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MECHANICS AND WAVE MOTION

PHO-1111

Self Learning Material

मङ्गलायतन
विश्वविद्यालय
॥ विश्वं ज्ञाने प्रतिष्ठितम् ॥

Directorate of Distance & Online Education

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CONTENTS

SYLLABUS

MECHANICS

CHAPTER 1 : VECTOR ANALYSIS

Scalars and vectors, dot and cross products, Triple and Quadruple product of vectors, Scalar and Vector fields, Gradient of a scalar field and its geometrical interpretation, Divergence and Curl of a vector field.

CHAPTER 2 : ROTATIONAL DYNAMICS

Rigid body motion, Rotational motion, torque and angular momentum. Moment of inertia and its calculations for disc, cylinder, spherical shell and solid sphere. Body rolling down on an inclined plane. Fly wheel, Motion of Top.

Concept of central force, Kepler's laws of planetary motion, Gravitational law, Gravitational potential and fields due to spherical shell and solid sphere. Two particle central force problem and reduced mass. Motion of planets and statellites.

CHAPTER 3 : PROPERTIES OF MATTER

Elasticity, Hook's law, Elastic constants and relation among them, Beam supported at both the ends, cantilever, Torsion cylinder, Maxwell's needle and Searl's method.

Streamline and turbulent flow, Equation of continuity, Viscosity, Poiseulle's law, Critical velocity, Reynold's number, Stoke's law and terminal velocity. Surface tension and surface energy, Molecular interpretation of surface tension, Pressure on a curved liquid surface.

UNIT 4 : RELATIVITY

Reference system, Inertial frames, Gallilean invariance, Michelson-Morley's experiment. Einstein's postulates for the special theory of relativity, Lorentz transformation equations, Length contraction and Time dilation, Concept of simultaneity, Relativistic addition of velocities. Variation of mass with velocity. Mass energy equivalence, Momentum-energy relations.

CHAPTER 5 : OSCILLATIONS

Potential well and periodic oscillations, Case of harmonic oscillation. Differential equation and solution of simple harmonic oscillations. Kinetic and potential energy. Examples of simple harmonic oscillations. Spring and mass system. Simple and compound pendulum. Torsional pendulum.

Superposition of two simple harmonic motions of same frequency along the same line Interference, Superposition of two mutually perpendicular simple harmonic vibrations of the same frequency, Lissajous figures, case of different frequencies.

UNIT

1

VECTOR ANALYSIS

LEARNING OBJECTIVES

LEARNING OBJECTIVES

After going through this unit you will learn :

- Scalar and vector quantities in detail along with their products.
- Graphical and geometrical representation of the quantities and the products.
- Various applications inform of field, Gradient, Curl Divergence.

1.1. SCALAR AND VECTOR QUANTITIES

Physical quantities are of two types :

(i) Scalar quantities (ii) Vector quantities

(i) Scalar Quantities : Scalar quantities are those which contain magnitude only and do not contain direction. Mass, length, time, density, speed, pressure, energy, work, temperature, specific heat, charge, current, potential, frequency etc. are scalar quantitites. Scalar quantities can be added, substracted and multiplied according to the rule of elementary algebra.

(ii) Vector Quantities : Vector quantities are those quantities which contain both magnitude and direction. Displacement, velocity, force, weight, momentum, stress, electric and magnetic intensities etc. are vector quantities.

For a quantity to be a vector, it is necessary that it satisfies the parallelogram law of addition.

(a) Graphical Representation of Vectors : Graphically, a vector is represented by a line with an arrow-head, in which the length of the line denotes the magnitude of the quantity and the arrow-head shows its direction, as shown in the fig. 1.

The vector is denoted by $\vec{A}$ and its magnitude is denoted by

$$A \quad \text{or} \quad |\vec{A}|$$

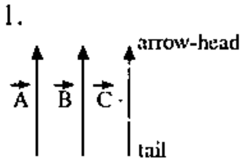


Fig. 1.

Equal Vectors : Two vectors are said to be equal if they have the same magnitude and direction.

In the above fig. two vectors $\vec{A}$ and $\vec{B}$ are equal *i.e.*

$$\vec{A} = \vec{B}$$

Null Vector or Zero Vector : A vector is said to be null vector if it has zero magnitude. It can be written as $\vec{0}$.

Addition of Vectors : Let us consider two vectors $\vec{A}$ and $\vec{B}$ as shown in fig. 2(a). Their sum is a vector $\vec{R}$, which is obtained by joining the tail of vector $\vec{A}$ to the head of vector $\vec{B}$ fig. 2(b).

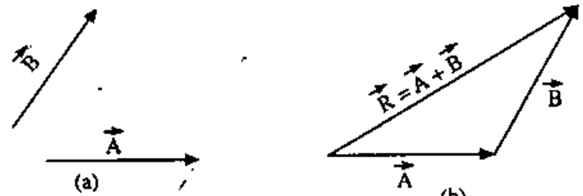


Fig. 2.

(a) **Properties of Vector Addition :**

(i) **Vector addition is commutative** : This means that addition of vectors is independent of order of vector added *i.e.*,

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

From the fig. 3

$$\vec{R}=\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

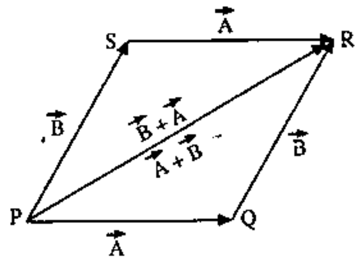


Fig. 3.

(ii) Vector Addition is Associative : This means that in addition of vectors more than two *i.e.* $\vec{A}$, $\vec{B}$ and $\vec{C}$, the resultant of adding $\vec{A}$ to the sum of $\vec{B}$ and $\vec{C}$ is equal to adding $\vec{C}$ to the sum of $\vec{A}$ and $\vec{B}$ *i.e.*

$$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$$

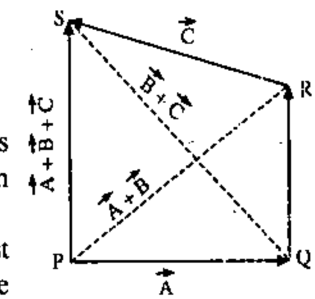


Fig. 4.

Substraction of Vectors : Let us consider two vectors $\vec{A}$ and $\vec{B}$ as shown in the fig. 5(a). We have to subtract $\vec{B}$ from $\vec{A}$.

Therefore, to subtract a vector $\vec{B}$ from a vector $\vec{A}$, we first convert $\vec{B}$ into $-\vec{B}$ by changing the direction of $\vec{B}$. Now join the head of $\vec{A}$ to tail of $\vec{B}$ and then tail of $\vec{A}$ to head of $\vec{B}$. It is the resultant and is denoted by $\vec{R}$ fig. 5(b).

