passes.
- If three branches of the curve passes through a point, then this point is called triple point.
- If $n$ branches passes through a point on the curve, then this point is called a multiple point of $n^{th}$ order.
- The point of inflexion and multiple points are also called the singular points. Or An unusual point on the curve is basically called a singular point.
- A cusp is said to be single if the curve lies entirely on one side of the common tangent.
- A cusp is said to be double if the curve lies on both sides of the common tangent.
- A cusp is said to be of first species if the two branches of the curve lie on opposite sides of common tangent.
- A cusp is said to be of second species if the two branches of the curve lie on same side of the common tangent.

## Objective Evaluation

### FILL IN THE BLANKS

1. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $m =$ _________ and $c =$ _________.
2. The equation $\phi_n(m) = 0$ gives the _________ of the asymptotes.
3. If one or more values of $m$ obtained from $\phi_n(m) = 0$ are such that $\phi'_n(m) = 0$ and $\phi_{n-1}(m)$, then the asymptotes _________.
4. If the coefficients of highest degree terms of $y$ are constant, then there are no asymptotes _________.
5. If the coefficients of highest degree terms of $x$ are not constant, then there will exist the asymptotes parallel to _________.
6. The number of asymptotes of $n^{th}$ degree curve cannot exceed _________.
7. The asymptotes parallel to $y$-axis of the curve $y^2(x^2 - a^2) = x$ are _________.
8. The curve $y^2 = 4ax$ has _________ asymptotes.
9. The $n$ asymptotes of a curve of the $n^{th}$ degree cut if in _________ points.
10. If $\alpha$ is a root of the equation $f(\theta) = 0$, then $r \sin(\theta - \alpha) =$ _________ is an asymptote of the curve $\frac{1}{r} = f(\theta)$.

### TRUE/FALSE

**Write 'T' for True and 'F' for False statement.**

1. The line $y = mx + c$ is an asymptote of the curve $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots$. **(T/F)**
2. The polynomial $\phi_n(m)$ is obtained by putting $y = m$ and $x = m$ in the $n^{th}$ degree terms of the curve. **(T/F)**
3. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$ then $m = \lim_{x \to \infty} \left(\frac{y}{x}\right)$. **(T/F)**
4. The curve $x^5 - y^5 = a^3xy$ has at most five asymptotes real as well as imaginary. **(T/F)**
5. The numbers of asymptotes of the curve of $n^{th}$ degree can exceed $n$. **(T/F)**
6. The one asymptote of a curve of the $n^{th}$ degree cuts it in $(n - 1)$ points. **(T/F)**
7. The curve $x^2/a^2 + y^2/b^2 = 1$ has no real asymptotes. **(T/F)**
8. The curve $y^2 = 4ax$ has two real asymptotes. **(T/F)**
9. The asymptote parallel to $x$-axis of the curve $xy = c^2$ is $y = 0$. **(T/F)**

Notes

10. If $\alpha$ is a root of the equation $f(\theta) = 0$, then $r \sin(\theta - \alpha) = f'(\alpha)$ is an asymptote of the curve $\frac{1}{r} = f(\theta)$. **(T/F)**

MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $\lim_{x\to\infty}(y/x)$ equals :
   (a) $c$ (b) $m$ (c) $-m$ (d) $-c$
2. If $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$, then $\lim_{x\to\infty, y/x\to m}(y - mx)$ equals :
   (a) $m$ (b) $-c$ (c) $c$ (d) $-m$
3. The $n$ asymptotes of a curve of the $n^{\text{th}}$ degree cut it in how many points :
   (a) 2 (b) $n$ (c) $n-1$ (d) $n(n-2)$
4. For non existence of the asymptotes of the curve for some values of $m$ obtained by $\phi_n(m) = 0$ such that $\phi_{n-1}(m) \neq 0$ and $\phi'_n(m)$ equals :
   (a) 0 (b) 1 (c) $m$ (d) non-zero
5. The number of asymptotes of a curve of the $n^{\text{th}}$ degree can not exceed :
   (a) $n-1$ (b) $n$ (c) $n-2$ (d) $n+1$
6. The asymptote of the curve $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots$ is :
   (a) $y = mx$ (b) $y = mx + c$ (c) $y = m$ (d) $y = c$
7. The curve $y^2 = 4ax$ has how many real asymptotes ?
   (a) 1 (b) 2 (c) Zero (d) none of these
8. The asymptotes of the curve $r(e^\theta - 1) = a(e^\theta + 1)$ are :
   (a) $r \sin\theta = 2a$ (b) $r\cos\theta = 2a$ (c) $r\sin\theta = a$ (d) $r\cos\theta = a$
9. The number of real asymptotes of the curve $y^3 = x^3 + 3$ are :
   (a) 1 (b) 0 (c) 3 (d) 2
10. For the curve $x^3 + y^3 - 3axy = 0$, $\phi_3(m)$ is :
   (a) $m^2 + 1$ (b) $m + 1$ (c) $m - 1$ (d) $m^3 + 1$

## ANSWERS

FILL IN THE BLANKS

1. $\lim_{x\to\infty} y/x$, $\lim_{x\to\infty y/x\to m}(y - mx)$ 2. slopes 3. will not exist
4. parallel to $y$-axis 5. $x$-axis 6. $n$ 7. $x = \pm a$ 8. No 9. $n(n-2)$
10. $1/f'(\alpha)$

TRUE/FALSE

1. T 2. F 3. T 4. T 5. F 6. F 7. T
8. F 9. T 10. F

MULTIPLE CHOICE QUESTIONS

1. (b) 2. (c) 3. (d) 4. (a) 5. (b) 6. (b) 7. (c)
8. (a) 9. (b) 10. (d)

□□□□

Chapter

# 7

# Differentiability

Notes

## STRUCTURE

- Introduction
- Deriuvative of a function
- Continuity and differentiability
- Algebra of derivatives
- Rolle's theorem
- Lagrange's mean value theorem
- Cauchy's mean value theorem
  - Summary
  - Objective Evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:

- The concept of left and right hand derivatives
- The continuous and differentiability of a function
- Rolle's, Lagrange's and Cauchy's mean value theorems

## 7.1 INTRODUCTION

If a function $f(x)$ is defined on nbd of a point $a$ and

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exist (finitely), then the function $f(x)$ is said to be differentiable at $a$ and this limit is called derivative of the function $f(x)$ at $a$.

In symbols, this derivative, is denoted by $f'(a)$ and in full read as the derivative of $f(x)$ at $x=a$ with respect to the variable $x$. The process of evaluating $f'(a)$ is called differentiation.

**Graphically**, $f'(a)$ means the gradient of the curve $y = f(x)$ at the point $(a, f(a))$.

**Quantitatively** $f'(a)$ means the rate of change of the function $f(x)$ at $a$, with respect to the variable $x$.

## 7.2 DERIUVATIVE OF A FUNCTION

### 7.2.1 LEFT HAND DERIVATIVE

The left hand derivative (regressive derivative) of $f$ at $x = a$ is given by

$$Lf'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

and, is denoted by $Lf'(a)$.

### 7.2.2 RIGHT HAND DERIVATIVE

The right hand derivative (progressive derivative) of $f$ at $x = a$ is given by

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The derivative $f'(a)$ exists when $Lf'(a) = Rf'(a)$.

### 7.2.3 DIFFERENTIABILITY IN AN INTERVAL

(i) A function $f : ]a, b[\to R$ is said to be differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.

Notes

(ii) A function $f : [a, b] \to R$ is said to be differentiable in $[a, b]$ iff $Rf'(a)$ and $Lf'(b)$ exists and $f$ is differentiable at every point of $]a, b[$.

(iii) Let $f$ be a function whose domain is an interval $I$. If $I_1$ be the set of all those points $x$ of $I$ at which $f$ is differentiable *i.e.*, $f(x)$ exists and if $I_1 \neq \phi$, we get another function $f'$ with domain $I_1$. It is called the first derivative of $f$. Similarly $2^{nd}, 3^{rd}, \ldots n^{th}$ derivative of $f$ are defined and one denoted by $f'', f''', \ldots, f^n$ respectively of course, in order that $f^n(x)$ may be defined, it is necessary (though not sufficient) that $f^{n-1}(x)$ may be defined for all $x$ in some open interval containing $a$.

## REMARKS

- $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ means the same thing as $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
- The derivative of a function at a point and the derivative of a function are two different but related concepts. The derivative of $f$ at a point $a$ is a number while the derivative of $f$ is a function. However, very often the term derivative of $f$ is used to denote both number and function and it is left to the context to distinguish what is intended.
- If $f(x)$ is derivable on internal $I$ then $f'(x)$ at end points of $I$ (if exists) would mean a left or right hand derivative of $f(x)$ according as it is a right or a left hand end point of $I$. Similar meaning holds for higher order derivatives.

## 7.3 CONTINUITY AND DIFFERENTIABILITY

**THEOREM 1. (A necessary condition for the existence of a finite derivative).** *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.*

**Proof.** Let $f$ be differentiable at $a$. Then $\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)}$ exists and equal to $f'(a)$.

Now we may write

$$f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{(x-a)}(x-a) \qquad (\text{If } x \neq a)$$

Taking limit as $x \to a$, we get

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left\{ \frac{f(x) - f(a)}{(x-a)}(x-a) \right\}$$

$$= \lim_{x \to a} \left\{ \frac{f(x) - f(a)}{x-a} \right\} . \lim_{x \to a} (x-a)$$

($\because$ limit of the product of two functions is equal to product of their limits)

$$= f'(a).0 = 0$$

so that $\lim_{x \to a} f(x) = f(a) \Rightarrow f(x)$ is continuous at $x = a$.

Hence, $f$ is continuous at $x = a$. Thus continuity is a necessary condition for differentiability.

## REMARKS

- While continuity is a necessary condition for the differentiability, it is not a sufficient condition as it is clear from the following examples :

  (i) Consider the function $f(x)$ defined on R by setting

  $$f(x) = 0 \quad \text{if} \quad x = 0$$
  $$f(x) = x \quad \text{if} \quad x \neq 0$$

  $f$ is obviously continuous as also derivative at every point except possibly at $x=0$. At $x=0$, $f$ is continuous but not derivable.

  (ii) Consider the function $f(x)$ such that

  $$\begin{cases} x \sin \frac{1}{x} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

Notes

this function is continuous at $x=0$ but not differentiable at $x=0$.

(iii) The function $f(x)=|x|$ is a continuous function, but not differentiable at $x=0$. ($\because Lf'(0)=-1$ and $Rf'(0)=1$)

- Continuity of a function even at every point of R has nothing to do with the differentiability of the function at any point.

## 7.4 ALGEBRA OF DERIVATIVES

**THEOREM 1.** *Let functions f and g be defined on an interval I. If f and g are differentiable at $x=a\in I$, then $f\pm g$ is also differentiable and*

$$(f\pm g)'(a)=f'(a)\pm g'(a)$$

**Proof.** Since, the functions $f$ and $g$ are differentiable at $a$, therefore

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a) \qquad ...(1)$$

and $$\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g'(a) \qquad ...(2)$$

Now, consider $\lim_{x\to a}\frac{(f\pm g)(x)-(f\pm g)(a)}{x-a}$

$$=\lim_{x\to a}\frac{[f(x)\pm g(x)]-[f(a)\pm g(a)]}{x-a}$$

$$=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\pm\frac{g(x)-g(a)}{x-a}\right]$$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\pm\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

$$=f'(a)\pm g'(a)$$

Hence $f\pm g$ is differentiable at $a$ and

$$(f\pm g)'(a)=f'(a)\pm g'(a)$$

**THEOREM 2.** *Let a function f(x) be differentiable at a point a and $c\in$ R, then the function cf is also differentiable at a and $(cf)'(a)=cf'(a)$*

**Proof.** By the defination of the derivative of a function at $x=a$, we have

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

Now, consider

$$\lim_{x\to a}\frac{(cf)(x)-(cf)(a)}{x-a}=\lim_{x\to a}\frac{c\,f(x)-c\,f(a)}{x-a}$$

$$=\lim_{x\to a}\left\{c\left(\frac{f(x)-f(a)}{x-a}\right)\right\}$$

$$=c\lim_{x\to a}\left\{\frac{f(x)-f(a)}{x-a}\right\}=cf'(a)$$

Hence, $cf$ is differentiable at $a$ and $(cf)'(a)=cf'(a)$

**THEOREM 3.** *Let the functions f and g be defined on an interval I. If f and g are differentiable at $a\in I$, then f. g is also differentiable and $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$*

**Proof.** Since, $f$ and $g$ are differentiable at a, we have

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a) \qquad ...(1)$$

and $$\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g'(a) \qquad ...(2)$$

Notes

Consider $\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}$

$$=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}$$

$$=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}.g(x)+f(a).\frac{g(x)-g(a)}{x-a}\right]$$

$$=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]\lim_{x\to a}g(x)+f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

$$=f'(a)g(a)+f(a)g'(a)$$

Hence, $fg$ is differentiable at $a$ and $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$

**THEOREM 4.** *If a function f is differentiable at $x=a$ and $f(a)\neq 0$, then the function $\frac{1}{f}$ is differentiable at a and* $\left(\frac{1}{f}\right)'(a)=-\frac{f'(a)}{[f(a)]^2}$

**Proof.** Since $f$ is differentiable at $a$, therefore, it is continuous also at $x=a$.

Also, since $f(a)\neq 0$

Consider
$$\frac{\frac{1}{f(x)}-\frac{1}{f(a)}}{x-a}=-\left[\frac{f(x)-f(a)}{x-a}\right].\frac{1}{f(x)}.\frac{1}{f(a)} \quad ...(1)$$

Since $f$ is differentiable at $x=a$, therefore,

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a) \quad ...(2)$$

Also, $f$ is continuous at $x=a$, therefore

$$\lim_{x\to a}f(x)=f(a)\neq 0 \quad ...(3)$$

By applying the theorem on the limits of a product to (1), and using (2) and (3), we find that

$$\lim_{x\to a}\frac{\frac{1}{f(x)}-\frac{1}{f(a)}}{x-a} \text{ exist and equal to } -\frac{f'(a)}{[f(a)]^2}$$

**THEOREM 5.** *Let f and g be defined on an interval I. If f and g are differentiable at $a\in I$, and if $g(a)\neq 0$, then the function $f/g$ is also differentiable at a.*

**Proof.** Let $F=f/g$. Then

$$F(x)-F(a)=(f/g)(x)-(f/g)(a)$$

$$=\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}=\frac{1}{g(x)g(a)}[f(x)g(a)-f(a)g(x)]$$

$$=\frac{1}{g(x)g(a)}[f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)]$$

$$\therefore \lim_{x\to a}\frac{F(x)-F(a)}{x-a}=\lim_{x\to a}\frac{1}{g(x)g(a)}\cdot\left[\left\{\frac{f(x)-f(a)}{x-a}\right\}g(a)-f(a)\left\{\frac{g(x)-g(a)}{x-a}\right\}\right]$$

or
$$F'(a)=\frac{1}{g(a)g(a)}[f'(a)g(a)-f(a)g'(a)]$$

$$\Rightarrow \left(\frac{f}{g}\right)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{[g(a)]^2}$$

**THEOREM 6.** *Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at a and g is differentiable at f(a), then gof is differentiable at a and $(g\circ f)'=g'(f(a)).f'(a)$ (This is known as Chain rule).*

Notes

**Proof.** Since, the range of $f$ contained in the domain of $g$, therefore, $gof$ has the same domain as that of $f$.

Now, let $y = f(x)$ and $y_0 = f(a)$

Since, $f$ is differentiable at $a$, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

or $$f(x) - f(a) = (x - a)[f'(a) + A(x)] \quad ...(1)$$

where $A(x) \to 0$ as $x \to a$.

Further since $g$ is differentiable at $y_0$, we have

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$$

or $$g(y) - g(y_0) = (y - y_0)[g'(y_0) + B(y)] \quad ...(2)$$

where $B(y) \to 0$ as $y \to y_0$

Now $(g \circ f)(x) - (g \circ f)(a) = g(f(x)) - g(f(a)) = g(y) - g(y_0)$

$$= (y - y_0)[g'(y_0) + B(y)] \quad \text{[By (2)]}$$
$$= [f(x) - f(a)][g'(y_0) + B(y)]$$
$$= (x - a)[f'(a) + A(x)][g'(y_0) + B(y)], \quad \text{[By (1)]}$$

Thus if $x \neq a$, then

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = [g'(y_0) + B(y)][f'(a) + A(x)] \quad ...(3)$$

Also $f$ being differentiable at $a$ is continuous at $a$ and hence $x \to a, f(x) \to f(a)$ i.e., $y \to y_0$. $\Rightarrow B(y) \to 0$ as $x \to 0$ and $A(x) \to 0$ as $x \to a$.

Now, taking the limit as $x \to a$, we get from (3)

$$\lim_{x \to a} \frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = g'(y_0) f'(x_0)$$

Hence the function is differentiable at $a$ and $(g \circ f)'(a) = g'(f(a)) f'(a)$

**THEOREM 7. (Derivative of the inverse function).** *If $f$ is differentiable at $x = a$ and is one-one function defined on interval $I$ with $f'(a) \neq 0$, then the inverse of the $f$ is differentiable at $f(a)$ and its derivative at $a$ is $\frac{1}{f'(c)}$.*

**Proof.** Let the domain of $f$ be $X$ and range $Y$.

If $g$ be the inverse of $f$, then $g$ is a function with domain $Y$ and range $X$ such that $f(x) = y \Leftrightarrow g(y) = x$.

Now, let us suppose $y = f(x)$ and $y_0 = f(a)$. Since, $f$ is differentiable at $a$, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

or $$f(x) - f(a) = (x - a)[f'(a) + A(x)] \quad ...(1)$$

where $A(x) \to 0$ as $x \to a$. Further, we have

$$g(y) - g(y_0) = x - a, \quad \text{[By definition of } g]$$

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - a}{y - y_0} = \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(a) + A(x)} \quad \text{[By (1)]}$$

It can be easily seen that if $y \to y_0$, then $x \to a$.

In fact, $f$ being differentiable at $a$, it is also continuous at $a$, which implies that $g = f^{-1}$ is continuous at $f(a) = y_0$ and consequently.

$g(y) \to g(y_0)$ as $y \to y_0$ i.e., $x \to a$ as $y \to y_0$, so that $A(x) \to 0$ as $y \to y_0$.

$$\therefore \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(a) + A(x)} = \frac{1}{f'(a)}$$

or $$g'(y_0) = \frac{1}{f'(a)} \text{ or } g'(\tilde{f}'(a)) = \frac{1}{f'(a)}$$

Notes

**THEOREM 8. (Darboux's Theorem or Intermediate Value Theorem).** *If $f$ is finitely differentiable in a closed interval $[a, b]$ and $f'(a)$, $f'(b)$ are of opposite sign, then there exist at least one* point $c \in ]a, b[$ such that $f'(c) = 0$.

**Proof.** Let us suppose that $f'(a) > 0$ and $f'(b) < 0$, then there exist intervals $]a, a + h[$ and $]b - h, b[$, $h > 0$ such that

$$f(x) > f(a)\ \forall x \in ]a, a+h[ \qquad \ldots (1)$$

$$f(x) > f(b)\ \forall x \in [b-h, b[ \qquad \ldots (2)$$

Now, since $f$ is finitely differentiable, then it is continuous in $[a, b]$ and hence it is bounded on $[a, b]$ and attains its supremum and infimum at least once in $[a, b]$. [$\because$ *A* continuous function attains its supremum and infimum at least once in $[a, b]$].

Thus if $M$ is the supremum of $f$ in $[a, b]$, then there exist $c \in [a, b]$ such that $f(c) = M$. It is clear from (1) and (2) that the upper bound is not attained at the end points $a$ and $b$ so that $c \in ]a, b[$.

Now we shall prove $f'(c) = 0$

If $f'(c) > 0$, then there exist an interval $]c, c + h]$, $h > 0$, such that $f(x) > f(c) = M$ $\forall x \in ]c, c + h[$, which is not possible, since $M$ is the supremum of the function $f(x)$ in $[a, b]$.

If $f'(c) < 0$ then there exist an interval $[c - h, c[$, $h > 0$ such that $f(x) > f(c) = M$ $\forall x \in [c - h, c[$, which is not possible.

Hence, we conclude that $f'(c) = 0$

## REMARK

- Darboux's theorem shows that derivative do share an important property of continuous functions. Since the image of an interval under a continuous function is an interval. Darboux's theorem essentially says that the result hold even if a function is not ,continuous, provided of course, it is a derivative. That is, if a function $g$ defined on an interval $I$ is the derivative of some function $f$, then $g(I)$ is an interval.

**THEOREM 9.** *Let $f$ be defined and differentiable on $[a, b]$, and if $c$ be any number between $f'(a)$ and $f'(b)$, then there exist a real number $k$ between $a$ and $b$ such that $f'(k) = c$.*

**Proof.** Let $g$ be the function defined on $[a, b]$ by setting

$$g(x) = f(x) - cx \text{ for all } x \in [a, b]$$

Now, $g$ is differentiable on $[a, b]$ and $g'(a) = f'(a) - c$, and $g'(b) = f'(b) - c$ since $c$ lies between $f'(a)$ and $f'(b)$. Therefore, it follows that $g'(a)$ and $g'(b)$ are of opposite signs.

Since $g$ is differentiable on $[a, b]$, and since $g'(a)\, g'(b) < 0$, therefore there exist a number $k$ between $a$ and $b$ such that $g'(k) = 0$ i.e., $f'(k) = c$.

**THEOREM 10.** *If $f$ is defined and differentiable on an interval, the range of $f'$ is an interval.*

**Proof.** Let the domain of $f$ (and therefore, that of $f'$) be an interval $X$ and let the range of $f'$ be $Y$. Also let $p$ and $q$ be two distinct points of $Y$. Then there exist two distinct points $a$ and $b$ in $X$ such that $f'(a) = p$ and $f'(b) = q$.

Assume that $a < b$.

Since $X$ is an interval and $a \in X$, $b \in X$, therefore $[a, b] \subset X$.

Now $f$ is defined and derivable on $[a, b]$. If $r$ be any real number between $p$ and $q$, then by theorem 9, there exists a real number $k$ between $a$ and $b$ such that $f'(k) = r$, that is $r \in Y$. Thus we find that if $p$ and $q$ are in $Y$, then every number between $p$ and $q$ is in $Y$, and this means that $Y$ is an interval.

Notes

### REMARKS

- If $Y$ does not contain at least two distinct elements, then it is a singleton.
- If $f$ is defined and differentiable on $[a, b]$ and $f'(x) \neq 0$ for any $x \in ]a, b[$ then $f'(x)$, retains the same sign, positive or negative in $]a, b[$ *i.e.*, $f(x)$ is either positive or negative for all values of $x \in ]a, b[$.

## Solved Examples

**Example 1.** *Prove that the function* $f(x) = |x| + |x-1|$ *is not differentiable at* $x = 0$ *and* $x = 1$.

**Solution.** Here, we observe that

(i) $|x| = -x$ and $|x-1| = 1-x$ when $x < 0$.

(ii) $|x| = x$ and $|x-1| = 1-x$ when $0 \le x \le 1$.

(iii) $|x| = x$ and $|x-1| = x-1$ when $x > 1$.

Hence, the given function can be rewritten as

$$f(x) = \begin{cases} -x+1-x = 1-2x, & x < 0 \\ x+1-x = 1, & 0 \le x \le 1 \\ x+x-1 = 2x-1, & x > 1 \end{cases}$$

Now, firstly we check the differentiability of $f(x)$ at $x = 0$.

We have
$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

and
$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$
$$= \lim_{h\to 0} \frac{1-2(-h)-1}{-h} = \lim_{h\to 0} \frac{2h}{-h} = -2$$

Thus $Rf'(0) \neq Lf'(0)$ Therefore, the given function is not differentiable at $x = 0$.

Now, we check the differentiability of $f(x)$ at $x = 1$.

We have
$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[2(1+h)-1]-1}{h}$$
$$= \lim_{h\to 0} \frac{2+2h-2}{h} = 2$$

and
$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

Thus $Rf'(1) \neq Lf'(1)$. Therefore, the given function is not differentiable at $x = 1$.

**Example 2.** *Prove that the function* $f(x) = |x|$ *is continuous at* $x = 0$, *but not differentiable at* $x = 0$, *where* $|x|$ *is the absolute value of* $x$.

**Solution.** Firstly, we check the continuity of the function $f(x)$ at $x = 0$.

We have
$$f(0) = |0| = 0$$
$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} |h| = \lim_{h\to 0} h = 0$$

and
$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} |-h| = \lim_{h\to 0} h = 0$$

$\therefore$
$$f(0+0) = f(0) = f(0-0)$$

Notes

Hence, $f(x)$ is continuous at $x = 0$.