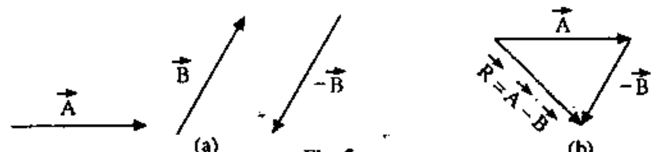


Fig. 5.

• 1.2. SCALAR PRODUCT OR DOT PRODUCT OF TWO VECTORS

The product of the magnitudes of two vectors and the smaller cosine of the angle between them is known as scalar or dot product of two vectors :

Let $\vec{A}$ and $\vec{B}$ be two vectors, then their scalar product is denoted by $\vec{A}\cdot\vec{B}$ and is defined as

$$\vec{A}\cdot\vec{B}=AB\cos\theta$$

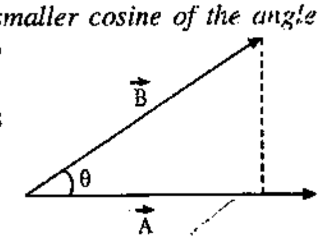


Fig. 6.

where θ is the angle between $\vec{A}$ and $\vec{B}$ and A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, respectively. The scalar product of two vectors is a scalar quantity and may be positive as well as negative.

Properties : **(i) The scalar product is commutative *i.e.***

$$\boxed{\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}}$$

(ii) The scalar product is distributive *i.e.*

$$\boxed{\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}}$$

(iii) The scalar product of two vectors vanishes when the vectors are at right angles *i.e.*

$$\boxed{\vec{A} \cdot \vec{B} = AB \cos 90^\circ = 0}$$

(iv) The scalar product of two vectors is equal to the product of their scalar magnitudes when the vectors are parallel *i.e.*

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

$$\because \quad \theta = 0^\circ$$

$$\therefore \quad \vec{A} \cdot \vec{B} = AB \cos 0^\circ$$

$$\boxed{\vec{A} \cdot \vec{B} = AB} \qquad [\because \ \cos 0^\circ = 1]$$

(v) The scalar product of a vector by itself is equal to the square of the magnitude of that vector *i.e.*

$$\boxed{\vec{A} \cdot \vec{A} = A^2}$$

(vi) The scalar products of unit vectors $\hat{i}, \hat{j}, \hat{k}$ have the following relations.

(a) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

(b) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(vii) The scalar product of two vectors is equal to the sum of the products of their corresponding x, y, z-components.

Proof : Let $\vec{A}$ and $\vec{B}$ be two vectors. Now, in components

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Their scalar product is given by

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

using the relations

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

and

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

we get

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

• 1.3. VECTOR PRODUCT OR CROSS PRODUCT OF TWO VECTORS

The product of the magnitudes of two vectors and sine of the angle between them in a direction perpendicular to the plane containing the two vectors, is known as vector product or cross product of two vectors.

The direction of this resultant vector will be perpendicular to the plane containing the two vectors. The cross product of two vectors is a vector quantity, so it is known as vector product.

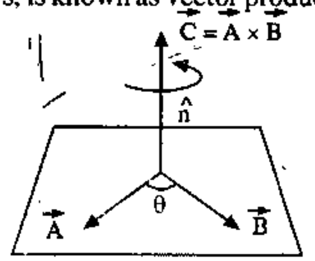


Fig. 7.

If $\vec{A}$ and $\vec{B}$ are two vectors then vector product is denoted by $\vec{A} \times \vec{B}$ and is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n} = \vec{C} \text{ (say)}$$

where $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. $\vec{C}$ is also perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Properties :

(i) The vector product is 'not' commutative *i.e.*

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

(ii) The vector product is distributive *i.e.*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) The magnitude of the vector product of two vectors at right angles is equal to the product of the magnitudes of the vectors. *i.e.*

$$\vec{A} \times \vec{B} = AB \sin 90^\circ\, \hat{n}.$$

$$\boxed{\vec{A} \times \vec{B} = AB\, \hat{n}}$$

(iv) The vector product of two parallel vectors is a null vector (or zero). *i.e.*

For parallel $\theta = 0^\circ$

$$\therefore \quad \vec{A} \times \vec{B} = AB \sin 0^\circ\, \hat{n}$$

$$\boxed{\vec{A} \times \vec{B} = 0}$$

(v) The vector product of a vector by itself is a null vector (zero) *i.e.*

$$\boxed{\vec{A} \times \vec{A} = 0}$$

(vi) The vector products of unit orthogonal vectors $\hat{i}, \hat{j}, \hat{k}$ have the following relations in the right-handed coordinate system.

(a) $\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$

$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$

and $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

(b) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

The cyclic order is to be strictly maintained.

(vii) The vector product of two vectors in terms of their x, y and z-components can be expressed as a determinant.

Proof : Let $\vec{A}$ and $\vec{B}$ be two vectors such that

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Their vector product is

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k}$$

$$+ A_y B_x \hat{j} \times \hat{i} + A_y B_y \hat{j} \times \hat{j} + A_y B_z \hat{j} \times \hat{k}$$

$$+ A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j} + A_z B_z \hat{k} \times \hat{k}$$

We know that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

and

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k},$$

$$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

then we get

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

• STUDENT ACTIVITY

1. When is the sum of two vectors maximum and when is it minimum ?

2. If a vector $\vec{A}$ is multiplied by a scalar m what will be the resultant vector ?

3. Give at least two examples, each of scalar and vector.

• 1.4. SCALAR TRIPLE PRODUCT $\vec{A}\cdot(\vec{B}\times\vec{C})$

It is a scalar product of one vector with vector product of two vectors. Obviously its resultant will be a scalar. It is denoted by $\vec{A}\cdot(\vec{B}\times\vec{C})$.

Now we represent the scalar triple product in the form of vector components as follows:

Let us consider $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

and $\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

$$\therefore \quad \vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \hat{i}(B_y C_z - B_z C_y) + \hat{j}(B_z C_x - B_x C_z) + \hat{k}(B_x C_y - B_y C_x)$$

and $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot[\hat{i}(B_y C_z - B_z C_y) + \hat{j}(B_z C_x - B_x C_z) + \hat{k}(B_x C_y - B_y C_x)]$

But $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

$$\therefore \quad \vec{A}\cdot(\vec{B}\times\vec{C}) = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(a) Geometrical Representation of Scalar Triple Product : The scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ geometrically represents the volume of the parallelopiped having its three sides as three given vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. This is proved as follows :

We know that

$$\vec{B}\times\vec{C} = \hat{n}\, BC \sin\phi$$

where $\hat{n}$ is a unit vector in the direction perpendicular to the plane of $\vec{B}$ and $\vec{C}$ and ϕ is the angle between $\vec{B}$ and $\vec{C}$.