Now, we check the differentiability of the function $f(x)$ at $x = 0$.

We have, $$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{|h|-0}{h} = 1$$

and $$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{|-h|-0}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$\Rightarrow \quad Rf'(0) \neq Lf'(0)$$

Hence, the function $f(x)$ is not differentiable at $x = 0$.

**Example 3.** *Let the function $f(x)$ satisfy the condition*

(i) $f(x+y) = f(x)\,f(y)\ \forall x, y$ (ii) $f(x) = 1 + x.g(x)$ *where* $\lim_{x\to 0} g(x) = 1$

*Show that the derivative $f'(x)$ exist and equal to $f(x)$ for all $x$.*

**Solution.** From condition (i), we have

$$f(x+\delta x) = f(x).f(\delta x)$$

Then $$f(x+\delta x) - f(x) = f(x)f(\delta x) - f(x)$$

$$\Rightarrow \quad \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{f(x)[f(\delta x)-1]}{\delta x} = \frac{f(x)\delta x\, g(\delta x)}{\delta x} \quad \text{[By (ii)]}$$

$$= f(x)g(\delta x)$$

$$\therefore \quad \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x\to 0} f(x)g(\delta x) = f(x).1$$

$$\therefore \quad f'(x) = f(x)$$

**Example 4.** *If $f(x)$ be an even function and $f'(0)$ exists, then find the value of $f'(0)$.*

**Solution.** Since $f(x)$ is an even function so $f(-x) = f(x)\forall x$

$$f'(0)\text{exist} \Rightarrow Rf'(0) = Lf'(0) = f'(0)$$

Now $$f'(0) = Rf'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}, h > 0$$

$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{h} \quad [\because f(-x) = f(x)]$$

$$= -\lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = -Lf'(0) = -f'(0)$$

$$\Rightarrow \quad 2f'(0) = 0 \Rightarrow f'(0) = 0$$

**Example 5.** *Show that the function* $f(x) = \begin{cases} x\tan^{-1}\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0\end{cases}$ *is not differentiable at $x = 0$.*

**Solution.** Here

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{0}$$

$$= \lim_{h\to 0}\frac{h.\tan^{-1}\frac{1}{h}-0}{h} = \lim_{h\to 0}\tan^{-1}\frac{1}{h} = \tan^{-1}\infty = \frac{\pi}{2}$$

and $$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-h\tan^{-1}\left(-\frac{1}{h}\right)}{-h} = \lim_{h\to 0}\tan^{-1}\left(-\frac{1}{h}\right)$$

Notes

$$= -\tan^{-1}\infty = -\frac{\pi}{2}$$

$\Rightarrow$ $Rf'(0) \neq Lf'(0)$

Hence, $f(x)$ is not differentiable at $x = 0$.

**Example 6.** *Test the continuity and differentiability of the following function in* $-\infty < x < \infty$

$$f(x) = \begin{cases} 1 & \text{if } -\infty < x < 0 \\ 1+\sin x & \text{if } 0 \le x < \dfrac{\pi}{2} \\ 2+\left(x-\dfrac{\pi}{2}\right)^2 & \text{if } \dfrac{\pi}{2} \le x < \infty \end{cases}$$

**Solution.** Firstly, we check the continuity and differentiability at $x = 0$.

(i) *Continuity of* $f(x)$ *at* $x = 0$.

$$f(0) = 1 + \sin 0 = 1$$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}(1+\sin h) = 1$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} 1 = 1$$

$\Rightarrow$ $f(0+0) = f(0) = f(0-0)$

Hence, $f(x)$ is continuous at $x = 0$.

(ii) *Differentiability of* $f(x)$ *at* $x = 0$.

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{(1+\sin h)-(1+\sin 0)}{h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

and $$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{1-(1+\sin 0)}{-h} = \lim_{h\to 0}\frac{0}{-h} = \lim_{h\to 0} 0 = 0$$

$\Rightarrow$ $Rf'(0) \neq Lf'(0)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Now, we shall check the continuity and differentiability at $x = \frac{\pi}{2}$.

(iii) *Continuity of* $f(x)$ *at* $x = \frac{\pi}{2}$

We have $$f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2 = 2$$

$$f\left(\frac{\pi}{2}+0\right) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0}\left[2\left\{\left(\frac{1}{2}\pi+h\right)-\frac{1}{2}\pi\right\}^2\right]$$

$$= \lim_{h\to 0}(2+h^2) = 2$$

and $$f\left(\frac{\pi}{2}-0\right) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right) = \lim_{h\to 0}\left[1+\sin\left(\frac{\pi}{2}-h\right)\right]$$

$$= \lim_{h\to 0}[1+\cos h] = 1+1 = 2$$

$\Rightarrow$ $$f\left(\frac{\pi}{2}+0\right) = f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}-0\right)$$

Hence, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

Notes

(iv) *Differentiability of f(x) at* $x=\frac{\pi}{2}$

$$Rf'\left(\frac{\pi}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2+\left\{\frac{\pi}{2}+h-\frac{\pi}{2}\right\}^2\right]-\left[2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2\right]}{h}$$

$$=\lim_{h\to 0}\frac{2+h^2-2}{h}=\lim_{h\to 0}h=0$$

$$Lf'\left(\frac{\pi}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\lim_{h\to 0}\frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}$$

$$=\lim_{h\to 0}\frac{-1+\cos h}{-h}=\lim_{h\to 0}\frac{1-\cos h}{h}=\lim_{h\to 0}\frac{2\sin^2(h/2)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\sin h/2}{h/2}.\sin h/2\right]$$

$$=\lim_{h\to 0}\left[\frac{\sin h/2}{h/2}\right].\lim_{h\to 0}[\sin h/2]=1\times 0=0$$

Therefore, $Rf'\left(\frac{\pi}{2}\right)=Lf'\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $f(x)$ is differentiable at $x=\frac{\pi}{2}$.

Since, here, we checked the continuity and differentiability at $x=0$ and $\frac{\pi}{2}$. It is obviously continuous and differentiable at all other points.

**Example 7.** *If* $f(x)=\begin{cases}x^2\ \sin\frac{1}{x} & ,\ if\ \ x\neq 0\\ 0 & ,\ if\ \ x=0\end{cases}$ *then, show that f(x) is continuous and differentiable everywhere.*

**Solution.** We have $f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}(0+h)^2\sin\frac{1}{0+h}=\lim_{h\to 0}h^2\sin\frac{1}{h}=0$

$$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}(0-h)^2\sin\frac{1}{0-h}=-\lim_{h\to 0}h^2\sin\frac{1}{h}=0$$

and $f(0)=0$

$\Rightarrow$ $f(0+0)=f(0)=(0-0)$

Hence, the function is continuous at $x=0$.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{(-h)^2\sin\left(-\frac{1}{h}\right)-0}{-h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$$

$\Rightarrow$ $Rf'(0)=Lf'(0)$

Hence, $f(x)$ is differentiable at $x=0$.

Notes

**Example 8.** *Let* $f(x) = \sqrt{(x)}\{1 + x\sin(1/x)\}$ *for* $x > 0$, $f(0) = 0$

*and* $f(x) = -\sqrt{(-x)}\{1 + x\sin(1/x)\}$ *for* $x < 0$.

*Show that* $f'(x)$ *exists every where and is finite except at* $x = 0$ where its value is $+\infty$.

**Solution.** We have

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{h})\{1+h\sin(1/h)\}-0}{h}$$

$$= \lim_{h\to 0}\left[\frac{1}{\sqrt{h}} + (\sqrt{h})\sin\left(\frac{1}{h}\right)\right] = \infty + 0 = \infty$$

and $$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-\sqrt{[-(-h)]}\left[1+(-h)\sin\frac{1}{-h}\right]-0}{-h}$$

$$= \lim_{h\to 0}\left[\frac{1}{\sqrt{h}} + \sqrt{h}\sin\frac{1}{h}\right] = \infty + 0 = \infty$$

$\Rightarrow$ $Rf'(0) = Lf'(0) = \infty$ $\therefore f'(0) = \infty$

Now, we have

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x}\sin\frac{1}{x} - \frac{1}{\sqrt{x}}\cos\frac{1}{x} \text{ for } x > 0$$

$$f'(x) = \frac{1}{2\sqrt{-x}} + \frac{3}{2}\sqrt{(-x)}\sin\frac{1}{x} - \frac{1}{\sqrt{(-x)}}\cos\frac{1}{x} \text{ for } x < 0$$

Hence, $f'(0)$ is finite for all $a \neq 0$.

**Example 9.** *Show that the function* $f : R \to R$ *defined by*

$$f(x) = \begin{cases} x\left[1+\frac{1}{3}\sin\log x^2\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

*is continuous everywhere but not differentiable at origin.*

**Solution.** Firstly, we check the continuity of $f(x)$ at $x = 0$. We have

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\left[(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}\right]$$

$$= \lim_{h\to 0}\left[h + \left(\frac{h}{3}\right)\sin\log h^2\right] = 0 + 0\times \text{ a finite quantity} = 0$$

Similarly, $f(0-0) = 0$

Hence, $f$ is continuous at $x = 0$.

Now we shall check the differentiability at $x = 0$. Therefore,

$$Rf'(0) = \lim_{h\to 0}\frac{(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}-0}{h} = \lim_{h\to 0}\left[1+\frac{1}{3}\sin\log h^2\right]$$

which does not exist, (since $\sin\log h^2$ oscillate between $-1$ and $1$ as $h \to 0$)

Similarly, $Lf'(0)$ = does not exist.

Hence, $f(x)$ is not differentiable at origin.

**Example 10.** *Draw the graph of the function* $y = |x-1| + |x-2|$ *in the interval* $[0, 3]$ *and*

Notes

*discuss the continuity and differentiability of the function in this interval.*

**Solution.** Here, we observe that

$y = 1-x+2-x = 3-2x$ when $x \le 1$

$= x-1+2-x = 1$ when $1 \le x \le 2$

$= x-1+x-2 = 2x-3$ when $x \ge 2$

Hence, the graph consists of the segments of the three straight lines $y = 3 - 2x$, $y = 1$ and $y = 2x - 3$ corresponding to the intervals [0, 1], [1, 2], [2, 3] respectively.

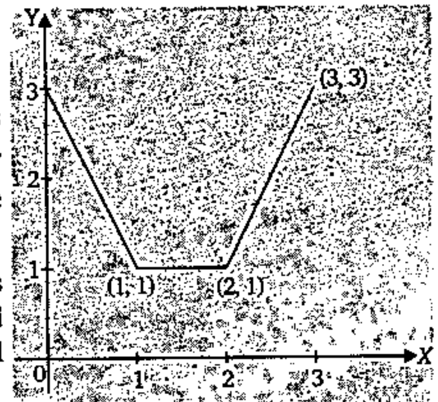

Fig. 1

The graph shows that the function is continuous throughout the interval and differentiable at all points of the interval [0, 3] except possibly at $x = 1$ and at $x = 2$.

(i) *Differentiability of $f(x)$ at $x = 1$.*

Here, $$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

and $$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{3-2(1-h)-1}{-h} = -2$$

$\Rightarrow$ $Rf'(1) \ne Lf'(1)$

$\Rightarrow f(x)$ is not differentiable at $x = 2$

(ii) *Differentiability of $f(x)$ at $x = 2$*

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{2(2+h)-3-1}{h} = 2$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$\Rightarrow$ $Rf'(2) \ne Lf'(2)$.

Hence, $f(x)$ is not differentiable at $x = 2$.

**Example 11.** *Show that the function*

$$f(x) = \begin{cases} x\left[\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right], & if\ x \ne 0 \\ 0 & ,if\ x = 0 \end{cases}$$

*is continuous but not differentiable at $x = 0$.*

**Solution.** (i) *Continuity of $f(x)$ at $x = 0$.*

We have

$$\text{RHL} = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} h\left[\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right] = \lim_{h\to 0} h\left[\frac{1-e^{-2/h}}{1+e^{-2/h}}\right]$$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0$$

and $$\text{LHL} = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} -h\left[\frac{e^{1/-h} - e^{-1/-h}}{e^{1/-h} + e^{-1/-h}}\right] = \lim_{h\to 0} -h\left[\frac{e^{-2/h}-1}{e^{-2/h}+1}\right]$$

$$= 0 \times \frac{0-1}{0+1} = 0$$

$\Rightarrow$ $f(0+0) = f(0-0) = f(0)$.

Hence, $f$ is continuous at $x = 0$.

Notes

(ii) *Differentiability of f(x) at x = 0.*
Here, we have

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\left[h\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}-0\right]}{h} = \lim_{h\to 0}\frac{1-e^{-2/h}}{1+e^{-2/h}} = \frac{1-0}{1+0} = 1$$

and $Lf'(0) = \lim_{h\to 0}\dfrac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\dfrac{f(-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{\left[(-h)\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}-0\right]}{-h} = \lim_{h\to 0}\frac{e^{-2/h}-1}{e^{2/h}+1} = \frac{0-1}{0+1} = -1$$

$\Rightarrow \quad Rf'(0) \neq Lf'(0)$

Hence, the function $f(x)$ is not differentiable at $x=0$.

**Example 12.** *Let* $f(x) = \begin{cases} e^{-1/x^2} \sin\dfrac{1}{x} & , \textit{when } x \neq 0 \\ 0 & , \textit{when } x = 0 \end{cases}$

*Show that at every point, f(x) is differentiable and f′ is continuous at x = 0.*

**Solution.** (i) *Differentiability at x = 0.*
Here, we have

$$Rf'(0) = \lim_{h\to 0}\frac{e^{-1/h^2}\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}\frac{\sin\frac{1}{h}}{he^{1/h^2}}$$

$$= \lim_{h\to 0}\frac{\sin 1/h}{h\left[1+\frac{1}{h^2}+\frac{1}{2!}\frac{1}{h^4}+\right]} = \lim_{h\to 0}\frac{\sin\frac{1}{h}}{h+\frac{1}{h}+\frac{1}{2!}\frac{1}{h^3}+\ldots}$$

$$= \frac{\text{a finite quantity lying between } -1 \text{ and } 1}{\infty} = 0$$

Similarly, $Lf'(0)=0$
Hence, the function $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$

(ii) *Continuity of f′*

$$f'(x) = \left(\frac{2}{x^3}\right)e^{-1/x^2}\sin\frac{1}{x} - \left(\frac{1}{x^2}\right)e^{-1/x^2}\cos(1/x)$$

$$= \left\{\left(\frac{2}{x}\right)\sin\frac{1}{x} - \cos\left(\frac{1}{x}\right)\right\}\left(\frac{1}{x^2}\right)\left(\frac{1}{e^{1/x^2}}\right) \quad \ldots(1)$$

Now $f'(0+0) = \lim_{h\to 0} f'(0+h) = \lim_{h\to 0}\left(\frac{2}{h}\sin\frac{1}{h} - \cos\frac{1}{h}\right).\frac{1}{h^2e^{1/h^2}}$

$$= \lim_{h\to 0}\left[\frac{2\sin(1/h)}{h^3e^{1/h^2}} - \frac{\cos(1/h)}{h^2e^{1/h^2}}\right]$$

$$= \lim_{h\to 0}\left[\frac{2\sin(1/h)}{h^3\left[1+\frac{1}{h^2}+\frac{1}{2!\,h^4}+\ldots\right]} - \frac{\cos(1/h)}{h^2\left[1+\frac{1}{h^2}+\frac{1}{2!\,h^4}+\ldots\right]}\right]$$

$$= \frac{\text{A finite quantity}}{\infty} - \frac{\text{A finite quantity}}{\infty} = 0$$

Similarly, $f'(0-0) = 0$
Hence $f'$ is continuous at $x = 0$.

Notes

**Example 13.** *Let* $f(x)=\begin{cases}-x-1 & , -2\le x\le 0\\ x-1 & , 0<x\le 2\end{cases}$ *and* $g(x)=f(|x|)+|f(x)|$.

*Test the differentiability of g(x) in the inteval ]–2, 2[*

**Solution.** Here, we have

$$|x|=-x, \text{ when } \quad -2\le x\le 0$$
$$|x|=x, \text{ when } \quad 0<x\le 2$$

Therefore, $$f(|x|)=\begin{cases}x-1 & , -2\le x\le 0\\ -x-1 & , 0<x\le 2\end{cases}$$

and $$|f(x)|=\begin{cases}1 & , -2\le x\le 0\\ -x+1 & , 0<x\le 1\\ x-1 & , 1<x\le 2\end{cases}$$

so $$g(x)=f(|x|)+|f(x)|=\begin{cases}-x & , -2\le x\le 0\\ 0 & , 0<x\le 1\\ 2x-2 & , 1<x\le 2\end{cases}$$

It is obvious that $g(x)$ is differentiable $\forall x\in\,]-2,2[$ except possibly at $x=0$ and 1.

At $x=0$ $\quad Rg'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{g(h)-g(0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$

and $\quad Lg'(0)=\lim_{h\to 0}\frac{g(0-h)-g(0)}{-h}=\lim_{h\to 0}\frac{g(-h)-g(0)}{-h}=\lim_{h\to 0}\frac{h-0}{-h}=-1$

Thus $\quad Rg'(0)\ne Lg'(0)$

Hence, $g(x)$ is not differentiable at $x=0$

At $x=1$. $Rg'(1)=\lim_{h\to 0}\frac{g(1+h)-g(1)}{h}=\lim_{h\to 0}\frac{2(1+h)-2-0}{h}=2$

$$Lg'(1)=\lim_{h\to 0}\frac{g(1-h)-g(1)}{-h}=\lim_{h\to 0}\frac{0-0}{-h}=0$$

Thus $Rg'(1)\ne Lg'(1)$. Therefore $g(x)$ is not differentiable at $x=1$.

**Example 14.** *Let* $f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}} & , \text{if } x\ne 0\\ 0 & , \text{if } x=0\end{cases}$

*Show that f is continuous at x= 0, but f′(0) does not exist.*

**Solution.** We have

$$\text{LHL}=f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h),h>0$$
$$=\lim_{h\to 0}\frac{-h}{1+e^{-1/h}}=\frac{0}{1+0}=0$$
$$\text{RHL}=f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h),h>0$$
$$=\lim_{h\to 0}\frac{h}{1+e^{1/h}}=0.\frac{0}{1+\infty}=0.0=0$$

and $\quad f(0)=0$ (given)

Therefore $f(0+0)=f(0)=f(0-0)$

Hence, $f(x)$ is continuous at $x=0$.

Now $\quad Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h},h>0$

$$=\lim_{h\to 0}\frac{\frac{h}{1+e^{1/h}}-0}{h}=\lim_{h\to 0}\frac{1}{1+e^{1/h}}=\frac{1}{1+\infty}=0$$

Notes

and $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{1+e^{1/h}}-0}{-h} = \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$\Rightarrow$ $Rf'(0) \neq Lf'(0)$

Hence, $f'(0)$ does not exist.

**Example 15.** *Show that the function $f(x) = x|x|$ is differentiable at the origin.*

**Proof.** Here, we have

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}, h > 0$$

$$= \lim_{h\to 0} \frac{h|h|-0}{h} = \lim_{h\to 0} h = 0$$

and $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-h|h|-0}{-h} = \lim_{h\to 0} h = 0$$

$\Rightarrow$ $Rf'(0) = Lf'(0)$

Hence, $f(x)$ is differentiable at $x = 0$.

**Example 16.** *Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \forall x$, and $y$. If $f'(0)$ exists and equal $-1$ and $f(0) = 1$, find $f(3)$.*

**Solution.** We have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \Rightarrow f\left(\frac{x+0}{2}\right) = \frac{f(x)+f(0)}{2}$$

$$\Rightarrow \quad f\left(\frac{x}{2}\right) = \frac{1}{2}[f(x)+f(0)] = \frac{1}{2}[f(x)+1]$$

$$\Rightarrow \quad f(x) = 2f\left(\frac{x}{2}\right) - 1 \qquad ...(1)$$

Now $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} f\left(\frac{2x+2h}{2}\right) - f(x)$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}[f(2x)+f(2h)]-f(x)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h} \qquad \text{[Using (1)]}$$

$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f'(0) = -1 \qquad \text{[Given]}$$

$$\Rightarrow \quad f(x) = -x + c \qquad ...(2)$$

Putting $x = 0$ in (2), we have $c = f(0) = 1$

Therefore, $f(3) = -3+1 = -2$

**Example 17.** *Test the continuity and differentiability $-\infty < x < \infty$, of the following function*

$$f(x) = \begin{cases} 1 & , \text{ if } -\infty < x < 0 \\ 1+\sin x & , \text{ if } 0 \le x < \pi/2 \\ 2+(x-\pi/2)^2 & , \text{ if } \pi/2 \le x < \infty \end{cases}$$

**Solution.** We shall test $f(x)$ for continuity and differentiability at $x = 0$ and $\pi/2$.

(i) *Continuity and differentiability of $f(x)$ at $x = 0$.*

We have $f(0) = 1 + \sin 0 \Rightarrow f(0) = 1$

Notes

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}(1+\sin h)=1$$

and $$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0} 1=1$$

Since $f(0)=f(0+0)=f(0-0), f(x)$ is continuous at $x=0$.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{(1+\sin h)-(1+\sin 0)}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{1-(1+\sin 0)}{-h}=\lim_{h\to 0}\frac{0}{-h}=0$$

Hence, $Rf'(0)\neq Lf'(0), f(x)$ is not differentiable at $x=0$.

(ii) *Continuity and differentiability of $f(x)$ at $x=\pi/2$.*

We have $f(\pi/2)=2+(\pi/2-\pi/2)^2=2$

$$f(\pi/2+0)=\lim_{h\to 0} f(\pi/2+h)=\lim_{h\to 0}[2+\{(\pi/2+h)-\pi/2\}^2]$$

$$=\lim_{h\to 0}(2+h^2)=2$$

and $$f(\pi/2-0)=\lim_{h\to 0} f(\pi/2-h)=\lim_{h\to 0}[1+\sin(\pi/2-h)]$$

$$=\lim_{h\to 0}(1+\cos h)=1+1=2$$

Hence, $f(\pi/2)=f(\pi/2-0)=f(\pi/2+0), f(x)$ is continuous at $x=\pi/2$

Now $$Rf'(\pi/2)=\lim_{h\to 0}\frac{f(\pi/2+h)-f(h/2)}{h}$$

$$=\lim_{h\to 0}\frac{[2+\{\pi/2+h-\pi/2\}^2]-[2+(h/2-\pi/2)^2]}{h}$$

$$=\lim_{h\to 0}\frac{2+h^2-2}{h}=\lim_{h\to 0} h=0$$

and $$Lf'(\pi/2)=\lim_{h\to 0}\frac{f(\pi/2-h)-f(\pi/2)}{-h}=\lim_{h\to 0}\frac{1+\sin(\pi/2-h)-2}{-h}$$

$$=\lim_{h\to 0}\frac{-1+\cos h}{-h}=\lim_{h\to 0}\frac{2\sin^2 h/2}{h}$$

$$=\lim_{h\to 0}\left[\frac{\sin(h/2)}{h/2}.\sin(h/2)\right]=1\times 0=0$$

Hence $Rf'(0)=Lf'(0), f(x)$ is differentiable at $x=\pi/2$.

**Example 18.** *Let* $f(x)=\sqrt{x}\{1+x\sin 1/x\}$ *for* $x\geq 0$
$f(0)=0$ *and* $f(x)=-\sqrt{(-x)}\{1+x\sin(1/x)\}$ *for* $x<0$

*Show that $f'(x)$ exists every where and is finite at $x=0$* where its value is $+\infty$.

**Solution.** We have

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\sqrt{h}\{1+h\sin 1/h\}-0}{h}$$

$$=\lim_{h\to 0}[1/\sqrt{h}+\sqrt{h}\sin(1/h)]=\infty+0=\infty$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

Notes

$$= \lim_{h\to 0} \frac{-\sqrt{(-h)}[1+(-h)\sin(-1/h)]-0}{-h}$$

$$= \lim_{h\to 0}[1/\sqrt{h}+\sqrt{h}\sin 1/h] = \infty + 0 = \infty$$

Hence, $Rf'(0) = Lf'(0) = \infty \quad \therefore f'(0) = \infty$

We have $f'(x) = \frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x}\sin\frac{1}{x} - \frac{1}{\sqrt{x}}\cos\frac{1}{x}$ for $x > 0$

and $f'(x) = \frac{1}{2\sqrt{-x}} + \frac{3}{2}\sqrt{(-x)}\sin\frac{1}{x} - \frac{1}{\sqrt{(-x)}}\cos\frac{1}{x}$ for $x > 0$

Hence, $f'(a)$ is finite for all $a \neq 0$.