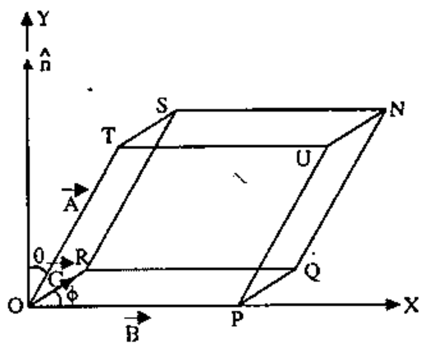


Fig. 8.

If the angle between $\hat{n}$ (*i.e.* direction of $\vec{B}\times\vec{C}$) and $\vec{A}$ be θ, then by definition of scalar product we can write.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A\,(\text{magnitude of } \vec{B}\times\vec{C})\cos\theta$$

$$= A\,(BC\sin\phi)\cos\theta \qquad [\because \text{ magnitude of unit vector is } 1]$$

Now by geometry of fig. 8

$BC\sin\phi =$ area of parallelogram (made by $\vec{B}$ and $\vec{C}$)

and $A\cos\theta =$ Projection of $\vec{A}$ along $\hat{n}$ *i.e.* the height of the parallelopiped.

Hence $\vec{A}\cdot(\vec{B}\times\vec{C}) =$ (base area × height of parallelopiped)

$=$ Volume of parallelopiped

(b) Properties of Scalar Triple Product : (i) In scalar triple product dot and cross can be interchanged.

i.e. $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

(ii) The value of scalar triple product does not change by changing the vectors in cyclic order while in non-cyclic order its value changes. *i.e.*

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

and $$\vec{A}\cdot(\vec{B}\times\vec{C})=-\vec{B}\cdot(\vec{A}\times\vec{C})=-\vec{A}\cdot(\vec{C}\times\vec{B})$$

(iii) Scalar triple product will be zero, if two vectors are equal *i.e.*,

$$\vec{A}\cdot(\vec{A}\times\vec{C})=0$$

or $$[\vec{A}\,\vec{A}\,\vec{C}]=0$$

(iv) If $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar then scalar triple product will be zero.

(v) If three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are such that $\vec{A}+\vec{B}+\vec{C}=0$ then

$$\vec{A}\cdot(\vec{B}\times\vec{C})=0$$

• 1.5. VECTOR TRIPLE PRODUCT $\vec{A}\times(\vec{B}\times\vec{C})$

It is the vector product of one vector with vector product of two other vectors. Obviously, its resultant will be a vector quantity. It is denoted by $\vec{A}\times(\vec{B}\times\vec{C})$ and is given by

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$$

Now we have to prove this retation :

L.H.S. $=\vec{A}\times(\vec{B}\times\vec{C})$

$$=(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z)\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$=(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z)\times[\hat{i}(B_y C_z-B_z C_y)+\hat{j}(B_z C_x-B_x C_z)+\hat{k}(B_x C_y-B_y C_x)]$$

Using relations

$$\hat{i}\times\hat{i}=0 \qquad \hat{i}\times\hat{j}=\hat{k} \qquad \hat{j}\times\hat{i}=-\hat{k}$$
$$\hat{j}\times\hat{j}=0 \qquad \hat{j}\times\hat{k}=\hat{i} \qquad \hat{k}\times\hat{j}=-\hat{i}$$
$$\hat{k}\times\hat{k}=0 \qquad \hat{k}\times\hat{i}=\hat{j} \qquad \hat{i}\times\hat{k}=-\hat{j}$$

We get

$$\vec{A}\times(\vec{B}\times\vec{C})=\hat{k}(B_z C_x-B_x C_z)A_x-\hat{j}(B_x C_y-B_y C_x)A_x$$
$$-\hat{k}(B_y C_z-B_z C_y)A_y+\hat{i}(B_x C_y-B_y C_x)A_y$$
$$+\hat{j}(B_y C_z-B_z C_y)A_z-\hat{i}(B_z C_x-B_x C_z)A_z$$
$$=\hat{i}[B_x A_y C_y+B_x A_z C_z-C_x A_y B_y-C_x A_z B_z]$$
$$+\hat{j}[B_y A_z C_z+B_y A_x C_x-C_y A_z B_z-C_y A_x B_x]$$
$$+\hat{k}[B_z A_y C_y+B_z A_x C_x-C_z A_x B_x-C_z A_y B_y] \quad \dots(1)$$

Now R.H.S. $=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

$$=(\hat{i}B_x+\hat{j}B_y+\hat{k}B_z)[A_x C_x+A_y C_y+A_z C_z]$$
$$-(\hat{i}C_x+\hat{j}C_y+\hat{k}C_z)[A_x B_x+A_y B_y+A_z C_z]$$
$$=[\hat{i}B_x(A_x C_x+A_y C_y+A_z C_z)-\hat{i}C_x(A_x B_x+A_y B_y+A_z B_z)]$$
$$+[\hat{j}B_y(A_x C_x+A_y C_y+A_z C_z)-\hat{j}C_y(A_x B_x+A_y B_y+A_z B_z)]$$
$$+[\hat{k}B_z(A_x C_x+A_y C_y+A_z C_z)-\hat{k}C_z(A_x B_x+A_y B_y+A_z B_z)]$$

$$= \hat{i}[B_x A_y C_y + B_x A_z C_z - C_x A_y B_y - C_x A_z B_z]$$
$$+ \hat{j}[B_y A_x C_x + B_y A_z C_z - C_y A_x B_x - C_y A_z B_z]$$
$$+ \hat{k}[B_z A_x C_x + B_z A_y B_y - C_z A_x B_x - C_z A_y B_y] \quad \ldots(2)$$

From equations (1) and (2) we find that

L.H.S. = R.H.S.

Hence $$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})}$$

In the determinent form it may be writen as

$$\begin{vmatrix} \vec{B} & \vec{C} \\ \vec{A} \cdot \vec{B} & \vec{A} \cdot \vec{C} \end{vmatrix}$$

This relation is known as **Lagrange's identity.**

(a) Properties of Vector Triple Product : (i) Vector triple product does not obey associative property *i.e.*

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

(ii) Vector triple product does not obey commutative law *i.e.*,

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq \vec{A} \times (\vec{C} \times \vec{B}).$$

• STUDENT ACTIVITY

4. Show that in a scalar triple product the dot and cross can be interchanged.

5. Prove that when two vectors are equal or parallel then the scalar triple product is zero.

• 1.6. FIELD

The region in space in which a function u is defined at all points, is known as field. Thus in cartesian coordinates $u = f(x, y, z)$ specifies a field.