**Example 19.** *Show that the function $f : \mathrm{R} \to \mathrm{R}$ defined by*

$$f(x) = x[1+(1/3)\sin\log x^2], x \neq 0 \text{ and } f(0) = 0$$

*is everywhere continuous but has no differential coefficient at the origin.*

**Proof.** Obviously the function $f(x)$ is continuous at every point of $R$ except possible at $x = 0$. Therefore, we have to check the continuity at $x = 0$. Given $f(0) = 0$

Now, $f(0+0) = \lim_{h\to 0}(0+h) = \lim_{h\to 0}[(0+h)]\{1+1/3\sin\log(0+h)^2\}$

$$= \lim_{h\to 0}[h+(h/3)\sin\log h^2] = 0 + 0 \times \text{a finite quantity} = 0.$$

Similarly, we can show that $f(0-0) = 0$.

Hence, $f$ is continuous at $x = 0$

Now $Rf'(0) = \lim_{h\to 0} \frac{(0+h)\{1+(1/3)\sin\log(0+h)^2\}-0}{h}$

$$= \lim_{h\to 0}\{1+1/3\sin\log h^2\} = \text{ which does not exist}$$

($\because$ $\sin\log h^2$ oscillates between $-1$ and 1 as $h \to 0$)

and $Lf'(0) = \lim_{h\to 0} \frac{(0-h)\{1+1/3\sin\log(0-h)^2\}-0}{-h}$

$$= \lim_{h\to 0}\{1+1/3\sin\log h^2\} = \text{ which does not exist as above.}$$

Hence, $f$ has no differential coefficient at $x = 0$.

**Example 20.** *Let $f(x) = e^{-1/x^2}.\sin 1/x$ when $x \neq 0$ and $f(0)=0$. Show that at every point, $f(x)$ has a differential coefficient and this is continuous at $x = 0$.*

**Solution.** *Differentiability at $x = 0$*

$$Rf'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}\sin 1/h - 0}{h} = \lim_{h\to 0} \frac{\sin 1/h}{he^{1/h^2}}$$

$$= \lim_{h\to 0} \frac{\sin 1/h}{h\left[1+\frac{1}{h^2}+\frac{1}{2!h^4}+\ldots\right]} = \lim_{h\to 0} \frac{\sin 1/h}{h+\frac{1}{h}+\frac{1}{2!}\frac{1}{h^3}+\ldots}$$

$$= \frac{\text{a finite quantity lying between } -1 \text{ and } +1}{\infty} = 0$$

Similarly, $Lf'(0) = 0$

Since $Rf'(0) = Lf'(0) = 0$. Hence, the function $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

If $x$ is any point other than zero, then

Notes

$$f'(x) = (2/x^3)e^{-1/x^2}\sin(1/x) - (1/x^2)e^{-1/x^2}\cos(1/x)$$

$$= \{(2/x)\sin(1/x) - \cos(1/x)\}(1/x^2)(1/e^{1/x^2}) \quad ...(1)$$

Now $f'(0+0) = \lim_{h\to 0} f'(0+h) = \lim_{h\to 0}\left(\frac{2}{«}\sin\frac{1}{-} - \cos\frac{1}{-}\right).\frac{1}{h^{«}e^{\;\;h^2}}$

$$= \lim_{h\to 0}\left(\frac{2\sin 1/h}{h^3 e^{1/h^2}} - \frac{\cos 1/h}{h^2 e^{1/h^2}}\right)$$

$$= \lim_{h\to 0}\left[\frac{2\sin 1/h}{h^3\left(1+\frac{1}{h^2}+\frac{1}{2!h^4}+\ldots\right)} + \left\{-\frac{\cos 1/h}{h^2\left(1+\frac{1}{h^2}+\frac{1}{2!h^4}+\ldots\right)}\right\}\right]$$

$$= \frac{\text{some finite quantity}}{\infty} - \frac{\text{some finite quantity}}{\infty} = 0$$

Similarly $f'(0-0) = 0$

Hence, $f'(x)$ is continuous at $x = 0$.

**Example 21.** *If $f$ is differentiable at a point $c$ then show that $|f|$ is also differentiable at $c$ provided $f(c) \neq 0$*

**Solution.** Since $f$ is differentiable at $c$ $\Rightarrow$ $f$ is continuous at $c$.

If $f(c) \neq 0$ then either $f(c) > 0$ or $f(c) < 0$.

If $f(c) > 0$ then there exists $\delta_1 > 0$ such that $f(x) > 0$ $\forall x \in ]c-\delta_1, c+\delta_1[$

If $f(c) < 0$ then there exists $\delta_2 > 0$ such that $f(x) < 0$ $\forall x \in ]c-\delta_2, c+\delta_2[$.

Therefore, we have

$$f(x) > 0 \;\forall x \in ]c-\delta_1, c+\delta_1[$$

$$f(x) < 0 \;\forall x \in ]c-\delta_2, c+\delta_2[$$

$$\Rightarrow \quad |f(x)| = \begin{cases} f(x) & ; \text{ if } x \in ]c-\delta_1, c+\delta_1[ \\ -f(x) & ; \text{ if } x \in ]c-\delta_2, c+\delta_2[ \end{cases}$$

Now since $f$ is given to be differentiable at $x = c$.

Hence, from above $|f|$ is alsodifferentiable at $x=c$.

## REMARK

- The above result does not hold if $f(c) = 0$.

## STUDENT ACTIVITY

**1.** Let $f(x) = \begin{cases} -1 & , -2 \le x \le 0 \\ x-1 & , 0 < x \le 2 \end{cases}$. Test the differentiability of $f(x)$.

**2.** Find $f'(1)$ if $f(x) = \begin{cases} \dfrac{x-1}{2x^2-7x+5} & , \text{when } x \neq 1 \\ -1/3 & , \text{when } x = 1 \end{cases}$.

**3.** Investigate the following function from the point of view of its differentiability. Does the differential coefficient of the function exist at $x = 0$ and $x = 1$?

$$f(x) = \begin{cases} -x & , \text{ if } x < 0 \\ x^2 & , \text{ if } 0 \le x \le 1 \\ x^3 - x + 1 & , \text{ if } x > 1 \end{cases}$$

## TEST YOURSELF

**1.** Determine the set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable.

**2.** Show that $f(x) = |x - 1|$, $0 \le x \le 2$ is not differentiable at $x = 1$.

**3.** Show that $f(x) = \begin{cases} -x & , \text{ when } x < 0 \\ x & , \text{ when } x \ge 0 \end{cases}$ is not differentiable at $x = 0$.

**4.** Show that the function $f(x) = \begin{cases} 2+x, & \text{if } x \ge 0 \\ 2-x, & \text{if } x < 0 \end{cases}$ is not differentiable at $x = 0$.

**5.** Show that the function $f(x) = |x-1| + 2|x-2| + 3|x-3|$ is not differentiable at the point 1,2 and 3.

**6.** Show that the function $f(x) = \begin{cases} x & , 0 \le x < 1 \\ 2-x & , x \ge 1 \end{cases}$ is not differentiable at $x = 1$.

**7.** The following limits are derivatives of certain functions at a certain point. Determine these functions and points.

(i) $\lim\limits_{x \to 2} \dfrac{\log x - \log 2}{x - 2}$ (ii) $\lim\limits_{h \to 0} \dfrac{\sqrt{(a+h)} - \sqrt{a}}{h}$

**8.** Let $f(x) = x^2 \sin(x^{-4/3})$ except when $x = 0$ and $f(0) = 0$. Prove that $f(x)$ has zero as a derivative at $x = 0$.

**9.** Discuss the existence of $f'(x)$ at $x = 0, 1, 2$, where $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1+x & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \le x \le 2 \\ 3x - x^2 & \text{for } x > 2 \end{cases}$$

### ANSWERS

**1.** Differentiable in $]-\infty, \infty[$

**7.** (i) $f(x) = \log x$, point is $x = 2$

(ii) $f(x) = \sqrt{x}$, point is $a = 2$

**9.** Not differentiable at $x = 0, 1, 2$

## 7.5 ROLLE'S THEOREM

*If a function f defined on $[a,b]$ is such that it is*

*(i) continuous in $[a,b]$, (ii) differentiable in $]a,b[$. (iii) $f(a) = f(b)$,*

*then there exists at least one vlaue of x, say c, $(a < c < b)$ such that $f'(c) = 0$*

**Proof.** Since, the function $f(x)$ is continuous on $[a, b]$

$\Rightarrow$ $f(x)$ is hounded [$\because$ Every continuous function is bounded.]

$\Rightarrow$ $f(x)$ attains its bounds [$\because$ A function, which is continuous on a closed bounded interval $[a, b]$, then it attains its bound on $[a, b]$.]

Let $M$ and $m$ are the supremum and infimum of $f(x)$ respectively.

Now there are two possibilities

Notes

**(i)** **If $M=m$,** then obviously $f(x)$ is a constant function, and therefore its derivative is zero, *i.e.*, $f'(x)=0\ \forall\, x \in\, ]a, b[$.

**(ii)** **If $M \neq m$,** then at least one of the numbers $M$ and $m$ must be different from the equal values $f(a)$ and $f(b)$.

Let us assume $\quad M \neq f(a)$.

Now, since, every continuous function on a closed interval attains its supremum, therefore, there exists a real number $c$ in $[a,b]$ such that $f(c)=M$. Also since $f(a) \neq M \neq f(b)$. Therefore $c \neq a$ and $c \neq b$, this implies that $c \in\, ]a,b[$.

Now, $f(c)$ is the supremum of $f$ on $[a, b]$

$\therefore \qquad f(x) \leq f(c)\ \forall x \in [a, b]$ [By the definition of supremum] ...(1)

In particular, $\quad f(c-h) \leq f(c),\ h>0.$

$$\Rightarrow \qquad \frac{f(c-h)-f(c)}{-h} \geq 0 \qquad \text{...(2)}$$

Since $f'(x)$ exists at each point of $]a, b[$, and hence, $f'(c)$ exists.

Therefore, from (2)

$$Lf'(c) \geq 0 \qquad \text{...(3)}$$

Similarly, from (1)

$$f(c+h) \leq f(c) \qquad h>0.$$

Then by the same arguments

$$Rf'(c) \leq 0. \qquad \text{...(4)}$$

Since $f(x)$ is differentiable in $]a, b[\ \Rightarrow f'(c)$ exist

$$\Rightarrow \qquad Lf'(c)=f'(c)=Rf'(c). \qquad \text{...(5)}$$

Now from (3), (4) and (5) $f'(c)=0$.

Similarly we can consider the case $M=f(a) \neq m$.

## REMARKS

- Converse of Rolle's theorem is not true *i.e.*, $f'(x)$ may vanish at a point $c \in\, ]a; b[$ without $f(x)$ satisfying the three conditions of Rolle's theorem.
- There may be more than one point like $c$ at which $f'(x)$ vanishes but Rolle's theorem ensures the existance of at least one such $c$.
- Rolle's theorem will not hold good if
  (a) $f(x)$ is discontinuous at some point in the interval $[a, b]$
  (b) $f'(x)$ does not exist at some point in the interval $]a, b[$
  (c) $f(a) \neq f(b)$.
- The hypothesis of Rolle's theorem cannot be weakened.
  **For example,** if $f(x)=1-|x|, -1 \leq x \leq 1$, then $f(-1)=f(1)=0$ and $f$ is continuous on $[-1,1]$. Also if $f'(x)$ exist $\forall\, x \in\, ]-1, 1[$ except at $x=0$. Then, $f$ satisfies all the condition of Rolle's theorem except that $f$ is not differentiable at $x=0$. For this $f$, there is no $c$ in $]-1,1[$ for which $f'(c)=0$.

## 7.5.1 GEOMETRICAL INTERPRETATION OF ROLLE'S THEOREM

Geometrically, Rolle's theorem means that if the curve $y=f(x)$ is continuous from $x=a$ to $x=b$, has a definite tangent at each point of $]a,b[$ and the ordinates at the extremities are equal, then there exists at least one point between $a$ and $b$ at which the tangent is parallel to $x$-axis.

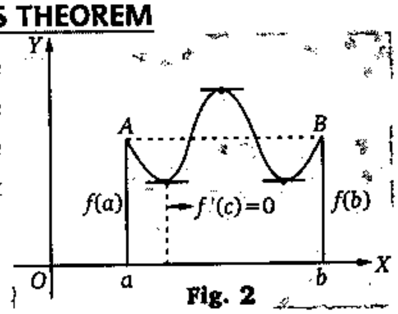


**Fig. 2**

## 7.5.2 ALGEBRAIC INTERPRETATION OF ROLLE'S THEOREM

Algebraically, Rolle's theorem means that if $f(x)$ is a polynomial function in $x$ and $x=a$ and $x=b$ are two roots of the equation $f(x)=0$, then, there is at least one root of the equation $f'(x)=0$ which lies between $a$ and $b$.

## 7.6 LAGRANGE'S MEAN VALUE THEOREM

*Let f be a function defined on [a, b] such that*

(i) *f is continuous on [a, b].* (ii) *f is differentiable on ]a, b[.*

*Then, there exists a real number c∈]a,b[ such that* $\frac{f(b)-f(a)}{b-a} = f'(c)$

**Proof:** Let us define a function $F(x)$ such that

$$F(x) = f(x) + Ax \ \forall\, x \in [a,b] \quad ...(1)$$

where $A$ is a constant to be suitably chosen such that $F(a)=F(b)$.

Now

(i) Since, $f$ is continuous on $[a,b]$ and $Ax$ is continuous on $[a,b]$ therefore, $F$ is continuous on $[a,b]$ [∵ sum of two continuous functions is again continuous.]

(ii) Similarly $F$ is differentiable on $(a, b)$

(iii) $F(a)=F(b) \Rightarrow \quad -A = \frac{f(b)-f(a)}{b-a}$ ...(2)

Hence, we find that $F$ satisfy all the conditions of Rolle's Theorem on $[a,b]$ and consequently, there exists a real number $c\in]a,b[$ such that $F'(c)=0$, this gives

$$f'(c)+A=0$$

$$\Rightarrow \quad -A=f'(c). \quad ...(3)$$

Now, from (2) and (3), we have

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

### REMARKS

- If we take $b=a+h$ and $c$ can be written as $a+\theta h$, where $\theta$ is some real number such that $0<\theta<1$. Lagrange's theorem then read as follows :
  "Let $f$ be defined and continuous on $[a, a+h]$ and differentiable on $]a, a+h[$, then for some real number $\theta(0<\theta<1)$
  $$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h).$$
- The hypothesis of the Lagrange's mean value theorem can not be weakened, as it is clear from the following examples :
  "Let $f$ be the function defined on $[-1,2]$ by setting $f(x)=|x|$, $\forall x\in[-1,2]$.
  Here, $f$ is continuous on $[-1,2]$ and differentiable at all points of $]-1, 2[$ except at $x=0$ (so that second condition is violated)
  Now $\quad f'(x) = \begin{cases} -1 & \text{if } x \in ]-1,0[ \\ 1 & \text{if } x \in ]0,2[ \end{cases}$
  Also $\quad \frac{f(2)-f(-1)}{2-(-1)} \neq f'(x)$ for any $x$ in $]-1, 2[$.
- Lagrange's mean value theorem is known as first mean value theorem.
- The result $f(b)-f(a)=f(b-a)f'(c)$ is also known as the formula for finite increment.
- For $f(a)=f(b)$, the Lagrange's mean value theorem yields Rolle's theorem.

## 7.6.1 GEOMETRICAL INTERPRETATION OF LAGRANGE'S MEAN VALUE THEOREM

If the curve $y=f(x)$ is continuous from $x=a$ and $x=b$ and has a tangent at each point on the curve between $x=a$ and $x=b$, then, geometrically, the first mean value theorem means that there is at least one point between $x=a$ and $x=b$ on the curve where the tangent to the curve parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$.

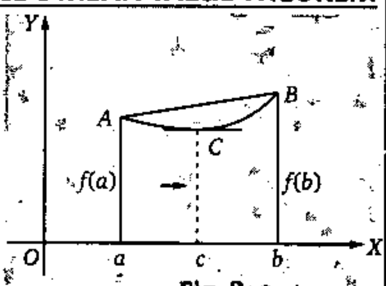

**Fig. 3**

Let $ACB$ be the graph of the function $y= f(x)$ then the co-ordinate of the points $A$ and $B$ are given by $(a, f(a))$ and $(b, f(b))$ respectively. If the chord $AB$ makes an angle $\theta$ with the $x$-axis, then

Notes

Notes

$$\tan\theta = \frac{f(b)-f(a)}{b-a} = f'(c), \text{ where } a < c < b.$$

## 7.6.2 DEDUCTION FROM THE FIRST MEAN VALUE THEOREM

**THEOREM 1.** *If a function f(x) satisfies the conditions of mean value theorem then*

(i) $f'(x) = 0\ \forall x \in ]a, b[ \Rightarrow f$ *is constant on* $[a, b]$,

(ii) $f'(x) > 0\ \forall x \in ]a,b[ \Rightarrow f$ *is strictly increasing on* $[a,b]$,

*and* (iii) $f'(x) < 0\ \forall x \in ]a,b[ \Rightarrow f$ *is strictly decreasing on* $[a,b]$.

**Proof.** (i) Let $x_1, x_2$ (where $x_1 > x_2$) be any two distinict points of $[a,b]$, then by Lagrange's mean value theorem,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0,\ x_1 < c < x_2 \qquad \ldots(1)$$

$\Rightarrow \quad f(x_2) = f(x_1).$

$\Rightarrow$ function keeps the same value. Therefore $f(x)$ is constant on $[a,b]$.

(ii) From (1), we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \text{ for some } c \in ]x_1,\ x_2[$$

But $\quad f'(c) > 0 \qquad [\because f'(x) > 0\ \forall x \in [a, b]]$

$\Rightarrow \quad f(x_2) - f(x_1) > 0.$

$\Rightarrow \quad f(x_2) > f(x_1).$

Thus $\quad x_2 > x_1 \Rightarrow f(x_2) > f(x_1)\ \forall x_1, x_2 \in [a,b]$

Hence, $f$ is strictly increasing on $[a,b]$.

(iii) Same as (ii).

### REMARK

- For a strictly increasing function $f$, the derivative $f'(x)$ need not be strictly positive. For example, consider $f(x) = x^3$, $x \in ]-1, 1[$. Here, $f(x)$ is strictly increasing but $f'(x) = 3x^2$, which is zero at $x = 0 \in ]-1, 1[$.

## Solved Examples

**Example 1.** *Determine whether* $f(x) = \frac{1}{x}, -1 < x < 0$ *is strictly increasing, decreasing or neither of these.*

**Solution.** Given that $\quad f(x) = \frac{1}{x}, \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2}$

For $-1 < x < 0 \quad f'(x) = -\frac{1}{x^2} < 0.$

So $f(x)$ is decreasing in $-1 < x < 0$.

## 7.7 CAUCHY'S MEAN VALUE THEOREM

*Let f and g be two functions defined on* $[a,b]$ *such that*

(i) *f and g are continuous on* $[a, b]$,

(ii) *f and g are differentiable on* $]a, b[$,

*and* (iii) $g'(x) \neq 0$ *for any point of* $]a, b[$.

*Then, there exists a real number* $c \in ]a, b[$ *such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

**Proof.** Let us define a function

$$F(x) = f(x) + A.g(x) \qquad \ldots(1)$$

where $A$ is a constant, to be suitably chosen such that

$$F(a) = F(b) \qquad \ldots(2)$$

Now, the function $F$ is the sum of two continuous and differentiable functions. Therefore

Notes

(i) $F$ is continuous on $[a,b]$,

(ii) $F$ is differentiable on $]a,b[$,

and (iii) $F(a)=F(b)$.

Then, by Rolle's theorem, there must exists a real number $c$ between $a$ and $b$ such that

$$F'(c)=0$$

Here,
$$F'(x)=f'(x)+Ag'(x)$$
$$F'(c)=0 \Rightarrow f'(c)+Ag'(c)=0$$

$$\Rightarrow \quad -A=\frac{f'(c)}{g'(c)} \qquad ...(3)$$

Now $\quad F(a)=F(b) \Rightarrow f(a)+Ag(a)=f(b)+Ag(b)$

$$\Rightarrow \quad -A=\frac{f(b)-f(a)}{g(b)-g(a)} \qquad ...(4)$$

From (3) and (4), we have

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

**REMARKS**

- If we put $b=a+h$, then $c$ can be written as $a+\theta h$, where $\theta \in R$ such that $0<\theta<1$, then Cauchy's mean value theorem can be restated as
"If $f$ and $g$ are continuous on $[a, a+h]$ and are differentiable on $]a, a+h[$ and $g'(x) \neq 0$ for any $x \in ]a, a+h[$ then, $\exists$ a $\theta \in R: 0<\theta<1$ such that
$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)}.$$
- If we take $g(a)=g(b)$, then the function $g$ would satisfy all the conditions of Rolle's theorem and, consequently for some $x$ in $]a,b[$, we would have $g'(x)=0$. In view of this we take $g(a) \neq g(b)$.
- In some cases, the Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem (*e.g.*, take $g(x)=k$).
- Cauchy's mean value theorem cannot be deduced by applying Lagrange's mean value theorem to two functions $f$ and $g$ seperately and then dividing. It can be easily seen that the desired result can not be obtained in this manner. In this way, we get
$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c_1)}{g'(c_2)}$$
where $a<c_1<b$, and $a<c_2<b$. But, it is not necessary that $c_1$ and $c_2$ are equal. Hence, Cauchy's means value theorem is not directly deducable from the first one.
- The conditions in the theorem are sufficient one. The conclusion may still hold even when the function involved do not satisfy the condition on $[a,b]$

## 7.7.1 GEOMETRICAL INTERPRETATION OF CAUCHY'S MEAN VALUE THEOREM

(1) Under suitable conditions, Cauchy's mean value theorem geometrically means that there is an ordinate $x=c$ between $x=a$ and $x=b$, such that the tangents at the points where $x=c$ cut the graphs of the function $f(x)$ and $\frac{f(b)-f(a)}{g(b)-g(a)}g(x)$ are mutually parallel.

(2) The ratio of the mean rates of increase of two functions in an interval is equal to the ratio of the actual rates of increase of the functions at some point within the interval.

## Solved Examples

**Example 1.** *Discuss the applicability of Rolle's theorem in the internal* $[-1,1]$ *to the function* $f(x)=|x|$.

**Solution.** Here, we have $\quad f(x)=|x|$

$$\Rightarrow \quad \left.\begin{array}{l} f(-1)=1 \\ f(1)=1 \end{array}\right\} \Rightarrow f(1)=f(-1)$$

and

Now, the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but $f(x)$

Notes

is not differentiable at $x=0 \in ]-1,1[$. Hence, Rolle's theorem is not satisfied (due to the second condition).

**Example 2.** *Verify Rolle's theorem the function* $f(x)=x^3-4x$ *on* $[-2, 2]$.

**Solution .** The function $f(x)=x^3-4x$ is a polynomial and so it is continuous and differentiable at all $x \in R$. In particular it is continuous in the closed interval $[-2,2]$ and differentiable in the open interval $]-2,2[$. Also $f(-2)=0=f(2)$.

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem in $[-2,2]$. Therefore, there must exist at least one real number 'x' in the open interval $]-2,2[$ for which $f'(x)=0$.

Also $\qquad f'(x)=x^3-4x$

Now $\qquad f'(x)=0$ gives $3x^2-4=0$ or $x=\pm\frac{2}{\sqrt{3}}=\pm 1.55$.

Both these values lie in the open interval $]-2, 2[$ and thus the conclusion of Rolle's theorem is verified.

**Example 3.** *Discuss the applicability of Rolle's theorem to the function*

$$f(x)=\log\left[\frac{x^2+ab}{(a+b)x}\right], \text{ in the interval } [a, b]$$

**Solution .** Here, we have

$$f(a)=\log\left[\frac{a^2+ab}{(a+b)a}\right]=\log 1=0$$

and

$$f(b)=\log\left[\frac{b^2+ab}{(a+b)b}\right]=\log 1=0$$

Also, it can be easily seen that $f(x)$ is continuous on $[a,b]$ and differentiable on $]a,b[$.

Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x)=0$ for at least one value of $x$ in $]a, b[$.

Now $\qquad f'(x)=0 \Rightarrow \frac{2x}{x^2+ab}-\frac{1}{x}=0$

$\Rightarrow \qquad 2x^2-(x^2+ab)=0$

$\Rightarrow \qquad x^2=ab$ or $x=\sqrt{ab}$.

Obviously $\qquad \sqrt{ab} \in ]a,b[ \qquad$ [being the geometric mean of $a$ and $b$]

Hence, the Rolle's theorem is verified.

**Example 4.** *Verify Rolle's theorem for the function* $f(x)=2x^3+x^2-4x-2$.

**Solution .** Since, $f(x)$ is a rational integral function of $x$, therefore it is continuous and differentiable for all real values of $x$.