(a) Scalar field : The region in space in which a scalar quantity is continuous and is defined by a single value at every point of the position variable is called a scalar field.

If ϕ is a scalar function of position variable $\vec{r}$ with a set of co-ordinates (x, y, z), then we denote the scalar field as $\phi(\vec{r}) = \phi(x, y, z)$.

Examples : (i) Variation of tempreture at various points along a metal rod one end of which is heated while the other end is kept cold is an example of scalar temperature field.

(ii) Variation of electric potential at various points surrounding a charged body is an example of **scalar potential field.**

(b) Vector field : The region in space in which a vector quantity is continuous and is defined by a single value at every point of the position variable is called a vector field.

If $\vec{V}$ is a vector function of position variable $\vec{r}$ with a set of coordinates (x, y, z), then we denote the vector field as

$$\vec{V}(\vec{r}) = \vec{V}(x, y, z)$$

Examples : (i) The electric field is an example of vector field. (ii) The gravitational field acting on a body is another example of vector field.

• 1.7. GRADIENT OF A SCALAR FIELD

Let us consider a point P in the scalar field having position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and let the value of the scalar field at this point be $\phi(\vec{r}) = \phi(x, y, z)$, where $\phi(x, y, z)$ is continuous differentiable function of x, y and z. Now

$$\frac{\partial \phi}{\partial x} = \text{rate of change of } \phi \text{ at point } P \text{ along the } X\text{-direction}$$

$$\frac{\partial \phi}{\partial y} = \text{rate of change of } \phi \text{ at point along the } Y\text{-direction}$$

and $\frac{\partial \phi}{\partial z}$ = rate of change of ϕ at point along the Z-direction

This means that ϕ has different rates of variation along different directions. The quantity $\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$ is known as gradient of ϕ or grad ϕ *i.e.* grad $\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \vec{\nabla}\phi$

Here $\vec{\nabla}$ (del) is called vector differential operator.

From grad $\phi = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$, the grad ϕ is a vector due to $\hat{i}, \hat{j}$ and $\hat{k}$.

(a) Physical significance : Now, let us have a scalar field and let ϕ and $\phi + d\phi$ be its two level surfaces, very close to each other as shown in the fig. 9.

Let the scalar quantity at these surfaces be ϕ and $\phi + d\phi$, respectively. Let $\vec{r}$ and $\vec{r} + d\vec{r}$ be the position vectors of points P and R with respect to origin O.

The distance between P and R is

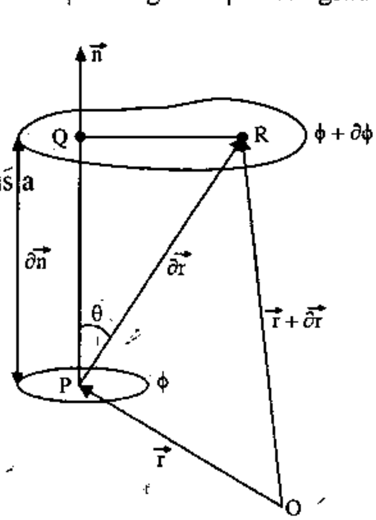


Fig. 9.

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$$
$$= \vec{r} + \overrightarrow{dr} - \vec{r} = \overrightarrow{dr}$$

Further let PQ represent the normal to the surface at point P, hence $\overrightarrow{PQ} = \partial\hat{n}$ represents the minimum possible distance between ϕ and $\phi + d\phi$. Also Let $\hat{n}$ be a unit vector along this direction PQ and θ be the angle between PR and PQ.

Now it is evident that the rate of change of ϕ at point P will be $\frac{\partial\phi}{\partial n}$ along PR and $\frac{\partial\phi}{\partial r}$ along PQ.

In ΔPRQ

$$PQ = PR\cos\theta \quad \text{or} \quad \partial n = \partial r\cos\theta \qquad \dots (1)$$

or

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial r\cos\theta} \quad \text{or} \quad \frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial n}\cos\theta \qquad \dots (2)$$

$\therefore$ $\frac{\partial\phi}{\partial r}$ will be maximum when

$$\cos\theta = 1 \quad \text{or} \quad \theta = 0$$

Thus, $\frac{\partial\phi}{\partial r}$ is maximum rate of change of ϕ with distance and it will be a vector having magnitude $\frac{\partial\phi}{\partial n}$ and direction along $\theta = 0$ *i.e.* along normal to the surface at that point. This vector is called the gradient (or grad) of a scalar at that point and is expressed as

$$\boxed{\text{grad }\phi = \frac{\partial\phi}{\partial n}\hat{n}}$$

Thus, **the gradient of a scalar field ϕ at any point is always a vector quantity,** whose magnitude is maximum rate of change of ϕ at that point and direction along the normal at that point.

It is also called geometrical interpretation of gradient of scalar field.

Example : The potential field produced by any charge and consider an equipotential level surface (of potential V). Then evidently, grad V at any point of this surface will represent a vector, which is equal to electric field intensity $\vec{E}$

i.e. Electric field $\vec{E} = -$ grad V

$$\boxed{\vec{E} = -\vec{\nabla}V}$$

(b) Gradient ϕ in Rectangular Coordinates : Let ϕ be a function of x, y, z then by definition of partial derivative

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \dots (3)$$

we know from the above

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\hat{n}$$

where $\hat{n}$ = unit vector

On multiplying by $\overrightarrow{dr}$ on both sides

$$(\text{grad }\phi)\, d\vec{r} = \frac{\partial\phi}{\partial x}\hat{n}\cdot\overrightarrow{dr}$$

$$= \frac{\partial\phi}{\partial x}dr\cos\theta \qquad [\because\ \hat{n}\cdot\overrightarrow{dr} = dr\cos\theta]$$

$$= \frac{\partial \phi}{\partial x} dx$$

$$= d\phi$$

From eq. (3)

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

Hence $$(\text{grad } \phi) \cdot d\vec{r} = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \quad \ldots (4)$$

Now

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Hence by eq. (4) we get

$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \quad \ldots (5)$$

which represents the grad ϕ in terms of rectangular coordinates.

Again, by definition operator

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

so by eq. (5), we get

$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = \vec{\nabla}\phi$$

i.e. $$\boxed{\text{grad } \phi = \vec{\nabla}\phi}$$

1.8. THE DIVERGENCE OF A VECTOR FIELD

The divergence of a vector at any point in a vector field is always a scalar quantity equal to the amount of flux per unit volume diverging from that point.

If $\vec{A}(x, y, z)$ be any vector field then the divergence of that vector $\vec{A}$ is denoted by div $\vec{A}$ or $\vec{\nabla} \cdot \vec{A}$ and given by

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (\hat{i}Ax + \hat{j}A_y + \hat{k}A_z)$$

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

div $\vec{A}$ is a scalar quantity and this is very useful in the principles of Hydrodynamics and Electromagnetism.

Now we have to calculate the value of div $\vec{A}$. For this, consider a very small parallelopiped as shown in the fig. 10. in a vector field. Let the sides of parallelopiped be along x, y and z-axes and their lengths be dx, dy, dz. Let $\vec{A}$ be the value of vector at centre O of parallelopiped and its components along x, y and z-axes are A_x, A_y and A_z respectively.

The rate of change of A_x along x-axis may be written as its derivative with respect to $\frac{\partial A_x}{\partial x}$ and consiquently, the increase in magnitude of A_x will be $\frac{\partial A_x}{\partial x}\frac{1}{2}dx$. Therefore the total magnitude of x component of $\vec{A}$ at point Q is $=$ x component at O + increase in it from O to Q

$$= A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx$$

Similarly, we can say that at point P, it will be $= A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx$

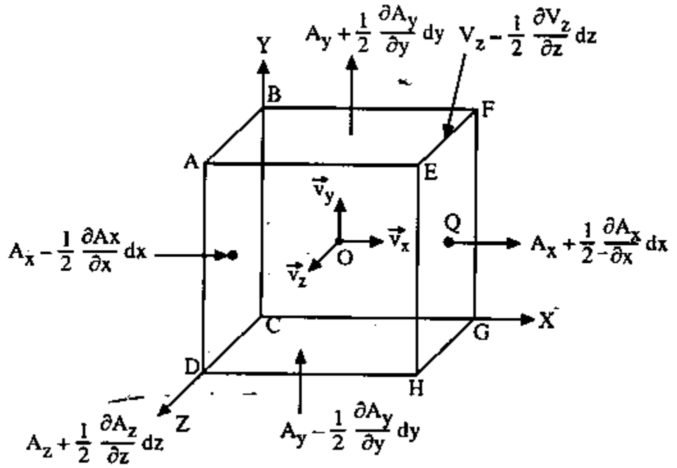


Fig. 10.

Here negative sign is taken due to point P which lies on negative direction of x.

Since the flux through any small area is defined as product of area and normal component of vector on this area, thus

(i) Flux through the face $EFGH$ (area $= dy\,dz$) will be

$$= \left[A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right]dy\,dz$$

(ii) Flux through the face $ABCD$ whose area $dy\,dz$ will be

$$= -\left[A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right]dy\,dz$$

Here positive and negative signs are taken according to the direction.

So, the net excess of flux leaving or diverging from the parallelopiped along x direction will be

$$= \left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz - \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz$$

$$= \frac{\partial A_x}{\partial x}dx\,dy\,dz$$

Similarly, we can say that the net excess of flux leaving the parallelopiped along y and z-directions will be

$$\frac{\partial A_x}{\partial y}dx\,dy\,dz \text{ and } \frac{\partial A_x}{\partial z}dx\,dy\,dz$$

Hence, the total flux diverging from the parallelopiped (of volume $dx\,dy\,dz$)

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)dx\,dy\,dz$$

or the flux diverging per unit volume will be

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \ldots (1)$$

This represents the div of the vector at that point *i.e.*

$$\boxed{\text{div } \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Example : In the case of flow of any fluid in any region, if $\vec{A}$ represents the velocity of fluid then div $\vec{A}$ gives the rate at which the fluid diverges per unit volume from that point.

If div $\vec{A}$ is positive then this means that the density of fluid is decreasing and if div $\vec{A}$ is negative then this shows that the density of fluid is increasing at that point.

Div $\vec{A}$ in terms of operator $\vec{\nabla}$: We know in terms of components $\vec{\nabla}$ and $\vec{A}$ may be written as

$$\vec{\nabla}=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

and
$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$

$\therefore$
$$\vec{\nabla}.\vec{A}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})$$
$$=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

Hence

$$\boxed{\text{div}\,\vec{A}=\vec{\nabla}.\vec{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}}$$

This is the desired result.

1.9. CURL OF A VECTOR FIELD

The curl of a vector function at any point in the non-lamellar vector field or non-curl field is the maximum line integral of the vector, computed per unit area at that point.

Consider a vector field $\vec{A}$, the magnitude and direction of which, is a function of position coordinates at a point, then the curl of the vector field $\vec{A}$ is denoted by curl $\vec{A}$ and given by

$$\text{curl}\,\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

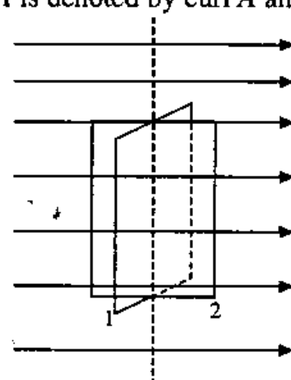


Fig. 11.

In the adjacent (fig. 11) a non-lamellar field is represented by several lines of flow. Now we consider a plane rectangular area in the field. When the area is perpendicular to the field (as shown by position 1) then the line integral round it is zero because no line of flow lies along its boundary. But when the area is parallel to the field (as shown by positions 2) then the line integral round the boundary has a finite value because the value of the vector at the upper edge would be different for different orientations. The maximum line integral computed per unit area along the boundary is the curl of the vector field.

(a) Physical Significance : (i) When a rigid body is in motion then the curl of its linear velocity at any point gives double its angular velocity in magnitude and direction.