Hence, the first two conditions of Rolle's theorem are satisfied in any interval.

Hence, $f(x)=0$ gives $2x^3+x^2-4x-2=0$

*i.e.,* $\qquad x=\pm\sqrt{2}, -\frac{1}{2}$

$\Rightarrow \qquad f\left(\sqrt{2}\right)=f\left(-\sqrt{2}\right)=f\left(-\frac{1}{2}\right)=0$

Now take the interval $\left[-\sqrt{2}, \sqrt{2}\right]$, then, all the conditions of Rolle's theorem are satisfied in this interval. Then, $\exists$ at least one value of $c$ in $]-\sqrt{2}, \sqrt{2}[$, such that $f'(c)=0$

$$f'(x)=0 \Rightarrow 6x^2+4x-4=0$$

$$\Rightarrow x=-1, 2/3.$$

Since, both the points $-1$ and $2/3$ lies in the open interval $]-\sqrt{2}, \sqrt{2}[$,. Hence, Rolle's theorem is verified.

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**Example 5.** *Verify Rolle's theorem for* $f(x)=x(x+3)e^{-x/2}$ *in* [–3, 0].

**Solution.** Here, we have

$$f(x)=x(x+3)e^{-x/2}$$

$$\therefore \quad f'(x)=(2x+3)e^{-x/2}+\left(x^2+3x\right)e^{-x/2}\cdot\left(-\frac{1}{2}\right)$$

$$=e^{-x/2}\left[2x+3-\frac{1}{2}\left(x^2+3x\right)\right]=-\frac{1}{2}\left[x^2-x-6\right]e^{-x/2}$$

$\Rightarrow$ $f'(x)$ exist for every value of $x$ in the interval [–3, 0]. Hence, $f(x)$ is differentiable and continuous in the interval [–3, 0]. Also, we have

$$f(-3)=f(0)=0$$

$\Rightarrow$ All the three conditions of Rolle's theorem are satisfied. So

$$f'(x)=0 \Rightarrow \frac{1}{2}\left(x^2-x-6\right)e^{-x/2}=0$$

$$\Rightarrow x^2-x-6=0 \Rightarrow x=3,-2.$$

Since, the values $x=-2$ lies in the open interval ]–3, 0[, the Rolle's theorem is verified.

**Example 6.** *Show that there is no real number p for which the equation* $x^3-3x+p=0$, *has two distinct roots in* ]0,1[.

**Solution.** Let, if possible, there are two distinct roots $a$ and $b$ of the given equation in ]0, 1[, such that $0<a<b<1$.

Now, let $f(x)=x^3-3x+p$

Obviously, $f(x)$ is continuous and differentiable for all values of $x$ (being a polynomial)

Also, we have $f(a)=f(b)=0$

$\Rightarrow f$ satisfies all the conditions of Rolle's theorem in $[a,b]$ hence, $\exists$ a point $c\in]a,b[$ such that $f'(c)=0$.

Now $f'(x)=0 \Rightarrow 3x^2-3=0$

$\Rightarrow x=\pm 1$

which is a contradiction $(\because a<c<b \text{ as } 0<a<b<1)$

$\Rightarrow$ our assumption is wrong. Hence, there cannot be two distinct roots of $f(x)=0$ in ]0, 1[ for any value of $p$.

**Example 7.** *Verify the Rolle's theorem for the function* $f(x)=x^2$ *in* [–1, 1].

**Solution.** Here, it can be easily seen that the function $f(x)=x^2$ is continuous as well as differentiable on R.

$\Rightarrow$ $f(x)$ is continuous and differentiable in [–1,1].

Also, we have $f(1)=f(-1)=1$.

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem in [–1,1].

$\Rightarrow$ $\exists$ at least one number, say $c$, in ]–1,1[ such that $f'(c)=0$.

Now $f'(x)=2x$

$f'(x)=0 \Rightarrow x=0$.

Since, the root $x=0$ lies in the interval ]–1, 1[. Hence, the Rolle's theorem is satisfied.

**Example 8.** *Verify Rolle's theorem for the function* $f(x)=x^2-3x+2$ *on the interval* [1,2].

**Solution.** Here, it can be easily seen that $f(x)=x^2-3x+2$ is continuous as well as differentiable on R (being a polynomial)

$\Rightarrow$ $f(x)$ is continuous in [1, 2] and differentiable in ]1, 2[.

Also, we have $f(1)=f(2)=0$.

Thus, $f(x)$ satisfies all the conditions of Rolle' theorem in [1, 2]

$\Rightarrow$ $\exists$ at least one number, say $c$, in ]1, 2[ such that $f'(c)=0$.

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Now, $f'(x) = 2x-3$

$f'(x) = 0 \quad \Rightarrow \quad x = 3/2.$

Since, the root $x = 3/2$ lies in the interval (1, 2). Hence, Rolle's theorem is verified.

**Example 9.** *If $a+b+c = 0$, then show that the quadratic equation $3ax^2+2bx+c = 0$ has at least one root in ]0, 1[.*

**Solution.** Let us define a function $f(x)$ such that $f(x) = ax^3+bx^2+cx+d$.

Here we have $f(0) = d$ and $f(1) = a+b+c+d = d \quad (\because a+b+c = 0)$

Obviously, $f(x)$ is continuous and differentiable in ]0, 1[ (being a polynomial).

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem in [0, 1]. Hence, there is at least one value of $x$ in the open interval ]0, 1[ where $f'(x) = 0$

i.e., $3ax^2+2bx+c = 0$ has at least one root in ]0, 1[.

**Example 10.** *Discuss the applicability of Rolle's Theorem to the function $f(x)=x^{2/3}$ in (–1,1)*

**Solution.** We have $f(x)= x^{2/3}$

$$\Rightarrow \quad f'(x)=\frac{2}{3}x^{-1/3}$$

$$\therefore \quad \lim_{x\to 0} f'(x)=\lim_{h\to 0}\frac{2}{3}(0+h)^{-1/3}=+\infty$$

Now, $$Rf'(0)=\lim_{h\to 0}\left\{\frac{f(0+h)-f(0)}{h}\right\}=\lim_{h\to 0}\left\{\frac{h^{2/3}-1}{h}\right\}=+\infty$$

and $$Lf'(0)=\lim_{h\to 0}\left\{\frac{f(0-h)-f(0)}{-h}\right\}=\lim_{h\to 0}\left\{\frac{(-h)^{2/3}}{-h}\right\}=-\infty$$

$$\therefore \quad Lf'(0)\neq Rf'(0).$$

$\therefore f'(0)$ does not exist showing that $f'(x)$ does not exist in the open interval (–1, 1). Hence, Rolle's Theorem is not applicable although $f'(-1) = f(1) = 1$ and $f(x)$ is continuous in the closed interval (–1,1).

**Example 11.** *Discuss the applicability of Rolle's theorem to the function* $f(x)=\begin{cases}x^2+1, & when\ 0\le x\le 1\\ 3-x, & when\ 1<x\le 2\end{cases}$

**Solution.** Here $f(0)=0^2+1$ and $f(2)=3-2=1$.

We shall show that $f(x)$ is continuous for all $x$ in the range (0,2)

Also $f(1)=1^2+1=2$

Again, $$f(1+0)=\lim_{x\to 1+0}(3-x)=\lim_{x\to 1+h}\left[3-(1+h)\right], \text{when } h\to 0$$

$$=\lim_{h\to 0}(2-h)=2$$

and $$f(1-0)=\lim_{x\to 1-0}\left(x^2+1\right)=\lim_{x\to(1-h)}\left[(1-h)^2+1\right], \text{when } h\to 0$$

$$=\lim_{h\to 0}\left(2-2h+h^2\right)=2$$

Hence, $f(1-0)=f(1)=f(1+0)$ and so the function $f(x)$ is continuous at $x=1$ and the continuous in the whole interval (0,2).

Again, $$f'(x)=\begin{cases}2x, & \text{when } 0\le x<1\\ -1, & \text{when } 1<x\le 2\end{cases}$$

$\therefore$ $f(x)$ is differentiable in the interval (0,2) except at $x=1$.

Now $$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\{3-(1+h)\}-2}{h}$$

$$=\lim_{h\to 0}\frac{2-h-2}{h}=\lim_{h\to 0}(-1)=-1$$

And $$Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{\left[(1-h)^2+1\right]-2}{-h}$$

Notes

$$= \lim_{h\to 0} \frac{2h-h^2}{h} = \lim_{h\to 0}(2-h) = 2$$

∴ Thus $Rf'(1) \neq Lf'(1)$ and so $f'(1)$ does not exist.

Hence, the function $f(x)$ is not differentiable in the entire range (0, 2) and therefore Rolle's theorem is not applicable to the given function $f(x)$ in (0, 2).

**Example 12.** *Verify Rolle's theorem for the function* $f(x) = x^3 - 6x^2 + 11x - 6$

**Solution.** Here, we have $f(x) = x^3 - 6x^2 + 11x - 6$, if $f(x) = 0$. Then $x^3 - 6x^2 + 11x - 6 = 0$

$\Rightarrow \quad (x-1)(x-2)(x-3) = 0 \quad \therefore \quad x = 1, 2, 3.$

$\Rightarrow \quad f(1) = 0 = f(2) = f(3)$

Also $\quad f'(x) = 3x^2 - 12x + 11$

Now $$Rf'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^3 - 6(x+h)^2 + 11(x+h) - 6\right] - \left[x^3 - 6x^2 + 11x - 6\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left\{(x+h)^3 - x^3\right\} - 6\left\{(x+h)^2 - x^2\right\} + 11\left\{(x+h) - x\right\}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} - 6\lim_{h\to 0} \frac{\left\{(x+h)^2 - x^2\right\}}{h} + 11\lim_{h\to 0} \frac{(x+h) - x}{h}$$

$$= 3x^2 - 12x + 11$$

Similarly $$Lf'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{(x-h)^3 - x^3}{-h} - 6\lim_{h\to 0} \frac{(x-h)^2 - x^2}{-h} + 11\lim_{h\to 0} \frac{(x-h) - x}{-h}$$

$$= 3x^2 - 12x + 11$$

Since $Lf'(x) = Rf'(x)$, therefore $f'(x)$ exists for all values of $x$ in [1, 3].

Also $f(x)$ is continuous. Hence, all conditions of Rolle's Theorem are satisfied, and so $f'(x) = 0$ for at least one value of $x$ in [1, 3].

From (1), equating $f'(x) = 0$ where $3x^2 - 12x + 11 = 0$, we get

$$x = 2 \pm \frac{\sqrt{3}}{3}$$

$$\therefore \quad x = 2.577, 1.423.$$

Both these above values lie in [1, 3].

**Example 13.** *Verify Rolle's Theorem for the function* $f(x) = 10x - x^2$.

**Solution.** Here $\quad f(x) = 0 \Rightarrow 10x - x^2 = 0 \quad \Rightarrow \quad x(10-x) = 0$

$\Rightarrow \quad x = 0, 10.$

Now, $\quad f(0) = 0, f(10) = 0 \quad \Rightarrow \quad f(0) = 0 = f(10).$

Also, $\quad f'(x) = 10 - 2x. \quad$ ...(1)

Now $$Rf'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[10(x+h) - (x+h)^2\right] - \left(10x - x^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(10x + 10h - x^2 - 2xh - h^2\right) - \left(10x - x^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{10h - 2xh - h^2}{h} = \lim_{h\to 0}(10 - 2x - h) = 10 - 2x$$

Notes

Similarly, $Lf'(x)=\lim_{h\to0}\frac{f(x-h)-f(x)}{-h}=\lim_{h\to0}\frac{\left[10(x-h)-(x-h)^2\right]-\left(10x-x^2\right)}{-h}$

$$=\lim_{h\to0}\frac{-10h+2xh-h^2}{-h}=10-2x$$

Thus $Lf'(x)=Rf'(x)$. Therefore $f'(x)$ exists for all values of $x$ in [0, 10]. Also $f(x)$ is continuous for all values of $x$ in [0,10].

Now, since every differentiable function is continuous. Hence, all the conditions of Rolle's Theorem are satisfied.

$\therefore$ $f'(x)=0$ for at least one value of $x$ in [0,10].

From (1), equating $f'(x)=0 \Rightarrow 2x=10 \Rightarrow x=5$ which lies in [0, 10].

**Example 14.** *Find 'c' of the mean value theorem, if $f(x)=x(x-1)(x-2); a=0, b=1/2$*

**Solution.** Here, we have $f(a)=f(0)=0$

$$f(b)=f\left(\frac{1}{2}\right)=\frac{3}{8}$$

$$\therefore \quad \frac{f(b)-f(a)}{b-a}=\frac{\frac{3}{8}-0}{\frac{1}{2}-0}=\frac{3}{4}$$

Now $f(x)=x^3-3x^2+2x$

$\therefore$ $f'(x)=3x^2-6x+2 \Rightarrow f'(c)=3c^2-6c+2$

Putting all these values in the Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a}=f'(c),(a<c<b)$$

we get $\frac{3}{4}=3c^2-6c+2$ or $c=1\pm\frac{\sqrt{21}}{6}$

Hence $c=\frac{1-\sqrt{21}}{6}$ lies in the open interval $]0,\frac{1}{2}[$ which is the required value.

**Example 15.** *If $f(x)=\log x$, find all numbers strictly between $e^2$ and $e^3$ such that $f'(x)=\frac{f(e^3)-f(e^2)}{e^3-e^2}$*

**Solution.** Obviously $f(x)=\log x$ is continuous in $[e^2, e^3]$ and differentiable in $]e^2, e^3[$. Then by Lagrange's mean value theorem. There exist $c\in]e^2, e^3]$, such that

$$f'(c)=\frac{f(e^3)-f(e^2)}{e^3-e^2} \Rightarrow \frac{1}{c}=\frac{3-2}{e^3-e^2}$$

$\therefore$ $c=(e^3-e^2)$.

There exist only one value $c=(e^3-e^2)$ in $]e^2, e^3[$.

**Example 16.** *Show that any chord of the parabola $y=Ax^2+Bx+C$ is parallel to the tangent at the point whose abscissa is same as that of the middle point of the chord.*

**Solution.** Let $a$ and $b$ (where $a<b$) be the abscissae of the ends of the chord and let $f(x)=Ax^2+Bx+C$. Obviously, $f(x)$ is continuous on $[a,b]$ and differentiable in $]a,b[$ (being a polynomial).

By Lagrange's mean value theorem there exists $c\in]a,b[$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

i.e., $Ab^2+Bb+C-Aa^2-Ba-C=(b-a)(2Ac+B)$

which gives $c=\frac{1}{2}(a+b)$ i.e., abscissa of the point at which the tangent is parallel to the chord is same as that of the middle point of the chord.



**Example 17.** *Separate the intervals in which the polynomial $2x^3-15x^2+36x+1$ is increasing or* *decreasing.*

**Solution.** Here, we have $f(x)=2x^3-15x^2+36x+1$

$\therefore$ $f'(x)=6x^2-30x+36=6(x-2)(x-3).$

Here $f'(x)>0$ for $x<2$ or for $x>3$.

$f'(x)<0$ for $2<x<3$

and $f'(x)=0$ for $x=2,3$

Clearly, $f'(x)$ is positive in the intervals $]-\infty,2]$ and $[3,\infty[$ and negative in the interval $]2,3[$

Hence, the function $f(x)$ is monotonically increasing in the interval $]-\infty,2]$, $[3,\infty[$ and monotonically decreasing in $]2, 3[$.

**Example 18.** *Use the function* $f(x)=x^{1/x}$, $x>0$ *show that* $e^{\pi}>\pi^e$.

**Solution.** Here $f(x)=x^{1/x},\ x>0$

$\therefore$ $$\log f(x)=\frac{1}{x}\log_e x$$

Differentiating w.r.t. $x$, we get

$$\frac{1}{f(x)}f'(x)=\frac{1}{x}.\frac{1}{x}-\frac{1}{x^2}\log_e x$$

$$f'(x)=\frac{x^{1/x}}{x^2}\left[1-\log_e x\right].$$

For $x>e,\ f'(x)<0$ $[\because \log_e x>1 \text{ for } x>e]$

$\therefore$ $f(x)$ is a decreasing function of $x$ for $x>e$.

Hence $$\pi>e\Rightarrow f(\pi)<f(e)\Rightarrow \pi^{1/\pi}<e^{1/e}$$

$$\Rightarrow \left(\pi^{1/\pi}\right)^{e\pi}<\left(e^{1/e}\right)^{e\pi}$$

$$\Rightarrow \pi^e<e^{\pi}$$

$$\Rightarrow e^{\pi}>\pi^e.$$

**Example 19.** *Show that* $\frac{x}{1+x}<\log(1+x)<x$, *for* $x>0$.

**Solution.** Let, $$f(x)=\log(1+x)-\frac{x}{1+x}$$

Obviously, $f(0)=0$.

and $$f'(x)=\frac{1}{1+x}-\frac{1.(1+x)-x.1}{(1+x)^2}=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{x}{(1+x)^2}$$

Here, we observe that $f'(x)>0$, for $x>0$.

$\Rightarrow$ $f(x)$ is monotonically increasing in the interval $[0,\infty[$. Therefore

$f(x)>f(0)$, for $x>0$

$\Rightarrow$ $\left[\log(1+x)-\frac{x}{1+x}\right]>0$, for $x>0$

$\Rightarrow$ $\log(1+x)>\frac{x}{1+x}$, for $x>0$ ...(1)

Now let $F(x)=x-\log(1+x)$.

Obviously $F(0)=0$

Then $$F'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}$$

Here, we observe that $F'(x)>0$, for $x>0$. Hence $F(x)$ is monotonically increasing in the interval $[0,\infty[$.

$\therefore$ $F(x)>F(0)$, for $x>0$

$\Rightarrow$ $[x-\log(1+x)]>0$, for $x>0$

$\Rightarrow$ $x>\log(1+x)$, for $x>0$ ...(2)

Now from (1) and (2), we get

Notes

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for } x>0$$

**Example 20.** *Prove that* $(1+x)<e^x<1+xe^x$, $\forall\ x>0$.

**Solution.** Let us consider the function $f(x)=e^x$ in $[0,x]$.

Obviously $f(x)$ is continuous as well as differentiable in $]0,x[$.

Then, by Lagrange's theorem $\exists\ c \in\ ]0, x[$, such that

$$f'(c)=\frac{f(x)-f(0)}{x-0}$$

or $$e^c=\frac{e^x-1}{x} \quad ...(1)$$

$$0<c<x \quad \Rightarrow \quad e^0<e^c<e^x \quad (\because e^x \text{ is an increasing function}) \quad ...(2)$$

Now, from (1) and (2), we have

$$e^0<\frac{e^x-1}{x}<e^x, \forall x>0$$

$$\Rightarrow \quad 1<\frac{e^x-1}{x}<e^x$$

$$\Rightarrow \quad x<e^x-1<xe^x$$

$$\Rightarrow \quad (1+x)<e^x<1+xe^x.$$

**Example 21.** *Let f be continuous on* $[a-h,a+h]$ *and differentiable in* $]a-h,a+h[$. *Prove that there is a real number* $\theta$ *between 0 and 1 such that*

$$f(a+h)-2f(a)+f(a-h)=h[f'(a+\theta h)-f'(a-\theta h)].$$

**Solution.** Consider the function $\phi$ defined on $[0, 1]$ by $\phi=f(a+ht)+f(a-ht)$ $\forall t\in[0, 1]$.

Obviously $\phi$ is continuous on $[0, 1]$ and differentiable on $]0, 1[$.

Then, by Lagrange's mean value theorem, there is a number $\theta$ lying between 0 and 1 such that $\phi(1)-\phi(0)=(1-0)\phi'(\theta)$

*i.e.,* $f(a+h)-2f(a)+f(a-h)=h[f'(a+\theta h)-f'(a-\theta h)]$.

which is the required result.

**Example 22.** *Show that Lagrange's mean value theorem does not holds for the function* $f(x)=|x|$ *in the interval* $[-1,1]$.

**Solution.** Since $f(x)=|x|$ is a continuous function on $[-1,1]$ but it is not differentiable at $x=0\in]-1,1[$. Hence, Lagrange's mean value theorem does not hold for the function $f(x)=|x|$ in the interval $[-1,1]$.

**Example 23.** *Verify Lagrange's mean value theorem for the function* $f(x)=\sin x$ *in* $\left[0,\frac{\pi}{2}\right]$.

**Solution.** The function $f(x)=\sin x$ is continuous and differentiable on R. Hence it is continuous as well as differentiable in $[0, \pi/2]$. Then, by Lagrange's mean value theorem, there must exists at least one $c$ in $]0,\pi/2[$ such that

$$\frac{f(\pi/2)-f(0)}{\pi/2-0}=f'(c) \quad ...(1)$$

Here $f(0)=0, f(\pi/2)=1$

$$f'(x)=\cos x \quad \Rightarrow \quad f'(c)=\cos c.$$

Put all these values in (1), we have

$$\frac{1-0}{\pi/2}=\cos c \Rightarrow \cos c=\frac{2}{\pi} \Rightarrow c=\cos^{-1}\left(\frac{2}{\pi}\right)$$

Since, $0<2/\pi<1$, therefore the value of $c=\cos^{-1}\left(\frac{2}{\pi}\right)$ lies in $\left]0,\frac{\pi}{2}\right[$, which is the required value of $c$. Hence, Lagrange's mean value theorem is verified.

**Example 24.** *If $f''(x)$ exist for all points in $[a, b]$ and $\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}$ where $a < c < b$, then, there is a number $l$ such that $a<l<b$ and $f''(l)=0$.*

**Solution.** Since $f''(x)$ exist for all points in $[a, b]$,

$\Rightarrow$ $f'(x)$ is continuous in $[a, b]$

$\Rightarrow$ $f(x)$ is continuous in $[a, b]$.

Now, applying Lagrange's mean value theorem to $f(x)$ in $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c)-f(a)}{c-a}=f'(l_1),\ a<l_1<c \quad ...(1)$$

and $$\frac{f(b)-f(c)}{b-c}=f'(l_2),\ c<l_2<b \quad ...(2)$$

Then, from (1) and (2), we get

$$f'(l_1)=f'(l_2) \qquad \left[\because \frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}\right]$$

Now $f'(x)$ satisfies all the conditions of Rolle's theorem in the interval $[l_1, l_2]$.

Hence $f''(l)=0$ where $l \in ]l_1, l_2[$ and $l \in ]a, b[$.

**Example 25.** *If $f(x)=(x-1)(x-2)(x-3)$ and $a=0$, $b=4$, find 'c' using Langrange's mean value theorem.*

**Solution.** We have $f(x)=(x-1)(x-2)(x-3)=x^3-6x^2+11x-6$

$f(a)=f(0)=-6$ and $f(b)=f(4)=6$

$$\therefore \quad \frac{f(b)-f(a)}{b-a}=\frac{6-(-6)}{4-0}=\frac{12}{4}=3.$$

Also $f'(x)=3x^2-12x+11$ gives $f'(c)=3c^2-12c+11$.

Putting these values in Lagrange's mean value theorem,

$$\frac{f(b)-f(a)}{b-a}=f'(c) \text{ where } a<c<b$$

we get $3=3c^2-12c+11$ or $3c^2-12c+8=0$

$$c=\frac{12\pm\sqrt{(144-96)}}{6}=2\pm\frac{2\sqrt{3}}{3}$$

As the value of $c$ lies in the open interval $]0,4[$. Hence both of these are the required values of $c$.

**Example 26.** *Examine if mean value theorem applies to $f(x)=x^3+3x^2-5x$ in the interval $[1,2]$. If it does, then find the intermediate point whose existence is asserted by the theorem.*

**Solution.** Given $f(x)=x^3+3x^2-5x$. ...(1)

$\therefore$ $f'(x)=3x^2+6x-5$ and $f'(c)=3c^2+6c-5$. ...(2)

Let $a=1$ and $b=2$, then from (1), we have

$$f(a)=f(1)=1^3+3(1)^2-5(1)=-1.$$

$$f(b)=f(2)=2^3+3(2)^2-5(2)=10.$$

From mean value theorem, we have

$f(b)-f(a)=(b-a).f'(c)$ $\Rightarrow$ $f(2)-f(1)=(2-1).f'(c)$

$\Rightarrow$ $10-(-1)=(2-1)f'(c)$ $\Rightarrow$ $3c^2+6c-5=11$ [using(2)]

$\Rightarrow$ $3c^2+6c-16=0$.

$\therefore$ $c=-1\pm 2.55$ *i.e.,* $c=-3.55, 1.55$.

**Example 27.** *Verify Cauchy's mean value theorem for the functions $f(x)=x^2-2x+3$, $g(x)=x^3-7x^2+26x-5$ in the interval $[-1, 1]$.*

**Solution.** Since $f(x)$ and $g(x)$ are polynomial in $x$, so these are continuous in the closed interval $[-1, 1]$ and also differentiable and continuous in the open interval $(-1,1)$.