(ii) When a current is passed through a conductor, then the curl of magnetic field at a nearby point represents the current flowing per unit area at that point. So curl of magnetic field is called **magnetomotive force.**

(b) Expression for Curl of a Vector Field : Let us consider an infinitesimal rectangular area *PQRS* at point *O*, where $\vec{A}$ has the components of magnitudes $A_x, A_y,$ and A_z as shown in the fig. 12. Let the sides dx, dy be parallel to the x- and y-axis, so that the normal to the area is along the z-aris. The arrows on the sides show the directions in which the components of $\vec{A}$ act. Since the rectangle is very small, therefore the components of $\vec{A}$ at the middle of any side may be taken as the average value along that side. *i.e.*

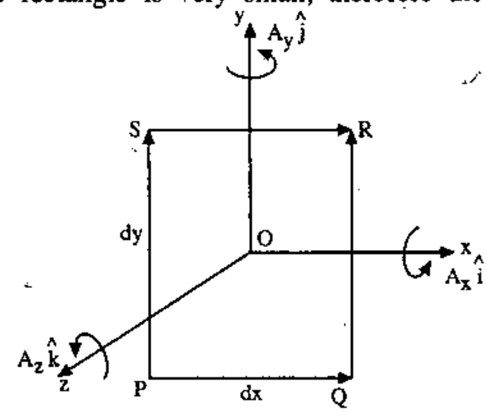


Fig. 12.

along PQ, $A_x - \frac{1}{2}\frac{\partial A_x}{\partial y}dy$

along QR, $A_y + \frac{1}{2}\frac{\partial A_y}{\partial x}dx$

along SR, $A_y + \frac{1}{2}\frac{\partial A_x}{\partial y}dy$

along PS, $A_y - \frac{1}{2}\frac{\partial A_z}{\partial x}dx$

Therefore, the line integral along the boundary $PQRS$

$$= \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial y}dy\right)dx + \left(A_y + \frac{1}{2}\frac{\partial A_y}{\partial x}dx\right)dy$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dx\,dy$$

Here $dx\,dy$ is the area of the element, so the line integral per unit area is

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

By definition, this is the magnitude of the component curl of $\vec{A}$ along z-axis *i.e.*

$$\text{curl}_z \vec{A} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

Similarly, along y-axis $\text{curl}_y \vec{A} = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j}$

and along x-axis $\text{curl}_x \vec{A} = \left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right)\hat{i}$

On adding these three components, we get

$$\text{curl}\,\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

(c) Curl $\vec{A}$ in terms of operator $\vec{\nabla}$: Now we have to prove that

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A}$$

we have

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

using

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$

$$\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k},$$

$$\hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i}$$

and $$\hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}$$

$$\because \quad \vec{\nabla}\times\vec{A}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})$$

$$\therefore \quad \vec{\nabla}\times\vec{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$\boxed{\vec{\nabla}\times\vec{A}=\text{curl}\,\vec{A}}$$

Thus, **the curl of a vector is always a vector quantity.**

A vector $\vec{A}$ is said to be irrotational if its curl is equal to zero *i.e.* $\vec{\nabla}\times\vec{A}=0$.

• SUMMARY

- Physical quantities are of two types : (i) Scalar quantities (ii) Vector quantities
- Scalar quantities are those which contain magnitude only and do not contain direction.
- Vector quantities are those quantities which contain both magnitude and direction.
- A vector is said to be null vector if it has zero magnitude. It can be written as $\vec{0}$.
- The product of the magnitudes of two vectors and the smaller cosine of the angle between them is known as scalar or dot product of two vectors.
- The product of the magnitudes of two vectors and sine of the angle between them in a direction perpendicular to the plane containing the two vectors, is known as vector product or cross product of two vectors.
- The scalar triple product $\vec{A}.(\vec{B}\times\vec{C})$ geometrically represents the volume of the parallelopiped having its three sides as three given vectors $\vec{A}, \vec{B}$ and $\vec{C}$.
- The region in space in which a function u is defined at all points, is known as field.
- The region in space in which a scalar quantity is continuous and is defined by a single value at every point of the position variable is called a scalar field.
- The region in space in which a vector quantity is continuous and is defined by a single value at every point of the position variable is called a vector field.
- The gradient of a scalar field ϕ at any point is always a vector quantity.
- The divergence of a vector at any point in a vector field is always a scalar quantity equal to the amount of flux per unit volume diverging from that point.
- The curl of a vector function at any point in the non-lamellar vector field or non-curl field is the maximum line integral of the vector, computed per unit area at that point.
- The relation $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$ is known as Lagrange's identity.

• TEST YOURSELF

1. Explain scalar and vector quantities. Explain the laws involved in addition and subtraction of vectors.
2. What do you understand by the scalar product of two vectors ?
3. What do you understand by the vector product of two vectors ?
4. What do you mean by scalar triple product of vectors ? Derive its expression and its properties.
5. Derive an expression for vector triple product.

6. What is field ? What are scalar and vector fields ? Give one example of each.
7. Define gradient of a scalar field. The gradient of a scalar field is a vector, explain. Give its physical significance.
8. Define the divergence of a vector field and find an expression for it.
9. Define the Curl of a vector field. Give the physical significance of the curl of a vctor field. Derive an expression for it.
10. If $\vec{A} \times \vec{B} = 0$, then
(a) A and B, both are zero
(b) one out of A and B must be zero
(c) $\vec{A}$ and $\vec{B}$ should be parallel
(d) $\vec{A}$ and $\vec{B}$ should be perpendicular to each other.
11. If $\vec{A} \times \vec{B} = \vec{C}$ then $\vec{C}$:
(a) perpendicular only to $\vec{A}$ or only to $\vec{B}$
(b) parallel to $\vec{A}$ only or $\vec{B}$ only
(c) parallel to both $\vec{A}$ and $\vec{B}$
(d) perpendicular to both $\vec{A}$ and $\vec{B}$
12. The dot product of two vectors vanishes when :
(a) one vector is null vector
(b) both vectors are null vectors
(c) the angle between both vectors is 90°
(d) All above
13. Two vectors $\vec{A}$ and $\vec{B}$ are perpendicular if
(a) $\vec{A} \times \vec{B} = 0$ (b) $\vec{A} \cdot \vec{B} = 0$ (c) $\vec{A} \times \vec{B} = 1$ (d) $\vec{A} \cdot \vec{B} = 1$
14. Which equation is correct?
(a) $\vec{A} \times \vec{B} = AB \sin\theta$ (b) $\vec{A} \times \vec{B} = AB \cos\theta$
(c) $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$ (d) $\vec{A} \times \vec{B} = AB$
15. The scalar triple product is represented as
(a) $\vec{A} . (\vec{B} . \vec{C})$ (b) $\vec{A} . (\vec{B} \times \vec{C})$
(c) $\vec{A} \times (\vec{B} \times \vec{C})$ (d) All above
16. The value of $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B})$ is :
(a) 0 (b) $\vec{A} \times \vec{B}$ (c) $\vec{A} . \vec{B}$ (d) None of these
17. The divergence of a vector field *i.e.*, div $\vec{A}$ is equal to :
(a) $\vec{\nabla} . \vec{A}$ (b) ∇A (c) $\vec{\nabla} \times \vec{A}$ (d) $\nabla . A$.
18. If $\vec{r}$ is the position vector then div r is equal to :
(a) zero (b) 3 (c) $\hat{r}$ (d) none of these.
19. If a vector $\vec{A}$ is solenoidal then :
(a) div $\vec{A} \neq 0$ (b) div $\vec{A} = 0$ (c) curl $\vec{A} = 0$ (d) curl $\vec{A} \neq 0$.
20. The divergence of a vector field is :
(a) always a vector field (b) alway a scalar field
(c) can both scalar and vector field (d) always zero.
21. Which of the following is not true :
(a) curl grad $S = 0$ (b) div curl $\vec{A} = 0$
(c) curl curl $\vec{A} = 0$ (d) $\nabla^2 \left(\frac{1}{r}\right) = 0$.
22. A vector is irrotational, then which is not true ?
(a) curl $\vec{A} = 0$ (b) div $\vec{A} = 0$
(c) div curl $\vec{A} = 0$ (d) curl grad $S = 0$.