Notes

Also $g'(x)=3x^2-14x+26$

$g'(-1)=3(-1)^2-14(-1)+26=43=+\text{ve}$

$g'(1)=3(1)^2-14(1)+26=15=+\text{ve}.$

Therefore, $g'(x)\neq 0$ for any value of $x$ in $(-1,1)$.

Hence all the conditions of Cauchy Mean Value Theorem are satisfied.

Then, using, $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$

Putting $a=-1$, $b=1$ (given), we have

$$\frac{f(1)-f(-1)}{g(1)-g(-1)}=\frac{f'(c)}{g'(c)}$$

$$\frac{\left[1^2-2(1)+3\right]-\left[(-1)^2-2(-1)+3\right]}{\left[1^3-7(1)^2+26(1)-5\right]-\left[(-1)^3-7(-1)^2+26(-1)-5\right]}=\frac{2c-2}{2c^2-14c+26}$$

$[\because f'(x)=2x-2]$

or $$\frac{2-6}{15-(-39)}=\frac{2c-2}{3c^2-14c+26}$$

or $-4(3c^2-14c+26)=54\times 2(c-1)$

or $3c^2+14c+26=-27(c-1)$

or $3c^2+13c-1=0$

$\Rightarrow$ $$c=\frac{-13\pm\sqrt{(181)}}{6}=\frac{-13\pm 13.454}{6}$$

i.e., $c=0.076,-4.409$

Since the value 0.076 lies in $[-1,1]$. Hence, Cauchy mean value theorem is verified.

**Example 28.** *Verify Cauchy's mean value theorem for the function $x^2$ and $x^3$ in the interval* $[1,2]$.

**Solution .** Let us suppose $f(x)=x^2$ and $g(x)=x^3$.

Then, obviously $f(x)$ and $g(x)$ are continuous in $[1,2]$ and differentiable in $]1,2[$.

Also $g'(x)=3x^2\neq 0$ for any point in $]1,2[$.

Then, by Cauchy's mean value theorem there exist at least one real number $c\in]1,2[$, such that

$$\frac{f(2)-f(1)}{g(2)-g(1)}=\frac{f'(c)}{g'(c)} \quad \text{....(1)}$$

After solving, we get $c=\dfrac{14}{9}$, which lies in the open interval $]1,2[$. Hence, Cauchy's mean value theorem is verified.

**Example 29.** *Use Cauchy's mean value theorem, to evaluate* $\lim\limits_{x\to 1}\left[\dfrac{\cos\frac{\pi x}{2}}{\log(1/x)}\right]$.

**Solution .** Let us suppose

$$f(x)=\cos\left(\frac{1}{2}\pi x\right),\ g(x)=\log x$$

$$a=x \quad \text{and} \quad b=1$$

Putting all these values in Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)},\ a<c<b$$

we get $$\frac{\cos\frac{\pi}{2}-\cos\frac{n\pi}{2}}{\log 1-\log x}=\frac{-\frac{1}{2}\pi\sin\left(\frac{\pi c}{2}\right)}{1/c};\ x<c<1$$

Now, taking the limit as $x\to 1$, which give that $c\to 1$, therefore

$$\lim_{x\to 1}\left\{\frac{0-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\lim_{c\to 1}\left\{\frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{(1/c)}\right\}$$

or $\lim_{x\to 1}\left\{\dfrac{-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=-\dfrac{1}{2}\pi$ $\left(\because \sin\frac{1}{2}\pi c\to 1 \text{ as } c\to 1\right)$

or $\lim_{x\to 1}\left\{\dfrac{\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\dfrac{\pi}{2}.$

**Example 30.** *If in the Cauchy's mean value theorem, we write $f(x)=e^x$ and $g(x)=e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

**Solution.** Since, we have

$$f(x)=e^x \text{ and } g(x)=e^{-x}$$

$$\therefore \quad \frac{f(b)-f(a)}{g(b)-g(a)}=\frac{e^b-e^a}{e^{-b}-e^{-a}}=-e^ae^b=-e^{a+b}$$

and $\dfrac{f'(x)}{g'(x)}=\dfrac{e^x}{-e^{-x}}$ so that $\dfrac{f'(c)}{g'(c)}=\dfrac{e^c}{-e^{-c}}=-e^{2c}$

After putting all these values in Cauchy's mean value theorem, we get

$$-e^{a+b}=-e^{2c} \quad\Rightarrow\quad a+b=2c$$

$$\Rightarrow \quad c=\frac{a+b}{2}$$

Hence, $c$ is the arithmetic mean between $a$ and $b$.

**Example 31.** *If $f(x)$, $g(x)$ and $h(x)$ are functions such that*

(i) *$f(x)$, $g(x)$ and $h(x)$ are continuous on $[a,b]$*

(ii) *$f(x)$, $g(x)$ and $h(x)$ are differentiable on $]a,b[$,*

*then* $\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix}=0$ *where $c\in\,]a,b[$*

**Solution.** Consider the function $F(x)$ such that

$$F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix}=0 \quad ...(1)$$

Obviously, $F(x)$ is of the form $A\,f(x)+B\,g(x)+C\,h(x)$, where $A$, $B$, $C$ are some real numbers. From the condition (i) and (ii), $F(x)$ is continuous on. $[a,b]$ and differentiable on $]a,b[$.

Also $F(a)=F(b)=0$.

$\Rightarrow F(x)$ satisfies all the conditions of Rolle's theorem. Hence, there exists a $c\in\,]a,b[$ such that $F'(c)=0$

*i.e.,* $$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix}=0.$$

**Example 32.** *Verify Cauchy's mean value for $f(x)=\sin x$ and $g(x)=\cos x$ in $\left[-\frac{\pi}{2},0\right]$*

**Solution.** It can be easily seen that $f(x)$ and $g(x)$ both are continuous on $\left[-\frac{\pi}{2},0\right]$ and differentiable on $\left]-\frac{\pi}{2},0\right[$.

Also, $g'(x)=-\sin x\neq 0$ for any point in the interval $\left]-\frac{\pi}{2},0\right[$.

Then, by Cauchy's mean value theorem, $\exists$ at least one $c\in\left]-\frac{\pi}{2},0\right[$ such that

Notes

$$\frac{f(0)-f\left(-\frac{\pi}{2}\right)}{g(0)-g\left(-\frac{\pi}{2}\right)}=\frac{f'(c)}{g'(c)}$$

Putting all the values and after simplification, we have $\cot c=-1 \Rightarrow c=-\pi/4$.

Since $c=-\pi/4$ lies in $]-\pi/2,0[$, hence, Cauchy mean value theorem is verified.

**Example 33.** *Show that* $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$

**Solution.** Let $f(x)=\sin x$ and $g(x)=\cos x$, for $x\in[\alpha,\beta]$, where $0<\alpha<\beta<\pi/2$.

$\therefore$ $f'(x)=\cos x$ and $g'(x)=-\sin x$.

It can be easily seen that both the function $f(x)$ and $g(x)$ are continuous in the closed interval $[\alpha,\beta]$ and differentiable in the open interval $]\alpha,\beta[$.

Hence, by cauchy's mean value theorem there exist at least one $\theta\in R$, $\theta\in]\alpha,\beta[$ such that

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=\frac{\cos\theta}{-\sin\theta}=-\cot\theta$$

$$\Rightarrow \frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta, \text{ where } 0<\alpha<\theta<\beta<\pi/2.$$

**Example 34.** *Show that the function f and g defined on* $\left[0,\frac{1}{2}\right]$, *by* $f(x)=x(x-1)(x-2)$ *and* $g(x)=x(x-2)(x-3)$ *satisfy the condition of Cauchy's mean value theorem.*

**Solution.** Here, we have

$$f(x)=x(x-1)(x-2)=x^3-3x^2+2x$$

and $$g(x)=x(x-2)(x-3)=x^3-5x^2+6x$$

$$\Rightarrow f'(x)=3x^2-6x+2 \text{ and } g'(x)=3x^2-10x+6$$

By Cauchy's mean value theorem, we have

$$\frac{f'(c)}{g'(c)}=\frac{f\left(\frac{1}{2}\right)-f(0)}{g\left(\frac{1}{2}\right)-g(0)},\ c\in\left]0,\frac{1}{2}\right[$$

or $$\frac{3c^2-6c+2}{3c^2-10c+6}=\frac{\frac{3}{8}-0}{\frac{15}{8}-0}=\frac{1}{5}$$

$$\Rightarrow 12c^2-20c+4=0$$

$$\Rightarrow c=\frac{5\pm\sqrt{13}}{6}$$

The value $\frac{5-\sqrt{13}}{6}$ of c belongs to $\left]0,\frac{1}{2}\right[$.

Hence, the Cauchy mean value theorem is satisfied.

**Example 35.** *Find 'c' of Cauchy's mean value theorem for the functions*

$$f(x)=\sqrt{x},\ \phi(x)=\frac{1}{\sqrt{x}} \text{ in } [a,b]$$

*and show that it is the G.M. of a and b.*

**Solution.** We have

(i) $f(x)$ and $\phi(x)$ are continuous in the closed interval $[a,b]$.

(ii) $f'(x)$ and $\phi'(x)$ exist in the open interval $(a,b)$.

(iii) $\phi'(x)=-1/2\,x^{-3/2}\neq 0$ for any $x$ in $]a,b[$.

Therefore $f(x)$ and $\phi(x)$ satisfies all the conditions of Cauchy's mean value theorem.

Notes

Hence there exist a point $c \in ]a,b[$ such that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \quad \ldots(1)$$

Also here $$f'(x) = \frac{1}{2}x^{-1/2}, \phi'(x) = -\frac{1}{2}x^{-3/2}$$

From (1), we get

$$\frac{\sqrt{b}-\sqrt{a}}{(1/\sqrt{b})-(1/\sqrt{a})} = \frac{1/2c^{-1/2}}{-1/2c^{-3/2}}$$

or $$\frac{(\sqrt{b}-\sqrt{a})\sqrt{a}.\sqrt{b}}{\sqrt{a}-\sqrt{b}} = -\frac{c^{3/2}}{c^{1/2}}$$

$\therefore$ $$c = \sqrt{ab}\,.$$

## STUDENT ACTIVITY

**1.** Verify the Rolle's theorem for the following functions:

(a) $f(x)=x^4-1$ on the interval $[-1,1]$ (b) $f(x)=e^x(\sin x-\cos x)$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

**2.** Find the value of $c$, of mean value theorem, when

(a) $f(x)=\sqrt{x^2-4}$ in the interval $[2,4]$ (b) $f(x)=2x^2+3x+4$ in the interval $[1,2]$

(c) $f(x)=x(x-1)$ in the interval $[1,2]$

**3.** (a) If $f(x)=\sqrt{x}$ and $g(x)=1/\sqrt{x}$, then show by Cauchy's mean value theorem that $c$ is the geometric mean between $a$ and $b$.

(b) If $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{x}$, then show that $c$ is the harmonic mean between $a$ and $b$.

## TEST YOURSELF

**1.** Discuss the applicability of Rolle's theorem of the following functions :

(a) $f(x)=2+(x-1)^{2/3}$ in the interval $[0,2]$ (b) $f(x)=x^2$ in $2 \le x \le 3$

(c) $f(x)=\tan x$ in $0 \le x \le \pi$ (d) $f(x)=x^4-3x^2+4$ in the interval $[-4,4]$

(e) $f(x)=1/(x^2+1)$ in the interval $[-3,3]$ (f) $f(x)=e^x \sin x$ in the interval $[0,\pi]$

(g) $f(x)=|x|$ in the interval $[-1,1]$ (h) $f(x)=(x-2)\sqrt{x}$ in the interval $[0,2]$

Notes

(i) $f(x)=(x-a)^m(x-b)^n, m,n\in Z^+$ in the interval $[a,b]$.

**2.** Show that between any two roots of $e^x\cos x=1$, there exists at least one root of $e^x\sin x-1=0$.

**3.** Let $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+...+\frac{a_{n-1}}{2}+a_n=0$. Show that there exists at least one real $x$ between 0 and 1 such that $a_0x^n+a_1x^{n-1}+...+a_n=0$.

**4.** If $f(x)=\begin{vmatrix}\sin x & \sin\alpha & \sin\beta\\ \cos x & \cos\alpha & \cos\beta\\ \tan x & \tan\alpha & \tan\beta\end{vmatrix}$ where $0<\alpha<\beta<\frac{\pi}{2}$. Show that $f'(l)=0$, where $\alpha<l<\beta$.

**5.** Verify the Lagrange's mean value theorem for the following functions :

(a) $f(x)=x^3$ in $[-1,1]$ (b) $f(x)=\sin x$ in $[0,\pi/2]$

(c) $f(x)=x^{2n}$ in $[-1,1]$, $n\in Z^+$ (d) $f(x)=2x^2-7x+10, x\in[2,5]$

## ANSWERS

**1.** (a) Not applicable (b) Not applicable (c) Not applicable (d) Verified (e) Verified (f) Verified (g) Not applicable (h) Not applicable (i) Verified

**5.** (a) Verified (b) Verified (c) Verified (d) Verified

## Summary

- Let $f(x)$ be a function defined on nbd of a point $a$ and $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists finitely then the function $f(x)$ is said to be differentiable at $a$ and this limit is called the derivative of the function $f(x)$ at $x=a$.
- The left hand derivative (regressive derivative) of $f$ at $x=a$ is given by $Lf'(a)=\lim_{h\to\infty}\frac{f(a-h)-f(a)}{-h}, h>0$
- The right hand derivative (Progressive derivative) of $f$ at $x=a$ is given by $Rf'(a)=\lim_{h\to\infty}\frac{f(a+h)-f(a)}{h}, h>0$
- A function $f(x)$ is said to be differentiable at $x=a$ if $Rf'(a)=Lf'(a)$.
- Every differentiable function is continuous.
- Let $f$ be a function defined on $[a,b]$ and $f'(c)$ exists for any point $c\in]a,b[$ such that $f'(c)>0$ then $f$ is increasing at $c$ and if $f'(c)<0$ then $f$ is decreasing at $c$.
- Let $f$ be a function defined and derivable on $[a,b]$ such that $f'(a)f'(b)<0$ then there exists some $c\in]a,b[$ such that $f'(c)=0$.
- If $f$ is defined and derivable on $[a,b]$ and $f'(a)\neq f'(b)$ then for each real number $k$ lying between $f'(a)$ and $f'(b)$ $\exists$ some $c\in]a,b[$ such that $f'(c)=k$.
- If $f$ is differentiable in $[a,b]$ such that $f'(x)\neq 0\ \forall\ x\in]a,b[$ then $f'(x)$ retains the same sign positive or negative in $]a,b[$.
- If $f$ is differentiable at a point $c$ then $|f|$ is also differentiable at $c$ provided $f(c)\neq 0$.
- **Rolle's Theorem:** If a function $f$ defined on $[a,b]$ is :
  (i) continuous on $[a,b]$. (ii) differentiable on $]a,b[$. (iii) $f(a)=f(b)$.
  then $\exists\ c\in]a,b[$ such that $f'(c)=0$.
- Geometrically Rolle's theorem states that 'Between two points with equal ordinates on the graph of $f$, there exists at least one point where the tangent is parallel to $x$-axis'.
- Algebraically, Rolle's theorem states that 'Between two zeroes of $f(x)$ there exists at least one zero of $f'(x)$'.
- Between two consecutive zeroes of $f'(x)$ there exists atmost one zero of $f(x)$.
- **(Lagrange's mean value theorem)** If a function $f$ defined on $[a,b]$ is
  (i) continuous on $[a,b]$ (ii) differentiable on $]a,b[$
  then there exists at least one real number $c\in]a,b[$ such that $\frac{f(b)-f(a)}{b-a}=f'(c)$

Notes

- If a function $f(x)$ satisfies the condition of mean value theorem and $f'(x)=0$ for all $x\in]a,b[$ then $f(x)$ is constant on $[a,b]$.
- If two functions have equal derivatives at all points of $]a,b[$ then they differ only by a constant.
- If a function $f$ is continuous on $[a,b]$, differentiable on $]a,b[$ and $f'(x)>0\ \forall x\in]a,b[$, then $f$ is strictly increasing function.
- If $f'$ exists and is bounded on some interval $I$ then $f$ is uniformly continuous on $I$.
- Geometrically, Lagrange's theorem state that between two points of the graph $f$ there exists at least one point where the tangent is parallel to the chord.
- **Cauchy's mean value theorem:** If two functions $f$ and $g$ defined on $[a,b]$ are (i) continuous on $[a,b]$ (ii) differentiable on $]a,b[$ (iii) $g'(x)\neq 0$ for any $x\in]a,b[$

  then there exists at least one $c\in]a,b[$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$
- Lagrange's mean value theorem can be deduce by Cauchy's mean value theorem as a particular case for $g(x)=x$.
- Geometrically, Cauchy's mean value theorem states that the mean rates of increase of two functions in an interval is equal to the ratio of actual rates of increase of the functions at some points within the interval.

## Objective Evaluation

### FILL IN THE BLANKS

1. Every differentiable function is __________ .
2. Every continuous function is __________ .
3. Sum and difference of two differentiable functions is again __________ .
4. The first mean value theorem is also known as __________ .
5. If $f'(x)>0$ then $f(x)$ is known as __________ .
6. If $f'(x)$ is positive at a point $x=a$, then in the neighbourhood of $x=a$, then function $f(x)$ is __________ .
7. The function $f(x)=x|x|$ is __________ .
8. If $f$ is a function, differentiable on an interval $I$, than $f'(I)$ is either interval or a __________ .
9. If $f$ is finitely differentiable in a closed interal $[a, b]$ and $f'(a), f'(b)$ are of opposite sign then $f'(c)=$ __________ for at least one value of $c\in]a, b[$.
10. If $f(x)$ is an even function. Then value of $f'(0)$ (if exist) is equal to __________ .

### TRUE/ FALSE

**Write 'T' for true and 'F' for false statement.**

1. Every continuous function is differentiable. **(T/F)**
8. Every differentiable function is continuous. **(T/F)**
9. Every differentiable function is bounded. **(T/F)**
3. A function is said to be differentiable if $Lf'(x)=Rf'(x)$. **(T/F)**
4. If $f'(x)>0$. Then $f(x)$ is an increasing function. **(T/F)**
5. The function $f(x)=|x|$ is differentiable everywhere. **(T/F)**
6. If $f(x)=0$ at each point in $]a, b[$ then $f(x)$ is a constant function. **(T/F)**
7. If $f$ is differentiable at $c$ and $f(c)\neq 0$ then $\dfrac{1}{f}$ is not necessarily differentiable. **(T/F)**
8. If two functions have equal derivative at all points in $(a, b)$ then they must be equal. **(T/F)**
9. If $f(x)$ is continuous at $x=0$, then the function $x f(x)$ is differentiable at $x=0$. **(T/F)**

### MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one**

1. A function $f$: $[a, b]\to R$ is said to be differentiable if $f$ is:

   (a) differentiable at each point of $[a, b]$

Notes

(b) differentiable at the ends points only
(c) differentiable at each point of $[a, b]$ except the end points
(d) none of the above

2. A function $f(x)$ is said to be differentiable at $x = a$, if:
(a) right hand and left hand derivative at $a$ exist and equal
(b) only right hand derivative must exist
(c) only left hand derivative must exist
(d) none of the above

3. Every differentiable function is:
(a) necessarily continuous (b) never continuous
(c) may or may not be continuous (d) none of the above

4. If $f$ is finitely differentiable in a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite sign, then:
(a) $f'(c) = 0 \ \forall\ c \in [a, b]$ (b) $f'(c) = 0$ for at least one $c \in ]a, b[$
(c) $f'(c) = 0 \ \forall\ c \in ]a, b[$ (d) none of the above

5. Every continuous function is:
(a) necessarily differentiable (b) never differentiable
(c) may or may not be differentiable (d) none of the above

6. If $f(x)$ is an even function. Then the value of $f(0)$ (if exist) is equal to:
(a) 1 (b) 0 (c) $+\infty$ (d) $-\infty$

7. If a function $f$ is continuous on $[a, b]$, differentiable on $]a, b[$ and if $f'(x) = 0 \ \forall\ x \in ]a, b[$ then $f(x)$ has a:
(a) constant value throughtout $[a, b]$ (b) constant value only at the end points
(c) constant value through out $]a, b[$ (d) none of the above

8. If $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on $]a, b[$ and if $f'(x) = g'(x)$ throughout the interval $]a, b[$, then:
(a) $f(x) = g(x) \ \forall\ x \in ]a, b[$ (b) $f(x) \neq g(x) \ \forall\ x \in ]a, b[$
(c) $f(x)$ and $g(x)$ differ only by a constant (d) none of the above

9. If $f$ is continuous on $[a, b]$ and $f'(x) \geq 0$ on $]a, b[$, then:
(a) $f$ is decreasing on $]a, b[$ (b) $f$ is decreasing on $[a, b]$
(c) $f$ is increasing on $]a, b[$ (d) $f$ is increasing on $[a, b]$

10. If $y = f(x)$ be an increasing function of $x$, then:
(a) $f'(x) \leq 0$ (b) $f'(x) = 0$ (c) $f'(x) > 0$ (d) none of these

## ANSWERS

**FILL IN THE BLANKS**

1. continuous 2. not necessarily differentiable 3. differentiable
4. Lagrange's mean value theorem 5. increasing function 6. increasing
7. differentiable at origin 8. singleton 9. 0 10. 0

**TRUE OR FALSE**

1. F 18. T 19. T 20. T 21. T 22. F 23. T
24. F 25. F 26. T

**MULTIPLE CHOICE QUESTIONS**

1. (a) 2. (a) 3. (a) 4. (b) 5. (c) 6. (b) 7. (a)
8. (c) 9. (d) 10. (c)

# Chapter 8 Taylor's theorem

## STRUCTURE

- Taylor's Theorem
- Maclaurin Theorem
- Power Series
  - Summary
  - Objective Evaluation

## LEARNING OBJECTIVES

After reading this chapter, you should be able to learn:

- The concepts of Taylor's and Maclaurin's series
- The concepts of Remainder terms
- The power series of some standard functions

## 8.1 INTRODUCTION

In this chapter we shall discuss the most important theorems namely Taylor's theorem. We shall also discuss Maclaurin's series expansion of some standard functions like $e^x$, $\log(1+x)$, $\sin x$, $\cos x$ etc.

## 8.2 TAYLOR'S THEOREM

*Let f(x) be a single valued function defined on* $[a,a+h]$ *such that*

(i) *all the derivative of f(x) upto* $(n-1)^{th}$ *order are continuous in* $[a, a+h]$, *and*

(ii) $f^n(x)$ *exists in* $(a,a+h)$

*then there exists a real number* $\theta, 0<\theta<1$, *such that*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

*where p is a given positive integer.*

**Proof.** Since, $f^n$ exists, all the derivative $f', f'' \ldots f^{n-1}$ exist and continuous on $[a, a+h]$, consider a function $f$ defined on $[a, a+h]$ such that

$$\phi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\ldots$$
$$+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+A(a+h-x)^p \quad \ldots(1)$$

where $A$ is a constant to be determined such that $\phi(a+h)=\phi(a)$

Now $\quad \phi(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p$

and $\quad \phi(a)=f(a+h)$

$$\Rightarrow \quad f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p \quad \ldots(2)$$

Now

(i) $f, f', f'', \ldots, f^{n-1}$ being all continuous on $[a, a+h]$ the function $\phi$ is continuous on $[a, a+h]$,

(ii) Similarly the function $\phi$ is differentiable on $]a, a+h[$,

Notes

and (iii) $\phi(a+h)=\phi(a)$.

Thus, the function $\phi$ satisfies all the conditions of Rolle's theorem and hence $\exists$ a real number $\theta(0<\theta<1)$ such that

$$\phi'(a+\theta h)=0.$$

Here
$$\phi'(x)=f'(x)+(-f'(x)+(a+h-x)f''(x)] +\frac{1}{2!}\left[-2(a+h-x)f''(x)+(a+h-x)^2 f'''(x)\right]+\ldots +\frac{1}{(n-1)!}\left[-(n-1)(a+h-x)^{n-2}f^{n-1}(x) +(a+h-x)^{n-1}f^n(x)\right]-Ap(a+h-x)^{p-1}$$

$$=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-Ap(a+h-x)^{p-1}$$

[Other terms canceled in pairs]

$$\therefore 0=\phi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)-Aph^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow A=\frac{h^{n-1}(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h), h\neq 0, \theta\neq 1$$

Now, putting the values of A in (2), we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

## 8.2.1 FORMS OF REMAINDER AFTER N TERMS

(i) The term $R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$ which occur after $n$ terms, is called the Taylor's remainder after $n$ terms. The theorem with this form of remainder is called Taylor's theorem with Scholomilch and Roche form of remainder.

(ii) For $p=1$, we get

$$R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$$

Then, $R_n$ is called Cauchy's form of remainder.

(iii) For $p=n$, we get

$$R_n=\frac{h^n}{n!}f^n(a+\theta h)$$

then, $R_n$ is called Lagrange's form of remainder.

## 8.2.2 ANOTHER FORM OF TAYLOR'S THEOREM

Replacing $h$ by $(x-a)$ in Taylor's theorem, we get

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$

The remainder, after $n$ terms can be written as

$$R_n=\frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(c), a<c<x.$$

### Deductions

Putting $a=0$ in second form of Taylor's theorem, we get (Maclaurin's theorem)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+R_n \quad \ldots(1)$$

(i) If $R_n=\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$, then (1) is known as Maclaurin's theorem with Schlomilch and Roche's form of remainder.

Notes

(ii) For $p=1$, $R_n=\dfrac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$ is called Cauchy's form of remainder.

(iii) For $p=n$, $R_n=\dfrac{x^n}{n!}f^n(\theta x)$, is called Lagrange's form of remainder.

## 8.2.3 TAYLOR'S SERIES

*Let f(x) possesess continuous derivatives of all orders in the interval $[a, a+h]$, then for every positive integral value of n, we have*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$

*where,* $$R_n=\frac{h^n}{n!}f^n(a+\theta h),(0<\theta<1). \quad \ldots(1)$$

Equation (1) can also be written as

$$S_n=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

Then $f(a+h)=S_n+R_n$.

Let us suppose $R_n \to 0$ as $n\to\infty$, then $\lim_{n\to\infty} S_n=f(a+h)$

*i.e.*, the series $f(a)+hf'(a)+\dfrac{h^2}{2!}f''(a)+\ldots+\dfrac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\ldots$ converges to $f(a+h)$.

Thus,

(i) If $f$ possess a continuous derivatives of every order in $[a, a+h]$.

(ii) The remainder after $n$ terms $R_n\to 0$ as $n\to\infty$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^n}{n!}f^n(a)+\ldots$$

This series is known as Taylor's series for the expansion of $f(a+h)$ as a power series in $h$.

## 8.2.4 MACLAURIN'S SERIES

If we put $a=0$ and replace $h$ by $x$ in Taylor's series, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

This series is known as Maclaurin's series for the expansion of $f(x)$ as a power series in $x$.

### REMARKS

- Maclaurin's series is a particular case of Taylor's series.
- Maclaurin's expansions of $f(x)$ fails if any of the functions $f(x), f'(x), f''(x)\ldots$ becomes infinite or discontinuous at any point of the interval $[0, x]$ or if $R_n$ does not tends to zero as $n$ tends to infinity.

## 8.3 MACLAURIN'S THEOREM

*Let f(x) be a function of x which possesses continuous derivatives of all orders in the interval [0,x] and can be expanded as an infinite series in x, then*

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

**Proof.** Let us define

$$f(x)=A_0+A_1x+A_2x^2+A_3x^3+\ldots \quad \ldots(1)$$

Let the expression (1) be differentiable term by term any number of times. Then by successive differentiation, we have

$$f'(x)=A_1+2A_2x+3A_3x^2+4A_4x^3+\ldots$$

$$f''(x)=2.1.A_2+3.2.A_3x+4.3.A_4x^2+\ldots$$

$$f'''(x)=3.2.A_3+4.3.2.A_4x+\ldots$$

..........................................................................

Notes

Putting $x=0$, we get

$$f(0)=A_0, f'(0)=A_1, f''(0)=2!A_2, f'''(0)=3!A_3....$$

$$\Rightarrow \quad A_0=f(0), A_1=f'(0), A_2=\frac{f''(0)}{2!}, A_3=\frac{f'''(0)}{3!}...$$

Substitute all these values in (1), we get

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$$

**REMARKS**

- The Maclaurin's theorem is a particular case of Taylor's Theorem, and can be obtained by replacing $a=0$ and $h=x$ in Taylor's theorem.
- If the function $f(x)$ is denoted by $y$, then the expansion may be written in the form

$$y=y(0)+x.y_1(0)+\frac{x^2}{2!}y_2(0)+...+\frac{x^n}{n!}y_n(0)+...$$

where $y(0)$, $y_1(0)$,$y_2(0)$,...,$y_n(0)$ etc. denotes values of $y$,$y_1$,$y_2$,...,$y_n$ respectively for $x=0$.

## 8.4 FAILURE OF TAYLOR'S AND MACLAURIN'S THEOREM

(a) *Taylor's theorem fails to expand $f(a+h)$ in an infinite power series in the following cases* :

- If any of the function $f(x), f'(x), f''(x)$... become infinite or does not exists for any value of $x$ in the given interval.
- If $R_n$ does not tends to zero as $n\to\infty$.

(b) *Maclaurin's theorem fails to expand $f(x)$ in an infinite power series in the following cases* :

- If any of the function $f(x), f'(x), f''(x)$... becomes infinite or does not exist in interval $[0, x]$.
- If $R_n$ does not tends to zero as $n\to\infty$.

**REMARK**

- Before expanding a given function as an infinite Taylor's or Maclaurin's series, it is essential to examine the behaviour of $R_n$ as $n\to\infty$, which is not simple in many cases. We, therefore, generally obtain the expansion by assuming the possibility of expanding it in an infinite series by assuming that $R_n\to 0$ as $n\to\infty$.

## 8.5 POWER SERIES EXPANSIONS OF SOME STANDARD FUNCTIONS

To find the power series expansion we shall use the following procedure.

**Step (1)** *Put the given function equal to $f(x)$.*

**Step (2)** *Differentiate $f(x)$, a number of times and obtain $f'(x), f''(x), f'''(x)$... and so on.*

**Step (3)** *Put $x = 0$ and find $f(0), f'(0), f''(0)$... and so on.*

**Step (4)** *Substitute the values of $f(0), f'(0), f''(0), f'''(0)$,... in*

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+...$$

We shall now consider Maclaurin's series expansions of the function $e^x$,sin $x$,cos $x$, $(1+x)^m$ and log $x$.

**(i) Expansion of $e^x$.** Let $f(x)=e^x \ \forall x\in R$.

Then $\quad f^n(x)=e^x \ \forall x\in R$.

Thus, for each positive $n$, $f^n$ is defined in the interval $[-h, h]$.

Writing, Lagrange's form of remainder, after $n$ terms

$$R_n(x)=\frac{x^n}{n!}f^n(\theta x),\ \theta\in R,\ 0<\theta<1$$

$$= \frac{x^n}{n!} e^{\theta x}$$

Now, we shall show that $\lim_{n\to\infty} R_n(x) = 0$. Here, it is enough to show that $e^{\theta x}$ is bounded in $[-h, h]$ and $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Since, $0<\theta<1$ and $x\in[-h,h]$, therefore $|\theta x|<h$ and consequently, $0<e^{\theta x}<e^h$, hence $e^{\theta x}$ is bounded.

Now, let us write $\quad a_n = \frac{x^n}{n!}\ \forall n\in N.$

Then $\quad \frac{a_{n+1}}{a_n} = \frac{x}{n+1} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

$\Rightarrow \quad \lim_{n\to\infty} a_n$ exists and equal to zero.

Now, $\quad \lim_{n\to\infty} R_n(x) = e^{\theta x}\left[\lim \frac{x^n}{n!}\right] = 0$

Hence, we find that the function $f(x)$ has a Maclaurin's series expansions for each $x\in[-h,h]$. This implies

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \ldots \quad \forall x\in R.$$

Substituting $f(x) = e^x, f'(x) = e^x, \ldots, f^n(x) = e^x$ at $x = 0$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots\ \forall x\in R$$

**(ii) Expansion of sin *x*.** Let $f(x) = \sin x,\ \forall x\in R$

$$\Rightarrow \quad f^n(x) = \sin\left(x + \frac{n\pi}{2}\right), \quad \forall x\in R$$

Writing, Lagrange's form of remainder after $n$ terms, we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } 0<\theta<1$$

$$= \frac{x^n}{n!}\sin\left(\theta x + \frac{n\pi}{2}\right)$$

Now, for all $x\in R$,

$$|R_n(x)| \le \left|\frac{x^n}{n!}\right|$$

and $\quad \lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad$ [as in (i)]

$$\lim_{n\to\infty} R_n(x) = 0$$

Thus, we find that, the function $f(x)$ has a Maclaurin's series expansions for each $x$ in $[-h,h]$. Hence, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \ldots \quad \forall x\in R.$$

Now, substituting $f(x) = \sin x, f^n(x) = \sin\frac{n\pi}{2}$, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ \forall x\in R.$$

**(iii) Expansion of cos *x*.**

Let $\quad f(x) = \cos x,\ \forall x\in R$

Then $\quad f^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$

Thus, for eaeh $n, f^n$ is defined in every interval $[-h, h]$.

Writing, Lagrange's remainder after $n$ terms, we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } 0<\theta<1$$

$$= \frac{x^n}{n!}\cos\left(\theta x + \frac{n\pi}{2}\right)$$

Notes

Now, for all $x \in R$,

$$|R_n(x)| \le \left|\frac{x^n}{n!}\right|$$

and $$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$ [as in (i)]

Thus, we find that, the function $f$ has a Maclaurin's series expansions for each $x \in [-h, h]$, which gives

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots \quad \forall x \in R.$$

Now, substituting $f(x) = \cos x \ldots, f^n(0) = \cos\frac{n\pi}{2}$, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \forall x \in R.$$

**(iv) Expansion of $(1+x)^m$.**

**Case (i).** Let $m$ is a positive integer, then letting

$$f(x) = (1+x)^m, \; \forall x \in R.$$

We find that for each $n \in N$, $f^n(x)$ exist for all $x \in R$, and whenever $n > m$, $f^n(x) = 0 \; \forall x \in R$.

$\Rightarrow$ $$R_n(x) = 0, \text{ whenever } n > m.$$

Hence, $\lim_{n\to\infty} R_n(x) = 0$ and for all $x \in R$, we have

$$f(x) = f(0) + x f'(0) + \ldots + \frac{x^m}{m!} f^m(0), \quad (\because \text{ All other terms must vanish.})$$

Substituting the value of $f(x), f(0), \ldots, f^m(0)$, We have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots + x^m$$

**Case (ii).** Let $m$ not be a positive integer (may be a fraction or negative integer).

Here, we find that, if we write

$$f(x) = (1+x)^m, \text{ whenever } x \ne -1$$

then $$f^n(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n}, \text{ whenever } x \ne -1.$$

Thus, for each positive integer $n$, $f^n$ is defined in $[-h,h]$ for each $h$ between 0 and 1.

Now, writing Cauchy's form of remainder after $n$ terms, we have

$$R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), \text{ where } 0 < \theta < 1$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$$

$$= \frac{m(m+1)\ldots(m+n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} . (1+\theta x)^{m-1}$$

Now, we observe that

(a) $$\lim_{n\to\infty} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n = 0$$

If we write $$a_n = \frac{m(m+1)\ldots(m-n+1)}{(n-1)!} x^n$$

Then, we have $\frac{a_{n+1}}{a_n} = \frac{(m-n)x}{n} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -x$

If follows that if $|x| < 1$, then $\lim_{n\to\infty} a_n = 0$

(b) $$\lim_{n\to\infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$$

In fact, since $0 < \theta < 1$ and $-1 < x < 1$, therefore, $0 < \left[\frac{1-\theta}{1+\theta x}\right] < 1$

and hence $$\lim_{n\to\infty} \left[\frac{1-\theta}{1+\theta x}\right]^{n-1} = 0$$

(c) If $m>1$, then $(1+\theta x)^{m-1}<(1-|x|)^{m-1}$

For (a), (b) and (c), we find that for all $x$ in $]-1,1[$ $\lim_{n\to\infty} R_n(x)=0$

Thus, we find that for each $h$ between 0 and 1, the function $f$ has Maclaurin's series expansion for all $x\in[-h,h]$.

Hence, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\ldots\quad \forall x\in]-1,1[.$$

Substituting the values of $f(x), f(0), f'(0), \ldots, f^{n-1}(0)$, we have

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots$$
$$+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots\text{whenever } -1<x<1$$

**(v) Expansion of $\log_e(1+x)$.**

Let $\qquad f(x)=\log(1+x), -1<x<1.$

Then $\qquad f^n(x)=\dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, whenever $x>-1$.

Now, we shall consider the following cases :

**Case (a)** Let $0\le x\le 1$. Writing Lagrange's form of remainder after $n$ terms, we have

$$R_n=\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n}=\frac{(-1)^{n-1}}{n}\cdot\left(\frac{x}{1+\theta x}\right)^n$$

Since, $0\le x\le 1, 0<\theta<1$, therefore

$$0<\frac{x}{1+\theta x}<1$$

$\therefore \qquad |R_n|<\dfrac{1}{n}$, and $\dfrac{1}{n}\to 0$ as $n\to\infty$

Therefore $\qquad \lim_{n\to\infty} R_n=0.$

**Case (b)** Let $-1<x<0$. Since in this case $\left|\dfrac{x}{1+\theta x}\right|$ need not be less than unity, therefore, we may not be able to show easily that $R_n\to 0$ as $n\to\infty$ by considering Lagrange's remainder.

Now, writing Cauchy's form of remainder, we have

$$R_n=\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$$
$$=(-1)^{n-1}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot\frac{1}{1+\theta x}$$

since $\qquad |x|<1$

therefore $\qquad \left|\dfrac{1-\theta}{1+\theta x}\right|<1$, so that $\left|\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}\right|<1$ and $\left|\dfrac{1}{1+\theta x}\right|<\dfrac{1}{1-|x|}$

Thus $\qquad |R_n|<\dfrac{|x|^n}{1-|x|}$

This implies that $\lim_{n\to\infty} R_n=0.$, since $|x|<1$. Thus we find that if $-1\le x\le 1$, then $\lim_{n\to\infty} R_n=0.$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\ldots\text{ whenever } -1<x\le 1.$$

Substituting the values of $f(x), f(0), f'(0), \ldots, f^{n-1}(0), \ldots$, we get

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,\text{ whenever } -1<x\le 1.$$

Notes

## Solved Examples

**Example 1.** *Show that*

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log a)^n, 0<\theta<1.$$

**Solution.** Let $f(x) = a^x$ ...(1)

Then $f^n(x) = a^x(\log a)^n \ \forall n\in N$ and $\forall x \in R$ ...(2)

Now, putting $x=0$, in (1) and (2), we get

$$f(0)=1, f^n(0)=(\log a)^n \ \ \forall\ n\in N.$$

From (2) $f^n(\theta x) = a^{\theta x}(\log a)^n$.

Now, by Maclaurin's series with Lagrange's form of remainder after $n$ terms we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}a^{\theta x}(\log a)^n \quad ...(3)$$

Now, substituting the above values in (3), we get

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log a)^n.$$

Here, Lagrange's form of remainder after $n$ terms

$$R_n = \frac{x^n}{n!}a^{\theta x}(\log a)^n \text{ where } 0<\theta<1.$$

**Example 2.** *Expand* $e^{a\sin^{-1}x}$ *by Maclaurin's series and find the general term. Hence, show that*

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \dots$$

**Solution.** Here $y = e^{a\sin^{-1}x}$ ...(1)

Then $y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$ ...(2)

$$\Rightarrow \left(\sqrt{1-x^2}\right)y_1 = ay \quad \Rightarrow \quad \left(1-x^2\right){y_1}^2 - a^2y^2 = 0 \quad ...(3)$$

Now, differentiating both the sides, we have

$$\left(1-x^2\right)2y_1y_2 - 2x{y_1}^2 - 2a^2yy_1 = 0$$

$$\Rightarrow \quad 2y_1\left[\left(1-x^2\right)y_2 - xy_1 - a^2y\right] = 0 \quad ...(4)$$

Since $2y_1 \neq 0$ hence $[(1-x^2)y_2 - xy_1 - a^2y] = 0$.

Now, differentiating $n$ times by Leibnitz theorem, we get

$$\left(1-x^2\right)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2) - y_{n+1}x - ny_n.1 - a^2y_n = 0$$

$$\Rightarrow \quad \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0 \quad ...(5)$$

Now, we can easily find, (from (1) to (5)) the following values

$$(y)_0 = 1, (y_1)_0 = a, \quad (y_2)_0 = a^2$$

$$(y_{n+2})_0 = (n^2+a^2)(y_n)_0. \quad ...(6)$$

Replacing $n$ by $(n-2)$ in (6), we get

$$(y_n)_0 = [(n-2)^2+a^2](y_{n-2})_0 = [(n-2)^2+a^2][(n-4)^2+a^2](y_{n-4})_0$$

If $n$ is odd, then

$$(y_n)_0 = [(n-2)^2+a^2][(n-4)^2+a^2]\dots(3^2+a^2)(1^2+a^2)(y_1)_0$$

$$= [(n-2)^2+a^2][(n-4)^2+a^2]\dots[(3^2+a^2)(1^2+a^2)].a$$

If $n$ is even, then

$$(y_n)_0 = [(n-2)^2+a^2][(n-4)^2+a^2]\dots(4^2+a^2)(2^2+a^2)(y_2)_0$$

$$= [(n-2)^2+a^2][(n-4)^2+a^2]\dots[(4^2+a^2)(2^2+a^2)].a^2$$

Hence, $y_n(0)=\begin{cases} a(1^2+a^2)(3^2+a^2)\ldots[(n-2)^2+a^2], & \text{if } n \text{ is odd} \\ a^2(2^2+a^2)(4^2+a^2)\ldots[(n-2)^2+a^2], & \text{if } n \text{ is even} \end{cases}$

Putting $n=1,2,3,4,\ldots$ in (6), we get

$$(y_3)_0=(3^2+a^2)(1^2+a^2)a, (y_6)_0=(4^2+a^2)(2^2+a^2)a^2 \text{ etc.}$$

Now putting all these values in the Maclaurin's theorem

$$y=(y)_0+x.(y_1)_0+\frac{x^2}{2!}(y_2)_0+\ldots+\frac{x^n}{n!}(y_n)_0+\ldots$$

We have $e^{a\sin^{-1}x}=1+ax+\frac{a^2}{2!}x^2+\frac{a(1^2+a^2)}{3!}x^3+\frac{a(2^2+a^2)}{4!}x^4+\ldots$

The general term is $\frac{x_n}{n!}(y_n)_0$.

Now putting $x=\sin\theta$ and $a=1$, in the above equation, we get

$$e^{\theta}=1+\sin\theta+\frac{1}{2!}\sin^2\theta+\frac{2}{3!}\sin^3\theta+\ldots$$

**Example 3.** *Expand log sin(x+h) in powers of h by Taylor's theorem.*

**Solution.** Let $f(x)=\log\sin(x)$

$\Rightarrow$ $f(x+h)=\log\sin(x+h)$.

Expanding $f(x+h)$ by Taylor's theorem in powers of $h$, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \quad \ldots(1)$$

Now $f(x)=\log\sin x \Rightarrow f'(x)=\cot x$

$f''(x)=-\operatorname{cosec}^2 x \Rightarrow f'''(x)=2\operatorname{cosec}^2 x\cot x$ etc.

Substituting all these values in equation (1), we get

$$\log\sin(x+h)=\log\sin x+h\cot x-\frac{h^2}{2!}\operatorname{cosec}^2 x+\frac{2h^3}{3!}\operatorname{cosec}^2 x\cot x+\ldots$$

**Example 4.** *Expand sin x in powers of* $\left(x-\frac{\pi}{2}\right)$ *with the help of Taylor's theorem.*

**Solution.** Let $f(x)=\sin x$.

Since, we want to expand $f(x)$ in powers of $\left(x-\frac{\pi}{2}\right)$, hence, we can write

$$f(x)=f\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$$

Now, expanding by Taylor's theorem, we get

$$f(x)=f\left[\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)\right]$$

$$=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 f''\left(\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 f'''\left(\frac{\pi}{2}\right)+\ldots \quad \ldots(1)$$

Now $f(x)=\sin x \Rightarrow f\left(\frac{\pi}{2}\right)=1$

$f'(x)=\cos x \Rightarrow f'\left(\frac{\pi}{2}\right)=0$

$f''(x)=-\sin x \Rightarrow f''\left(\frac{\pi}{2}\right)=-1$

$f'''(x)=-\cos x \Rightarrow f'''\left(\frac{\pi}{2}\right)=0$

and so on.

Substituting all these values in (1), we get

$$\sin x=1+\left(x-\frac{\pi}{2}\right).0+\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2.(-1)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3.0+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4+\ldots$$

$$=1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4+\ldots$$

Notes

**Example 5.** *If* $f(x)=(x-a)^{5/2}$ *and* $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h)$ *find the value of* $\theta$.

**Solution.** Here, we have

$$f(x)=(x-a)^{5/2} \quad\Rightarrow\quad f(x+h)=(x+h-a)^{5/2}$$

$$\Rightarrow \quad f'(x)=\frac{5}{2}(x-a)^{3/2} \text{ and } f''(x)=\frac{15}{4}(x-a)^{1/2}$$

$$\therefore \quad f''(x+\theta h)=\frac{15}{4}(x+\theta h-a)^{1/2}$$

Putting all these values in the given relation, we have

$$(x+h-a)^{5/2}=(x-a)^{5/2}+\frac{5}{2}h(x-a)^{3/2}+\frac{15}{4}(x+\theta h-a)^{1/2}\frac{h^2}{2!}$$

Now, taking limit as $x\to a$, we have

$$h^{5/2}=\frac{15}{4}(\theta h)^{1/2}\frac{h^2}{2!} \quad\Rightarrow\quad \theta=\frac{64}{225}.$$

**Example 6.** *Let f is twice differentiable function and* $|f|<\alpha$, $|f''|<\beta$, *for* $x>a$, *then show that* $|f'|<2\sqrt{\alpha\beta}\ \forall x>a$.

**Solution.** Let us suppose $x>a$ and $h>0$, then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h),\quad 0<\theta<1$$

$$\Rightarrow \quad hf'(x)=f(x+h)-f(x)-\frac{h^2}{2!}f''(x+\theta h)$$

$$\Rightarrow \quad |hf'(x)|=\left|f(x+h)-f(x)-\frac{h^2}{2!}f''(x+\theta h)\right|$$

$$\leq |f(x+h)|+|-f(x)|+\frac{h^2}{2!}|-f''(x+\theta h)|$$

[By using triangular inequality]

$$<\alpha+\alpha+\frac{h^2}{2}\beta=2\alpha+\frac{h^2}{2}\beta$$

$$\Rightarrow \quad |f'(x)|<\frac{2\alpha}{h}+\frac{h}{2}\beta=F(h)(\text{say})$$

Now, $|f'(x)|$ is independent of $h$ and also less than $F(h)$ for all values of $h$.

Therefore $|f'(x)|$ must be less than the minimum value of $F(h)$.

For, maxima or minima of $F(h)$, we have

$$F'(h)=0$$

$$\Rightarrow \quad -\frac{2\alpha}{h^2}+\frac{\beta}{2}=0 \quad\Rightarrow\quad h=\pm 2\sqrt{\frac{\alpha}{\beta}}$$

and $$F''(h)=\frac{2\alpha}{h^3}>0 \text{ for } h=2\sqrt{\frac{\alpha}{\beta}}$$

Hence $f(h)$ is minimum for $h=2\sqrt{\frac{\alpha}{\beta}}$,

the minimum value of $F(h)$ is $=2\alpha.\frac{1}{2}\sqrt{\frac{\alpha}{\beta}}+\frac{\beta}{2}.2\sqrt{\frac{\alpha}{\beta}}=2\sqrt{\alpha\beta}$

Hence $$|f'(x)|<2\sqrt{\alpha\beta}.$$

## TEST YOURSELF

1. If $f''$ exists and continuous on $[a,b]$ and differentiable on $]a,b[$, then prove that

$$f(b)-f(a)-\frac{1}{2}(b-a)\{f'(a)-f'(b)\}=-\frac{(b-a)^3}{12}f'''(d)$$

where $d \in R$ such that $d \in ]a, b[$.

**2.** Prove that

$$\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \dots + \frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}.\pi\right) + \frac{a^nx^n}{n!}\sin\left(a\theta x + \frac{n\pi}{2}\right)$$

**3.** If $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$, find the value of $\theta$ as $x \to 1$, $f(x)$ being $(1-x)^{5/2}$.

**4.** Show that the number $\theta$ which occurs in the Taylor's Theorem with Lagrange's form of remainder after $n$ terms approaches the limit $\frac{f^{n+1}(a)}{(n+1)}$ as $h \to 0$ provided that $f^{n+1}(x)$ is continuous and different from zero as $x \to a$.