23. Which one of the following is NOT correct?
(a) The gradient of a scalar field is always a vector field.
(b) A vector field can always be expressed as gradient of a scalar field.
(c) The divergence of a vector is always a scalar.
(d) The line integral round a close path in a non-curl vector field is zero.

24. The divergence of the curl of a vector quantity is :
(a) always positive (b) always negative
(c) zero (d) none of these

25. Which of the following is not possible?
(a) gradient of a scalar (b) divergence of a vector
(c) divergence of a scalar (d) curl of a vector

26. Gradient of a scalar is :
(a) vector (b) scalar (c) may be scalar or vector (d) zero.

ANSWERS

10. (c) **11.** (d) **12.** (d) **13.** (b) **14.** (c) **15.** (b) **16.** (a) **17.** (a) **18.** (b) **19.** (b)
20. (b) **21.** (c) **22.** (d) **23.** (b) **24.** (b) **25.** (c). **26.** (c)

❑❑❑

UNIT 2

ROTATIONAL DYNAMICS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- Rotational Motion in detail along with the torque and Angular Momentum acting on the rotating body.
- What is Moment of Inertia and its application on masses of different shape.
- Gravitational field and gravitational potential on different bodies.
- About the motion of Planets and Satellites their energy consideration and their weightlessness.

• 2.1. ROTATIONAL MOTION : TORQUE AND ANGULAR MOMENTUM

(i) Angular Displacement (θ) **:** Angular displacement is the angle described by the position vector $\vec{r}$ about the axis of rotation. It is denoted by θ and is measured in radian or degree.

If θ is positive then rotation will be anticlockwise and if θ is negative then rotation will be clockwise.

(ii) Angular velocity (ω) : The rate of change of angular displacement is known as angular velocity. It is denoted by ω and is defined as

$$\boxed{\omega = \frac{d\theta}{dt}}$$

It is measured in radian/sec. In rigid body the radial lines from all the particles of the body are perpendicular to the axis of rotation. These particles sweep out equal angles in equal time intervals so the angular velocity ω is same for every particle of the rigid body.

Angular velocity depends upon the point about which the rotation is considered.

(iii) Angular acceleration (α) : The rate of change of angular velocity of a body about the axis of rotation is known as angular acceleration. It is denoted by α and is defined as

$$\alpha = \frac{d\omega}{dt}$$

$$= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \qquad \left[\because \ \omega = \frac{d\theta}{dt}\right]$$

$$\boxed{\alpha = \frac{d^2\theta}{dt^2}}$$

It is measured in radian/sec^2. Angular acceleration is same for all particles of the rigid body.

(iv) Equations of rotational motion : There are three relations between rotational kinematic variables. These are :

(1) $\omega = \omega_0 = \alpha t$ (2) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (3) $\omega^2 = \omega_0^2 + 2\alpha\theta$

where ω_0 = initial angular velocity

ω = final angular velocity

α = angular acceleration

θ = angular displacement

t = time

These equations are known as **equations of rotational motion.**

When the motion is linear then above equations reduce to :

(1) $v = u + at$ (2) $s = ut + \frac{1}{2}at^2$ (3) $v^2 = u^2 + 2as$

Now we have to prove the equations of rotational motion.

Proof of : $\boxed{\omega = \omega_0 + \alpha t}$

Let a rigid body be rotating about an axis with a uniform angular acceleration α, then we know

$$\alpha = \frac{d\omega}{dt}$$

$$\therefore \qquad d\omega = \alpha\, dt \qquad \text{...(1)}$$

Let at $t = 0$ $\omega = \omega_0$

and at $t = t$ $\omega = \omega$

On integrating the eqn. (1) between above limit-, *i.e.,*

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha\, [t]_0^t$$

$$\therefore \qquad \omega - \omega_0 = \alpha\,(t - 0)$$

$$\boxed{\omega = \omega_0 + \alpha t} \qquad \text{... (2)}$$

This proves the relation first.

Proof of : $\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$

Let ω be the angular velocity of the rigid body at any time 't' then we know that

$$\omega = \frac{d\theta}{dt}$$

$$\therefore \quad d\theta = \omega\, dt \qquad \text{... (3)}$$

Let at $t = 0$ $\quad \theta = 0$

and at $t = t$ $\quad \theta = \theta$

So on integrating equation (3) we get

$$\int_0^\theta d\theta = \int_0^t \omega\, dt$$

$$[\theta]_0^\theta = \int_0^t (\omega_0 + \alpha t)\, dt \qquad \text{[by (2)]}$$

$$\theta - 0 = \left[\omega_0 t + \frac{\alpha t^2}{2}\right]_0^t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$$

This proves the relation second.

Proof of : $\omega^2 = \omega_0^2 + 2\alpha\theta$ ... (4)

We know that $\omega = \dfrac{d\theta}{dt}$

and $$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right)$$

$$\therefore \quad \alpha = \left(\frac{d\omega}{d\theta}\right)\omega$$

$$\omega\, d\omega = \alpha\, d\theta \qquad \text{... (5)}$$

when $\theta = 0$, $\omega = \omega_0$ initial angular velocity

and when $\theta = \theta_0$, $\omega = \omega$ final angular velocity.

∴ On integrating (5) we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^\theta \alpha\, d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha\, [\theta]_0^\theta$$

$$\frac{\omega^2 - \omega_0{}^2}{2} = \alpha(\theta - 0)$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha\theta} \qquad \text{... (6)}$$

This proves the relation third.

2.2. TORQUE ACTING ON A PARTICLE

(a) Torque Acting on a Partile : The torque acting on a particle can be explained easily by the following example. When we switch on a fan then the centre of the fan remains unmoved while the fan rotates with an equal acceleration. As the centre of mass of the fan remains at rest then the vector sum of external force acting on the fan must be zero. **This means that an angular acceleration is produced even when resultant external force is zero.** And we also know that we can not produce angular acceleration without applying an external force. Hence here the question arises or what is the reason for producing angular acceleration ?"