**5.** Show that the function $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.

**6.** Obtain by Maclaurin's theorem the expansion of $e^{\sin x}$.

**7.** If $f(x) = \exp\left[-\frac{1}{x^2}\right]$, for $x \neq 0$ and $f(0) = 0$, then show that :

(i) $f^n(0) = 0 \ \forall n = 0, 1, 2, \dots$

and (ii) The Taylor's series for $f$ about 0 agrees with $f(x)$ only at $x = 0$.

**8.** Expand "log sec $x$" by Maclaurin's series expansion, upto the term containing $x^6$.

**9.** If $x > 0$, show that $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$.

### ANSWERS

**3.** $\theta = \frac{9}{25}$ **7.** $y = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$ **9.** $y = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$

## 8.6 SOME MORE EXPANSIONS

**Example 1.** *Expand* $\tan^{-1} x$.

**Solution.** Let

$$f(x) = \tan^{-1} x \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$$

$$= (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots \quad \text{(By Binomial expansion)}$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \dots \Rightarrow f''(0) = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 + \dots \Rightarrow f'''(0) = -2$$

$$f^{iv}(x) = 24x - 120x^3 + \dots \Rightarrow f^{iv}(0) = 0$$

$$f^{v}(x) = 24 - 360x^2 + \dots \Rightarrow f^{v}(0) = 24$$

.....................................................................

Put all these values in Maclaurin's series, we get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

### REMARKS

- To expand an alone inverse function, find its first derivative, expand by Binomial theorem and then find other derivatives.
- The expansion of $\tan^{-1} x$ is valid only if $-1 < x < 1$.
- This expansion for $\tan^{-1} x$ known as Gregory's series, which is very useful in finding the value of $\pi$.
- In a like manner, we may get $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \dots$

Notes

**Example 2.** *If $y=\sin(m\sin^{-1}x)$, then show that*

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

*Hence, or otherwise expand $\sin m\theta$ in powers of $\sin\theta$.*

**Solution.** Here, we have

$$y=f(x)=\sin(m\sin^{-1}x) \quad \text{...(1)}$$

$$\Rightarrow \quad y_1=\cos\left(m\sin^{-1}x\right).\frac{m}{\sqrt{1-x^2}} \quad \text{...(2)}$$

$$\Rightarrow \quad (1-x^2)y_1^{\,2}=m^2\cos^2(m\sin^{-1}x)$$

$$\Rightarrow \quad (1-x^2)y_1^{\,2}=m^2[1-\sin^2(m\sin^{-1}x)]$$

$$\Rightarrow \quad (1-x^2)y_1^{\,2}=m^2(1-y^2) \qquad [\because y=\sin(m\sin^{-1}x)]$$

$$\Rightarrow \quad (1-x^2)y_1^{\,2}+m^2y^2-m^2=0 \quad \text{...(3)}$$

Differentiating w.r.t. $x$, we get

$$(1-x^2)2y_1y_2-2xy_1^{\,2}+2m^2yy_1=0$$

$$\Rightarrow \quad 2y_1[(1-x^2)y_2-xy_1+m^2y]=0$$

$$\Rightarrow \quad (1-x^2)y_2-xy_1+m^2y=0 \quad \text{...(4)}$$

Now, differentiating (4) $n$ times, we get

$$\left(1-x^2\right)y_{n+2}+n.y_{n+1}(-2x)+\frac{n(n-1)}{1.2}y_n(-2)-xy_{n+1}-n.y_n+m^2y_n=0$$

$$\Rightarrow \quad \left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2-m^2\right)y_n=0 \quad \text{...(5)}$$

Now, put $x=0$ in (1), (2), (4) and (5), we get

$$y(0)=0,\ y_1(0)=m,\ y_2(0)+m^2y(0)=0\Rightarrow y_2(0)=0$$

and $\quad y_{n+2}(0)=(n^2-m^2)y_n(0).$ ...(6)

Putting $n=2,4,6,\ldots$ in (6), we get

$$y_4(0)=(2^2-m^2)y_2(0)=0$$

$$y_6(0)=(4^2-m^2)y_4(0)=0$$

$$y_8(0)=0$$

.............. and so on.

Here, we observe that $y_n(0)=0$ if $n$ is even.

Now, putting $n=1,3,5,\ldots$ in (6), we get

$$y_3(0)=(1^2-m^2)y_1(0)=(1^2-m^2).m$$

$$y_5(0)=(3^2-m^2)y_3(0)=(3^2-m^2)(1^2-m^2).m$$

..............................................................

Putting all these values in Maclaurin's series, we get

$$\sin\left(m\sin^{-1}x\right)=mx+\frac{m\left(1^2-m^2\right)}{3!}x^3+\frac{m\left(1^2-m^2\right)\left(3^2-m^2\right)}{5!}x^5+\ldots$$

Let $\quad \theta=\sin^{-1}x\Rightarrow x=\sin\theta$

Then, we get

$$\sin m\theta=m\sin\theta+\frac{m(1^2-m^2)}{3!}\sin^3\theta+\frac{m(1^2-m^2)(3^2-m^2)}{5!}\sin^5\theta+\ldots$$

**Example 3.** *Expand $\tan x$ by Macluarin's theorem as far as $x^5$ and hence find the value of $\tan 46°30'$ upto four decimal places.*

**Solution.** Let $\quad f(x)=\tan x \qquad \Rightarrow f(0)=0$

$f'(x)=\sec^2x=1+\tan^2x \qquad \Rightarrow f'(0)=1$

Notes

$f''(x)=2\tan x\sec^2 x=2\tan x(1+\tan^2 x)=2\tan x+2\tan^3 x$ $\Rightarrow f''(0)=0$

$f'''(x)=2\sec^2 x+6\tan^2 x\sec^2 x=2(1+\tan^2 x)+6\tan^2 x(1+\tan^2 x)$

$=2+8\tan^2 x+6\tan^4 x$ $\Rightarrow f'''(0)=2$

$f^{iv}(x)=16\tan x\sec^2 x+24\tan^3 x\sec^2 x=8\sec^2 x(2\tan x+3\tan^3 x)$

$=8(1+\tan^2 x)(2\tan x+3\tan^3 x)$

$=16\tan x+40\tan^3 x+24\tan^5 x$ $\Rightarrow f^{iv}(0)=0$

and $f^{v}(x)=16\sec^2 x+120\tan^2 x\sec^2 x+120\tan^4 x\sec^2 x$

$=8\sec^2 x(2+15\tan^2 x+15\tan^4 x)$ $\Rightarrow f^{v}(0)=16$

Now, putting all these values in Maclaurin's series'

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\ldots$$

We get $\tan x=0+x+\frac{x^3}{3!}.2+\frac{x^5}{5!}.16+\ldots$

$\Rightarrow$ $\tan x=x+\frac{x^3}{3}+\frac{2}{5}x^5+\ldots$

**Deduction.** Here

$$x=46°30'=\left(46\frac{1}{2}\right)^\circ=\left(\frac{93}{2}\right)^\circ=\frac{93}{2}\times\frac{\pi}{180}\text{ Radians}$$

$$=\frac{31}{120}\times\frac{22}{7}=\frac{31\times 11}{60\times 7}=\frac{314}{420}=0.812$$

Now, putting $x=46°30'=0.812$ in (1), we get

$$\tan 46°30'=0.812+\frac{(0.812)^3}{3}+\frac{2}{15}(0.812)^5=0.812+0.1784+0.047=1.0374$$

**Example 4.** *Expand* $\log\{x+\sqrt{(1+x^2)}\}$ *in ascending powers of x and find the general term.*

**Solution.** Let $y=\log\{x+\sqrt{(1+x^2)}\}$ ...(1)

$\Rightarrow$ $y_1=\frac{1}{x+\sqrt{1+x^2}}.\left[1+\frac{2x}{2\sqrt{(1+x^2)}}\right]=\frac{1}{\sqrt{1+x^2}}$ ...(2)

$\Rightarrow$ $y_1^2(1+x^2)-1=0.$

Differentiating again w.r.t. $x$, we get

$2y_1[(1+x^2)y_2+xy_1]=0$

$\Rightarrow$ $[(1+x^2)y_2+xy_1]=0$ $(\because 2y_1\neq 0)$ ...(3)

Differentiating (3) $n$ times, we get

$$(1-x^2)y_{n+2}+n.y_{n+1}.2x+\frac{n(n-1)}{1.2}y_2.2+y_{n+1}.x+n.y_n=0$$

$\Rightarrow$ $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$ ...(4)

Putting $x=0$ in (1),(2),(3) and (4), we have

$$y(0)=0, y_1(0)=1, y_2(0)=0$$

$$y_{n+2}(0)=n^2y_n(0)$$ ...(5)

From (5), we have

$$y_3(0)=-1^2y_1(0)=-1^2$$

Notes

$$y_5(0)=(-3^2)y_3(0)=(-3^2)(-1^2)=3^2.1^2$$

$$y_7(0)=(-5^2)y_5(0)=(-5^2)(-3^2)(-1^2)=-5^2.3^2.1^2 \quad \text{.... and so on.}$$

Putting $n-2$ for $n$ in (5), we get

$$y_n(0)=\{-(n-2)^2\}y_{n-2}(0) \qquad ...(6)$$

$$=[-(n-2)^2][-(n-4)^2]y_{n-4}(0).$$

Here we observe that

If $n$ is odd, then $y_n(0)=[-(n-2)^2][-(n-4)^2]-...(-5^2)(-3^2)(-1^2).1$

$$=[-1]^{(n-1)/2}(n-2)^2(n-4)^2...5^2.3^2.1^2 \qquad ...(7)$$

Also from (5), we get $y_4(0)=-2^2.y_2(0)=0$

$$y_6(0)=-4^2.y_4(0)=0 \text{ ... and so on.}$$

If $n$ is even.

Then, $\quad y_n(0)=0.$

Putting all these values in Maclaurin's series

$$y= y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+...$$

We get $\log\left[x+\sqrt{(1+x^2)}\right]=x-\frac{x^3}{3!}.1^2+\frac{x^5}{5!}\left(3^2.1^2\right)-\frac{x^7}{7!}\left(5^2.3^2.1^2\right)+...$

*General term*. The general term $=\frac{x^n}{n!}y_n(0)$

where $\quad y_n(0)=\begin{cases}(-1)^{(n-1)/2}(n-2)^2(n-4)^2...5^2.3^2.1^2 & \text{, if } n \text{ is odd}\\ 0 & \text{, if } n \text{ is even}\end{cases}$

**Example 5.** *Prove by Maclaurin's theorem, that* $e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3.x^4}{1.2.3.4}+...$

**Solution.** Let $f(x)=e^{\sin x} \Rightarrow f(0)=e^0=1$

$$f'(x)=e^{\sin x}.\cos x \Rightarrow f'(0)=e^0\cos 0=1$$

$$f''(x)=e^{\sin x}(-\sin x)+\cos x\, e^{\sin x}\cos x$$

$$=e^{\sin x}[\cos^2 x-\sin x] \qquad \Rightarrow f''(0)=e^0[1-0]=1$$

$$f'''(x)=e^{\sin x}[2\cos x(-\sin x)-\cos x]+e^{\sin x}\cos x.[\cos^2 x-\sin x]$$

$$=e^{\sin x}\cos x[-2\sin x-1+\cos^2 x-\sin x]$$

$$=-e^{\sin x}\cos x[3\sin x+\sin^2 x] \quad \Rightarrow f'''(0)=0$$

$$f^{iv}(x)=-e^{\sin x}\cos x[3\cos x+2\sin x\cos x]$$

$$+e^{\sin x}\sin x[3\sin x+\sin^2 x]-[3\sin x+\sin^2 x]\cos x\, e^{\sin x}\cos x$$

$$\Rightarrow \quad f^{iv}(0)=-3.$$

Putting all these values in Maclaurin's theorem, given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+...$$

we get, $\quad e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3.x^4}{1.2.3.4}+...$

**Example 6.** (*i*) *If* $f(x)=x^3+8x^2+15x-24$, *calculate the valve of* $\left(\frac{11}{10}\right)$ *by Taylor's series.*

(*ii*) *If* $f(x)=x^3-2x+5$, *find the value of* $f(2.001)$ *with the help of Taylor's theorem. Find the approximate change in the value of* $f(x)$ *when* $x$ *changes from 2 to 2.001.*

Notes

$f''(x) = 2\tan x \sec^2 x = 2\tan x(1+\tan^2 x) = 2\tan x + 2\tan^3 x \quad \Rightarrow f''(0)=0$

$f'''(x) = 2\sec^2 x + 6\tan^2 x\sec^2 x = 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 8\tan^2 x + 6\tan^4 x \quad \Rightarrow f'''(0)=2$

$f^{iv}(x) = 16\tan x\sec^2 x + 24\tan^3 x\sec^2 x = 8\sec^2 x(2\tan x + 3\tan^3 x)$

$= 8(1+\tan^2 x)(2\tan x + 3\tan^3 x)$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x \quad \Rightarrow f^{iv}(0)=0$

and $f^{v}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x$

$= 8\sec^2 x(2 + 15\tan^2 x + 15\tan^4 x) \quad \Rightarrow f^{v}(0)=16$

Now, putting all these values in Maclaurin's series'

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots$$

We get $\tan x = 0 + x + \frac{x^3}{3!}.2 + \frac{x^5}{5!}.16 + \ldots$

$$\Rightarrow \qquad \tan x = x + \frac{x^3}{3} + \frac{2}{5}x^5 + \ldots$$

**Deduction.** Here

$$x = 46°30' = \left(46\frac{1}{2}\right)^\circ = \left(\frac{93}{2}\right)^\circ = \frac{93}{2}\times\frac{\pi}{180}\text{ Radians}$$

$$= \frac{31}{120}\times\frac{22}{7} = \frac{31\times 11}{60\times 7} = \frac{314}{420} = 0.812$$

Now, putting $x = 46°30' = 0.812$ in (1), we get

$$\tan 46°30' = 0.812 + \frac{(0.812)^3}{3} + \frac{2}{15}(0.812)^5 = 0.812 + 0.1784 + 0.047 = 1.0374$$

**Example 4.** *Expand* $\log\{x + \sqrt{(1+x^2)}\}$ *in ascending powers of x and find the general term.*

**Solution.** Let $\quad y = \log\{x + \sqrt{(1+x^2)}\} \quad \ldots(1)$

$$\Rightarrow \quad y_1 = \frac{1}{x+\sqrt{1+x^2}}.\left[1 + \frac{2x}{2\sqrt{(1+x^2)}}\right] = \frac{1}{\sqrt{1+x^2}} \quad \ldots(2)$$

$\Rightarrow \quad y_1^2(1+x^2) - 1 = 0.$

Differentiating again w.r.t. $x$, we get

$2y_1[(1+x^2)y_2 + xy_1] = 0$

$\Rightarrow \quad [(1+x^2)y_2 + xy_1] = 0 \qquad (\because 2y_1 \neq 0) \quad \ldots(3)$

Differentiating (3) $n$ times, we get

$$(1-x^2)y_{n+2} + n.y_{n+1}.2x + \frac{n(n-1)}{1.2}y_2.2 + y_{n+1}.x + n.y_n = 0$$

$$\Rightarrow \quad (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0 \quad \ldots(4)$$

Putting $x=0$ in (1),(2),(3) and (4), we have

$$y(0)=0,\ y_1(0)=1,\ y_2(0)=0$$

$$y_{n+2}(0) = n^2 y_n(0) \quad \ldots(5)$$

From (5), we have

$$y_3(0) = -1^2 y_1(0) = -1^2$$

Notes

$$y_5(0)=(-3^2)y_3(0)=(-3^2)(-1^2)=3^2.1^2$$

$$y_7(0)=(-5^2)y_5(0)=(-5^2)(-3^2)(-1^2)=-5^2.3^2.1^2 \quad \text{.... and so on.}$$

Putting $n-2$ for $n$ in (5), we get

$$y_n(0)=\{-(n-2)^2\}y_{n-2}(0) \qquad ...(6)$$

$$=[-(n-2)^2][-(n-4)^2]y_{n-4}(0).$$

Here we observe that

If $n$ is odd, then $y_n(0)=[-(n-2)^2][-(n-4)^2]-...(-5^2)(-3^2)(-1^2).1$

$$=[-1]^{(n-1)/2}(n-2)^2(n-4)^2...5^2.3^2.1^2 \qquad ...(7)$$

Also from (5), we get $y_4(0)=-2^2.y_2(0)=0$

$$y_6(0)=-4^2.y_4(0)=0 \text{ ... and so on.}$$

If $n$ is even.

Then, $\quad y_n(0)=0.$

Putting all these values in Maclaurin's series

$$y=y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+...$$

We get $\log\left[x+\sqrt{(1+x^2)}\right]=x-\frac{x^3}{3!}.1^2+\frac{x^5}{5!}\left(3^2.1^2\right)-\frac{x^7}{7!}\left(5^2.3^2.1^2\right)+...$

*General term.* The general term $=\frac{x^n}{n!}y_n(0)$

where $\quad y_n(0)=\begin{cases}(-1)^{(n-1)/2}(n-2)^2(n-4)^2...5^2.3^2.1^2 & \text{, if } n \text{ is odd}\\ 0 & \text{, if } n \text{ is even}\end{cases}$

**Example 5.** *Prove by Maclaurin's theorem, that* $e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3.x^4}{1.2.3.4}+...$

**Solution.** Let $f(x)=e^{\sin x} \Rightarrow f(0)=e^0=1$

$$f'(x)=e^{\sin x}.\cos x \Rightarrow f'(0)=e^0\cos 0=1$$

$$f''(x)=e^{\sin x}(-\sin x)+\cos x\, e^{\sin x}\cos x$$

$$=e^{\sin x}[\cos^2 x-\sin x] \qquad \Rightarrow f''(0)=e^0[1-0]=1$$

$$f'''(x)=e^{\sin x}[2\cos x(-\sin x)-\cos x]+e^{\sin x}\cos x.[\cos^2 x-\sin x]$$

$$=e^{\sin x}\cos x[-2\sin x-1+\cos^2 x-\sin x]$$

$$=-e^{\sin x}\cos x[3\sin x+\sin^2 x] \quad \Rightarrow f'''(0)=0$$

$$f^{iv}(x)=-e^{\sin x}\cos x[3\cos x+2\sin x\cos x]$$

$$+e^{\sin x}\sin x[3\sin x+\sin^2 x]-[3\sin x+\sin^2 x]\cos x\, e^{\sin x}\cos x$$

$$\Rightarrow \qquad f^{iv}(0)=-3.$$

Putting all these values in Maclaurin's theorem, given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+...$$

we get, $\quad e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3.x^4}{1.2.3.4}+...$

**Example 6.** (*i*) *If* $f(x)=x^3+8x^2+15x-24$, *calculate the valve of* $\left(\frac{11}{10}\right)$ *by Taylor's series.*

(*ii*) *If* $f(x)=x^3-2x+5$, *find the value of* $f(2.001)$ *with the help of Taylor's theorem. Find the approximate change in the value of* $f(x)$ *when* $x$ *changes from 2 to 2.001.*

**Solution.** (i) By Taylor's Theorem, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \quad \ldots(1)$$

We want to find $f\left(\frac{11}{10}\right)$ *i.e.*, $f\left(1+\frac{1}{10}\right)$

Put $x=1$ and $h=\frac{1}{10}$, and expand by Taylor's series, we get

$$f\left(\frac{11}{10}\right)=f\left(1+\frac{1}{10}\right)=f(1)+\frac{1}{10}f'(1)+\frac{1}{10^2}.\frac{1}{2!}f''(1)+\frac{1}{3!}\frac{1}{(10)^3}f'''(1)+\ldots \quad \ldots(2)$$

Now $f(x)=x^3+8x^2+15x-24 \Rightarrow f(1)=0$

$f'(x)=3x^2+16x+15 \Rightarrow f'(1)=34$

$f''(x)=6x+16 \Rightarrow f''(1)=22$

$f'''(x)=6 \Rightarrow f'''(1)=6$

$f^{iv}(x)=0 \Rightarrow f^{iv}(1)=0$

Put all these values in (2), we get

$$f\left(1+\frac{1}{10}\right)=0+\frac{1}{10}.34+\frac{11}{100}+\frac{1}{1000}=3.4+0.11+0.001=3.511.$$

(ii) Here put $x=2$ and $h=0.001$ in Taylor's series, we get

$$f(2.001)=f(2)+(0.001)f'(2)+\frac{(0.001)^2}{2!}f''(2)+\frac{(0.001)^3}{3!}f'''(2)+\ldots \quad \ldots(3)$$

Now $f(x)=x^3-2x+5 \Rightarrow f(2)=9$

$f'(x)=3x^2-2 \Rightarrow f'(2)=10$

$f''(x)=6x \Rightarrow f''(2)=12$

$f'''(x)=6 \Rightarrow f'''(2)=6$

$f^{iv}(x)=0 \Rightarrow f^{iv}(2)=0$

Put all these values in (2), we get

$$f(2.0001)=9+(0.001)10+\frac{1}{2!}(0.001)^2(12)+\frac{1}{3!}(0.001)^3.6+\ldots$$
$$=9+0.01+0.000006+0.000000001$$
$$=9.010006001=9.01 \text{ approximately.}$$

Approximate value of $f(2.001)-f(2)=9.01-9=0.01$ approximately.

**Example 7.** *Expand log $(1+\sin x)$ by Maclaurin's theorem in ascending power of x upto first five terms.*

**Solution.** Let $y=f(x)=\log(1+\sin x)$.

By Maclaurin's expansion for $f(x)$, we have

$$y=f(x)=(y)_0+\frac{x}{1!}(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\frac{x^4}{4!}(y_4)_0+\ldots \quad \ldots(1)$$

Now $y=\log(1+\sin x) \quad \therefore \quad (y)_0=0$

$$y_1=\frac{\cos x}{1+\sin x}\Rightarrow(y_1)_0=1$$

$$y_2=\frac{-\sin x(1+\sin x)-\cos^2 x}{(1+\sin x)^2}=-\frac{(1+\sin x)}{(1+\sin x)^2}=-\frac{1}{1+\sin x}$$

$\Rightarrow \quad (y_2)_0=-1$

$$y_3=\frac{\cos x}{(1+\sin x)^2}=-\frac{\cos x}{(1+\sin x)}.\frac{1}{(1+\sin x)}=-y_1y_2$$

Notes

$\Rightarrow \quad (y_3)_0 = -1(-1) = 1$

$y_4 = -y_1y_3 - y_2^2 \Rightarrow (y_4)_0 = -1.1 - (-1)^2 = -1 - 1 = -2$

$y_5 = -y_1y_4 - y_2y_3 - 2y_2y_3 = -y_1y_4 - 3y_2y_3$

$\Rightarrow \quad (y_5)_0 = -1.(-2) - 3(-1).1 = 2 + 3 = 5$ and so on.

Therefore, $\log(1+\sin x) = 0 + \frac{x}{1!}.1 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}.1 + \frac{x^4}{4!}.(-2) + \ldots$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{25} \ldots$$

**Example 8.** *Expand sin* $(\pi/4+\theta)$ *in powers of* $\theta$.

**Solution.** Let $f(\theta) = \sin(\pi/4+\theta) \Rightarrow f(0) = \sin \pi/4 = 1/\sqrt{2}$

$f'(\theta) = \cos(\pi/4+\theta) \Rightarrow f'(0) = \cos \pi/4 = 1/\sqrt{2}$

$f''(\theta) = -\sin(\pi/4+\theta) \Rightarrow f''(0) = -\sin \pi/4 = -1/\sqrt{2}$

$f'''(\theta) = -\cos(\pi/4+\theta) \Rightarrow f'''(0) = \cos \pi/4 = -1/\sqrt{2}$

$f^{iv}(\theta) = \sin(\pi/4+\theta) \Rightarrow f^{iv}(0) = 1/\sqrt{2}$ and so on.

The $n^{th}$ derivative of $f(\theta)$ is given by

$$f^n(\theta) = \sin\left(\theta + \frac{\pi}{4} + \frac{n\pi}{2}\right)$$

The Maclaurin's expansion of $f(\theta)$ with Lagrange's from of remainder is

$$f(\theta) = f(0) + \frac{\theta}{1!}f'(0) + \frac{\theta^2}{2!}f''(0) + \frac{\theta^3}{3!}f'''(0) + \ldots + \frac{\theta^{n-1}}{(n-1)!}f^{n-1}(0) + R_n \quad \ldots(1)$$

where $R_n = \frac{\theta^n}{n!}f^n(t\theta) = \frac{\theta^n}{n!}\sin\left(t\theta + \frac{\pi}{4} + \frac{n\pi}{2}\right), 0 < t < 1.$

Now $|R_n| = \left|\frac{\theta^n}{n!}\sin\left(t\theta + \frac{\pi}{4} + \frac{n\pi}{2}\right)\right| = \left|\frac{\theta^n}{n!}\right| \cdot \left|\sin\left(t\theta + \frac{\pi}{4} + \frac{n\pi}{2}\right)\right| \le \left|\frac{\theta^n}{n!}\right|.$

$\therefore \lim_{n\to\infty} |R_n| \le \lim_{n\to\infty}\left|\frac{\theta^n}{n!}\right| = 0 \qquad \left[\because \lim_{n\to\infty}\frac{\theta^n}{n!} = 0\right]$

$\therefore \quad \lim_{n\to\infty} R_n = 0$

Thus all the conditions of Maclaurin's series expansion are satisfied. Hence, from (1), the expansion of sin $(\theta+\pi/4)$ is given by

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{\theta}{1!}\frac{1}{\sqrt{2}} + \frac{\theta^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \ldots$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left[1 + \frac{\theta}{1!} - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} + \ldots\right]$$

## STUDENT ACTIVITY

1. Expand the following functions by Maclaurin's theorem : $\log_e(1+e^x)$

2. Expand the following functions by Maclaurin's theorem : $\log(1+\tan x)$.

**Solution.** (i) By Taylor's Theorem, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \quad \ldots(1)$$

We want to find $f\left(\frac{11}{10}\right)$ *i.e.*, $f\left(1+\frac{1}{10}\right)$

Put $x=1$ and $h=\frac{1}{10}$, and expand by Taylor's series, we get

$$f\left(\frac{11}{10}\right)=f\left(1+\frac{1}{10}\right)=f(1)+\frac{1}{10}f'(1)+\frac{1}{10^2}\cdot\frac{1}{2!}f''(1)+\frac{1}{3!}\frac{1}{(10)^3}f'''(1)+\ldots \quad \ldots(2)$$

Now $f(x)=x^3+8x^2+15x-24 \Rightarrow f(1)=0$

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$f''(x)=6x+16 \Rightarrow f''(1)=22$

$f'''(x)=6 \Rightarrow f'''(1)=6$

$f^{iv}(x)=0 \Rightarrow f^{iv}(1)=0$

Put all these values in (2), we get

$$f\left(1+\frac{1}{10}\right)=0+\frac{1}{10}.34+\frac{11}{100}+\frac{1}{1000}=3.4+0.11+0.001=3.511.$$

(ii) Here put $x=2$ and $h=0.001$ in Taylor's series, we get

$$f(2.001)=f(2)+(0.001)f'(2)+\frac{(0.001)^2}{2!}f''(2)+\frac{(0.001)^3}{3!}f'''(2)+\ldots \quad \ldots(3)$$

Now $f(x)=x^3-2x+5 \Rightarrow f(2)=9$

$f'(x)=3x^2-2 \Rightarrow f'(2)=10$

$f''(x)=6x \Rightarrow f''(2)=12$

$f'''(x)=6 \Rightarrow f'''(2)=6$

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Put all these values in (2), we get

$$f(2.0001)=9+(0.001)10+\frac{1}{2!}(0.001)^2(12)+\frac{1}{3!}(0.001)^3.6+\ldots$$
$$=9+0.01+0.000006+0.000000001$$
$$=9.010006001=9.01 \text{ approximately.}$$

Approximate value of $f(2.001)-f(2)=9.01-9=0.01$ approximately.

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**Solution.** Let $y=f(x)=\log(1+\sin x)$.

By Maclaurin's expansion for $f(x)$, we have

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Now $y=\log(1+\sin x) \quad \therefore \quad (y)_0=0$

$$y_1=\frac{\cos x}{1+\sin x}\Rightarrow (y_1)_0=1$$

$$y_2=\frac{-\sin x(1+\sin x)-\cos^2 x}{(1+\sin x)^2}=-\frac{(1+\sin x)}{(1+\sin x)^2}=-\frac{1}{1+\sin x}$$

$\Rightarrow \quad (y_2)_0=-1$

$$y_3=\frac{\cos x}{(1+\sin x)^2}=-\frac{\cos x}{(1+\sin x)}\cdot\frac{1}{(1+\sin x)}=-y_1y_2$$

Notes

$$\Rightarrow \quad (y_3)_0=-1(-1)=1$$

$$y_4=-y_1y_3-y_2^2 \quad \Rightarrow \quad (y_4)_0=-1.1-(-1)^2=-1-1=-2$$

$$y_5=-y_1y_4-y_2y_3-2y_2y_3=-y_1y_4-3y_2y_3$$

$$\Rightarrow \quad (y_5)_0=-1.(-2)-3(-1).1=2+3=5 \text{ and so on.}$$

Therefore, $\log(1+\sin x) = 0+\frac{x}{1!}.1+\frac{x^2}{2!}.(-1)+\frac{x^3}{3!}.1+\frac{x^4}{4!}.(-2)+...$

$$= x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{25}...$$

**Example 8.** *Expand sin (π/4+θ) in powers of θ.*

**Solution.** Let

$$f(\theta)= \sin(\pi/4+\theta) \quad \Rightarrow \quad f(0)=\sin \pi/4=1/\sqrt{2}$$

$$f'(\theta)=\cos(\pi/4+\theta) \quad \Rightarrow \quad f'(0)=\cos \pi/4=1/\sqrt{2}$$

$$f''(\theta)=-\sin(\pi/4+\theta) \quad \Rightarrow \quad f''(0)=-\sin \pi/4=-1/\sqrt{2}$$

$$f'''(\theta)=-\cos(\pi/4+\theta) \quad \Rightarrow \quad f'''(0)=\cos \pi/4=-1/\sqrt{2}$$

$$f^{iv}(\theta)=\sin(\pi/4+\theta) \quad \Rightarrow \quad f^{iv}(0)=1/\sqrt{2} \text{ and so on.}$$

The $n^{th}$ derivative of $f(\theta)$ is given by

$$f^n(\theta)=\sin\left(\theta+\frac{\pi}{4}+\frac{n\pi}{2}\right)$$

The Maclaurin's expansion of $f(\theta)$ with Lagrange's from of remainder is

$$f(\theta)=f(0)+\frac{\theta}{1!}f'(0)+\frac{\theta^2}{2!}f''(0)+\frac{\theta^3}{3!}f'''(0)+...+\frac{\theta^{n-1}}{(n-1)!}f^{n-1}(0)+R_n \quad ...(1)$$

where $R_n=\frac{\theta^n}{n!}f^n(t\theta)=\frac{\theta^n}{n!}\sin\left(t\theta+\frac{\pi}{4}+\frac{n\pi}{2}\right), 0<t<1.$

Now $|R_n|=\left|\frac{\theta^n}{n!}\sin\left(t\theta+\frac{\pi}{4}+\frac{n\pi}{2}\right)\right|=\left|\frac{\theta^n}{n!}\right|.\left|\sin\left(t\theta+\frac{\pi}{4}+\frac{n\pi}{2}\right)\right|\le\left|\frac{\theta^n}{n!}\right|.$

$$\therefore \lim_{n\to\infty}|R_n|\le\lim_{n\to\infty}\left|\frac{\theta^n}{n!}\right|=0 \qquad \left[\because \lim_{n\to\infty}\frac{\theta^n}{n!}=0\right]$$

$$\therefore \lim_{n\to\infty}R_n=0$$

Thus all the conditions of Maclaurin's series expansion are satisfied. Hence, from (1), the expansion of sin (θ+π/4) is given by

$$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}+\frac{\theta}{1!}\frac{1}{\sqrt{2}}+\frac{\theta^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)+...$$

$$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left[1+\frac{\theta}{1!}-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{\theta^5}{5!}-\frac{\theta^6}{6!}-\frac{\theta^7}{7!}+...\right]$$

## STUDENT ACTIVITY

1. Expand the following functions by Maclaurin's theorem : $\log_e(1+e^x)$

2. Expand the following functions by Maclaurin's theorem : $\log(1+\tan x)$.

Notes

3. If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$, show that $(n+1)a_{n+1} + (n-1)a_{n-1} = ma_n$.

4. If $e^{e^x} = a_0 + a_1x + a_2x^2 + ... + a_n x^n + ...$ show that

$$a_{n+1} = \frac{1}{n+1}\left[a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + ... + \frac{a_{n-r}}{r!} + ... \frac{a_0}{n!}\right]$$

## TEST YOURSELF

1. Expand the following functions by Maclaurin's theorem :
   (i) Sec $x$ (ii) $e^{x\cos x}$ (iii) $e^x \sec x$
2. Apply Maclaurin's theorem to prove that $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + ...$
3. If $y = \sin^{-1} x = a_0 + a_1x + a_2x^2 + ...$ Prove that $(n+1)(n+2)a_{n+2} = n^2 a_n$.
4. Show that :
   (i) $e^x \cos x = 1 + x - \frac{2x^3}{3!} + \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + ... + \cos\left(\frac{n\pi}{4}\right)\frac{2^{n/2}}{n!}x^n + ...$

   (ii) $e^x \sin x = x + x^2 - \frac{2x^3}{3!} + \frac{2^2x^5}{5!} - ... + \sin\left(\frac{n\pi}{4}\right)\frac{2^{n/2}}{n!}x^n + ...$

   (iii) $e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + ... + \frac{(a^2 + b^2)^{\frac{n}{2}}}{n!}x^n \sin\left(n\tan^{-1}\frac{b}{a}\right) + ...$

   (iv) $e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2!}x^2 +$
   $+ \frac{a(a^2 - 3b^2)}{3!}x^3 + ... + \frac{(a^2 + b^2)^{\frac{n}{2}}}{n!}x^n \cos\left(n\tan^{-1}\frac{b}{a}\right) + ...$
5. Expand the following :
   (i) $\tan^{-1}x$ in powers of $\left(x - \frac{\pi}{4}\right)$. (ii) $2x^3 + 7x^2 + x - 1$ in powers of $x-2$.
   (iii) $\sin^{-1}(x+h)$ in power of $x$. (iv) $\log \sin x$ in power of $(x-a)$.
6. Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - ...$
7. Use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta\frac{\sin\theta}{1}$$
$$-(h\sin\theta)^2\frac{\sin 2\theta}{2} + (h\sin\theta)^3\frac{\sin 3\theta}{3} + ... + (-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n} + ...$$

where $\theta = \cot^{-1} x$

Notes

8. If $y=e^{\tan^{-1}x}$, show that $(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$. Hence, or otherwise, find out the coefficient of $x^5$ if $e^{\tan^{-1}x}$ is expanded in powers of $x$.

9. Expand $(\sin^{-1}x)^2$ in ascending powers of $x$ and deduce that

$$\theta^2 = 2.\frac{\sin^2\theta}{2!}+2^2.\frac{2\sin^4\theta}{4!}+2^2.4^2\frac{2\sin^6\theta}{6!}+...$$

## ANSWERS

1. (i) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+...$ (ii) $1+x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{11x^4}{24}-\frac{x^5}{5}+...$ (iii) $1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+...$

5. (i) $\tan^{-1}\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)\Big/\left(1+\frac{\pi^2}{16}\right)-\pi\left(x-\frac{\pi}{4}\right)^2\Big/\left[4\left(1+\frac{\pi^2}{16}\right)^2\right]+...$

(ii) $45+53(x-2)+19(x-2)^2+2(x-2)^3+...$

(iii) $\sin^{-1}h+x\left(1-h^2\right)^{-1/2}+\frac{x^2}{2!}h\left(1-h^2\right)^{-3/2}+\frac{x^3}{3!}\left[\left(1-h^2\right)^{-5/2}\left(1+2h^2\right)\right]+...$

(iv) $\log\sin a+(x-a)\cot a-\frac{(x-a)^2}{2!}\operatorname{cosec}^2 a+\frac{(x-a)^3}{3!}2\operatorname{cosec}^2 a\cot a+...$

8. $\frac{1}{24}$  9. $2.\frac{x^2}{2!}+\frac{2.2^2}{4!}x^4+\frac{2.2^2.4^2}{6!}x^6+...+\frac{2.2^2.4^2...(2n-2)^2}{(2n)!}x^{2n}+...$

## Summary

- Let $f(x)$ be a single valued function defined on $[a, a+h]$ such that
  (i) all the derivative of $f(x)$ upto $(n-1)^{th}$ order are continuous in $[a, a+h]$, and
  (ii) $fn(x)$ exists in $(a,a+h)$ then there exists a real number $q, 0 < q < 1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

where $p$ is a given positive integer.

- The term $R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$ which occur after $n$ terms, is called the Taylor's remainder after n terms. The theorem with this form of remainder is called Taylor's theorem with Scholomilch and Roche form of remainder.
- For p=1, we get $R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$ Then, $R_n$ is called Cauchy's form of remainder.
- For $p=n$, we get $Rn=\frac{h^n}{n!}f^n(a+\theta h)$ then, $R_n$ is called Lagrange's form of remainder.
- Let $f(x)$ possesess continuous derivatives of all orders in the interval $[a, a+h]$, then for every positive integral value of $n$, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$

where, $R_n=\frac{h^n}{n!}.f^n(a+\theta h),(0<\theta<1)$.

- If we put $a=0$ and replace $h$ by $x$ in Taylor's series, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$$

This series is known as Maclaurin's series for the expansion of $f(x)$ as a power series in $x$.

- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^{n-1}}{(n-1)!}+...\ \forall x\in R$

3. If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$, show that $(n+1)a_{n+1} + (n-1)a_{n-1} = ma_n$.

4. If $e^{e^x} = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n + \ldots$ show that

$$a_{n+1} = \frac{1}{n+1}\left[a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \ldots + \frac{a_{n-r}}{r!} + \ldots \frac{a_0}{n!}\right]$$

## TEST YOURSELF

1. Expand the following functions by Maclaurin's theorem :
   (i) Sec $x$   (ii) $e^{x\cos x}$   (iii) $e^x\sec x$
2. Apply Maclaurin's theorem to prove that $\log\sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots$
3. If $y = \sin^{-1}x = a_0 + a_1x + a_2x^2 + \ldots$ Prove that $(n+1)(n+2)a_{n+2} = n^2a_n$.
4. Show that :

   (i) $e^x\cos x = 1 + x - \frac{2x^3}{3!} + \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \ldots + \cos\left(\frac{n\pi}{4}\right)\frac{2^{n/2}}{n!}x^n + \ldots$

   (ii) $e^x\sin x = x + x^2 - \frac{2x^3}{3!} + \frac{2^2x^5}{5!} - \ldots + \sin\left(\frac{n\pi}{4}\right)\frac{2^{n/2}}{n!}x^n + \ldots$

   (iii) $e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + \ldots + \frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{n!}x^n\sin\left(n\tan^{-1}\frac{b}{a}\right) + \ldots$

   (iv) $e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2!}x^2 + $

   $$+\frac{a\left(a^2 - 3b^2\right)}{3!}x^3 + \ldots + \frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{n!}x^n\cos\left(n\tan^{-1}\frac{b}{a}\right) + \ldots$$

5. Expand the following :
   (i) $\tan^{-1}x$ in powers of $\left(x - \frac{\pi}{4}\right)$.   (ii) $2x^3 + 7x^2 + x - 1$ in powers of $x-2$.
   (iii) $\sin^{-1}(x+h)$ in power of $x$.   (iv) $\log\sin x$ in power of $(x-a)$.
6. Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots$
7. Use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta\frac{\sin\theta}{1}$$
$$-(h\sin\theta)^2\frac{\sin 2\theta}{2} + (h\sin\theta)^3\frac{\sin 3\theta}{3} + \ldots + (-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n} + \ldots$$

where $\theta = \cot^{-1}x$

Notes

8. If $y=e^{\tan^{-1}x}$, show that $(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$. Hence, or otherwise, find out the coefficient of $x^5$ if $e^{\tan^{-1}x}$ is expanded in powers of $x$.

9. Expand $(\sin^{-1}x)^2$ in ascending powers of $x$ and deduce that

$$\theta^2=2.\frac{\sin^2\theta}{2!}+2^2.\frac{2\sin^4\theta}{4!}+2^2.4^2\frac{2\sin^6\theta}{6!}+...$$

## ANSWERS

1. (i) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+...$ (ii) $1+x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{11x^4}{24}-\frac{x^5}{5}+...$ (iii) $1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+...$

5. (i) $\tan^{-1}\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)\Big/\left(1+\frac{\pi^2}{16}\right)-\pi\left(x-\frac{\pi}{4}\right)^2\Big/\left[4\left(1+\frac{\pi^2}{16}\right)^2\right]+...$

(ii) $45+53(x-2)+19(x-2)^2+2(x-2)^3+...$

(iii) $\sin^{-1}h+x\left(1-h^2\right)^{-1/2}+\frac{x^2}{2!}h\left(1-h^2\right)^{-3/2}+\frac{x^3}{3!}\left[\left(1-h^2\right)^{-5/2}\left(1+2h^2\right)\right]+...$

(iv) $\log\sin a+(x-a)\cot a-\frac{(x-a)^2}{2!}\operatorname{cosec}^2 a+\frac{(x-a)^3}{3!}2\operatorname{cosec}^2 a\cot a+...$

8. $\frac{1}{24}$ 9. $2.\frac{x^2}{2!}+\frac{2.2^2}{4!}x^4+\frac{2.2^2.4^2}{6!}x^6+...+\frac{2.2^2.4^2...(2n-2)^2}{(2n)!}x^{2n}+...$

## Summary

- Let $f(x)$ be a single valued function defined on $[a, a+h]$ such that
  (i) all the derivative of $f(x)$ upto $(n-1)^{\text{th}}$ order are continuous in $[a, a+h]$, and
  (ii) $fn(x)$ exists in $(a,a+h)$ then there exists a real number $q, 0<q<1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

where $p$ is a given positive integer.

- The term $R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$ which occur after $n$ terms, is called the Taylor's remainder after n terms. The theorem with this form of remainder is called Taylor's theorem with Scholomilch and Roche form of remainder.
- For p=1, we get $R_n=\frac{h^n(1-\theta)^{n-1}}{P(n-1)!}f^n(a+\theta h)$ Then, $R_n$ is called Cauchy's form of remainder.
- For $p=n$, we get $Rn=\frac{h^n}{n!}f^n(a+\theta h)$ then, $R_n$ is called Lagrange's form of remainder.
- Let $f(x)$ possesess continuous derivatives of all orders in the interval $[a, a+h]$, then for every positive integral value of $n$, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$

where, $R_n=\frac{h^n}{n!}f^n(a+\theta h),(0<\theta<1)$.

- If we put $a=0$ and replace $h$ by $x$ in Taylor's series, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$$

This series is known as Maclaurin's series for the expansion of $f(x)$ as a power series in $x$.

- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^{n-1}}{(n-1)!}+...\ \forall x\in R$

Notes

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \ \forall x \in R.$
- $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + x^m$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots,$ whenever $-1 < x \le 1.$
- $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log a)^n, 0 < \theta < 1.$

## Objective Evaluation

### FILL IN THE BLANKS

1. Maclaurin's theorem is a particular case of __________.
2. For $p = 1$, $R_n = \frac{h^n(1-\theta)^{n-1}}{p(n-1)!} f^n(a+\theta h)$ then $R_n$ is called __________.

### TRUE/FALSE

**Write 'T' for true and 'F' for false statement.**

1. Maclaurin's theorem is a particular case of Taylor's theorem. **(T/F)**
2. Taylor's theorem is a particular case of Maclaurin's theorem. **(T/F)**

### MULTIPLE CHOICE QUESTIONS

**Choose the most appropriate one.**

1. If $f(x)$ is an even function then the value of $f'(0)$, if exists is equal to :
   (a) 1 (b) 0 (c) 2 (d) $\infty$
2. If a function $f$ is continuous on $[a, b]$, differentiable on $]a, b[$ and if $f'(x) = 0 \ \forall x \in ]a, b[$ then $f(x)$ has a :
   (a) constant value throughout $[a,b]$
   (b) constant value only on the end points
   (c) constant value throughout $]a,b[$
   (d) none of the above
3. If $f(x)$ and $g(x)$ are continuous on $[a,b]$ and differentiable on $]a,b[$ and if $f'(x)=g'(x)$ throughout the interval ]a, b[ then :
   (a) $f(x)=g(x) \ \forall x \in ]a,b[$
   (b) $f(x) \ne g(x) \ \forall x \in ]a,b[$
   (c) $f(x)$ and $g(x)$ differ only by a constant
   (d) none of the above
4. If $f$ is continuous on $[a,b]$ and $f'(x) \ge 0$ on $]a,b[$ then :
   (a) $f$ is decreasing on $]a,b[$
   (b) $f$ is decreasing on $[a,b]$
   (c) $f$ is increasing on $[a,b]$
   (d) none of the above
5. If $f(x)$ is an increasing function on $x$, then :
   (a) $f'(x) \le 0$ (b) $f'(x)=0$ (c) $f'(x)>0$ (d) none of the above
6. If $f'(x)$ is positive at a point $x=a$ then in the nbd of $a$ :
   (a) $f(x)$ is positive
   (b) $f(x)$ is increasing
   (c) $f(x)$ is negative
   (d) none of the above
7. The function $f(x)$ has equal values at the point $x=a$ and $x=b$ then :
   (a) there is a maximum of $f(x)$ between $a$ and $b$
   (b) there is a minimum of $f(x)$ between $a$ and $b$
   (c) there is a maximum or minimum of $f(x)$ between $a$ and $b$
   (d) none of the above
8. If $f''(x)>0$ at points in $]a,b[$ then the function $f$ is :
   (a) strictly increasing
   (b) strictly decreasing
   (c) constant
   (d) none of the above
9. If a function $f(x)$ satisfy the condition of mean value theorem and $f'(x)=0 \ \forall x \in ]a,b[$ then :
   (a) $f(x)=0$
   (b) $f(x)$ is an increasing function

Notes

(c) $f(x)$ is constant (d) none of the above

**10.** The value of $c$ of Rolle's theorem for the function $f(x)=\sin x$ in $[0,\pi]$ is given by :

(a) $\pi/3$ (b) $\pi/2$ (c) $\pi$ (d) none of the above

**11.** The value of $c$ of Lagrange's mean value theorem for $f(x)=x(x-1)$ in $[1,2]$ is given by :

(a) $\frac{1}{4}$ (b) $\frac{3}{2}$ (c) $\frac{5}{4}$ (d) none of the above

**12.** Lagrange's form of remainder after $n$ terms in Taylor's development of the function $ex$ in a finite form in the interval $[a,a+h]$ is :

(a) $\frac{h^n}{n!}e^{a+\theta h}$ (b) $\frac{h^{n+1}}{(n+1)!}e^{a+\theta h}$ (c) $\frac{h^n}{n!}e^{\theta h}$ (d) none of the above

## ANSWERS

### FILL IN THE BLANKS

**1.** Taylor's theorem **2.** Cauchy's form of remainder

### TRUE/FALSE

**1.** T **2.** F

### MULTIPLE CHOICE QUESTIONS

**1.** (b) **2.** (a) **3.** (c) **4.** (c) **5.** (c) **6.** (b) **7.** (c)
**8.** (a) **9.** (c) **10.** (b) **11.** (b) **12.** (a)

□□□□

➡ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \;\forall x \in R.$

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➡ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots,$ whenever $-1 < x \le 1$.

➡ $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \dots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log a)^n, 0 < \theta < 1.$

## Objective Evaluation

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1. Maclaurin's theorem is a particular case of ________.
2. For $p = 1$, $R_n = \frac{h^n(1-\theta)^{n-1}}{p(n-1)!} f^n(a+\theta h)$ then $R_n$ is called ________.

### TRUE/FALSE

**Write 'T' for true and 'F' for false statement.**

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2. Taylor's theorem is a particular case of Maclaurin's theorem. (T/F)

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   (c) there is a maximum or minimum of $f(x)$ between $a$ and $b$
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Notes

(c) $f(x)$ is constant (d) none of the above

10. The value of $c$ of Rolle's theorem for the function $f(x)=\sin x$ in $[0,\pi]$ is given by :

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11. The value of $c$ of Lagrange's mean value theorem for $f(x)=x(x-1)$ in $[1,2]$ is given by :

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(a) $\frac{h^n}{n!}e^{a+\theta h}$ (b) $\frac{h^{n+1}}{(n+1)!}e^{a+\theta h}$ (c) $\frac{h^n}{n!}e^{\theta h}$ (d) none of the above

## ANSWERS

**FILL IN THE BLANKS**

1. Taylor's theorem 2. Cauchy's form of remainder

**TRUE/FALSE**

1. T 2. F

**MULTIPLE CHOICE QUESTIONS**

1. (b) 2. (a) 3. (c) 4. (c) 5. (c) 6. (b) 7. (c)
8. (a) 9. (c) 10. (b) 11. (b) 12. (a)