The answer is **torque** due to the force. When an external force acts on a body then it has a tendency to rotate the body about a fixed axis. In this position the force acting on the body is known as torque on the body.

The torque acting on the body is equal to the product of the magnitude of force and perpendicular distance of the line of action of force from the axis of rotation. It is denoted by $\vec{\tau}$.

Thus torque = force × perpendicular distance

$$\vec{\tau} = \vec{r} \times \vec{F}$$

or

$$\boxed{\tau = Fr \sin\theta}$$

where F = magnitude of force

r = perpendicular distance

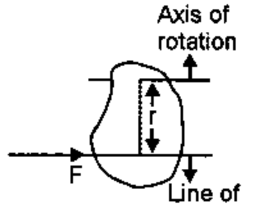


Fig. 1

Its unit is N-m. in S.I. system.

Angular momentum of a particle : Let us consider a particle of mass m whose position vector is $\vec{r}$ from the origin O as shown in the fig. 2. The linear momentum of the particle is given by

$$\vec{P} = m\vec{v}$$

where $\vec{v}$ = linear velocity of the particle.

The angular momentum of the particle about origin O is equal to the vector product of $\vec{r}$ and $\vec{p}$ *i.e.,*

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \qquad \dots (1)$$

In magnitude

$$L = rp \sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{p}$.

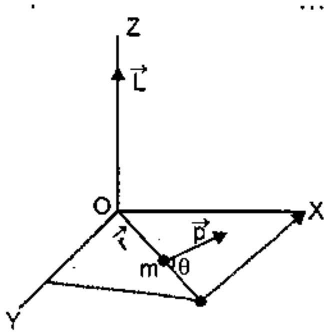


Fig. 2

From above it is clear that the angular momentum about O is zero when the line of action of $\vec{P}$ passes through O. In this position $\theta = 0$.

Relation between torque and angular momentum :

From eqn. (1)

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times p)$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= 0 + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \quad \ldots (2)$$

But according to Newton's 2^{nd} law

We have force $\vec{F} = \dfrac{d\vec{p}}{dt}$

$\therefore$ by (2) $\dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$ ...(3)

But $\vec{\tau} = \vec{r} \times \vec{F}$

so, by (3) $\dfrac{d\vec{L}}{dt} = \vec{\tau}$

Thus, **the rate of change of angular momentum of a particle is equal to the torque acting on the particle.**

(d) Angular momentum of a particle moving with constant velocity : From eqn. (2), we have

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{r} \times m\frac{d\vec{v}}{dt}$$

If $\vec{v}$ is constant, *i.e.*, $\dfrac{d\vec{v}}{dt} = 0$

Then $\dfrac{d\vec{L}}{dt} = 0$

$$\therefore \quad \boxed{\vec{L} = \text{constant}}$$

Hence the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

• STUDENT ACTIVITY

1. If a body is rotating, is it necessarily being acted upon by an external torque ?

2. Torque and work are both defined as force times distance. Explain, how do they differ?

2.3. MOMENT OF INERTIA

(a) Moment of inertia : "The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of the masses of the various particles and squares of their perpendicular distance from the axis of rotation."

Consider a body rotating about an axis OZ and let m_1, m_2, m_3 .... be the masses of the particles of the body and r_1, r_2, r_3 be the distances from the axis of rotation OZ. Then moment of inertia of the body about axis of rotation is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

$$\boxed{I = \Sigma mr^2}$$

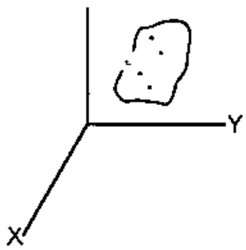


Fig. 3

The unit of moment of inertia in cgs system is **g cm²** and **kg m²** in S.I. system. Its dimensional formula is $[ML^2T^0]$, It is **tensor** quantity.

Moment of inertia depends upon the following factors :

1. Mass of the body.
2. Distribution of the mass of the body.
3. Distance from the axis of rotation.

(b) Physical Significance :

We know that

$$\text{K.E. of translation of body} = \frac{1}{2} mv^2$$

$$\text{K.E. of rotation of body} = \frac{1}{2} I\,\omega^2$$

On comparing it is clear that u is similar to ω therefore m is similar to I. Hence **moment of inertia (I) plays the same role in rotational motion as mass (m) plays in linear motion.**

This is the physical significance of moment of inertia.

(c) Radius of Gyration : The distance from the axis of rotation for every body may always be found, at which if whole mass of the body is concentrated then the moment of inertia of the body about that axis remains same. This distance from the axis of rotation is called radius of gyration about the axis.

Let M be the mass of the body which is concentrated at distance K from the axis of rotation, then moment of inertia is given by

$$I = MK^2$$

so $$\boxed{K = \sqrt{\frac{I}{M}}}$$

Hence it may be defined as "the perpendicular distance from the axis of rotation, the square of which when multiplied with total mass of the body, gives the moment of inertia of the body about that axis."

Unit in M.K.S. system is meter and dimensional formula is [L].

The radius of gyration is not a constant quantity.

2.4. KINETIC ENERGY OF A ROTATING BODY

Let us consider a body of mass M rotating with an angular velocity ω about an axis whose kinetic energy is to be determined. Let m_1, m_2, m_3 ... be the masses of the particles of the body and r_1, r_2, r_3 ... be the distances of these particles from the axis of rotation. When the body rotates then all the particles of the body rotate with the same angular velocity ω but move with different linear velocities. Let v_1, v_2, v_3 ... be the linear velocities of the different particles.

In this position

Kinetic energy of the particle of mass m_1

$$K_1 = \frac{1}{2} m_1 v_1^2$$

or $$K_1 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly, kinetic energy of the particle of mass m_2

$$K_2 = \frac{1}{2} m_2 r_2^2 \omega^2$$

therefore kinetic energy of the body

$$K = K_1 + K_2 + K_3 \ldots$$

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \ldots$$

$$K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots$$

$$K = \frac{1}{2} \omega^2 \Sigma mr^2$$

$$K = \frac{1}{2} I \omega^2 \qquad [\because I = \Sigma mr^2]$$

This is the required expression for the kinetic energy of the body in terms of moment of inertia.

From above $$I = \frac{2K}{\omega^2}$$

If $\omega = 1$ Radian

then $$\boxed{I = 2K}$$

Hence, "The moment of inertia of a body about a given axis is equal to double of the kinetic energy of the body rotating with unit angular velocity about the given axis."

(b) Angular momentum of a rotating body : It is defined as "The sum of the moments of linear momentum of all the particles of a rotating rigid body about the axis of rotation and is called angular momentum about that axis."

Consider a rigid body which is moving around an axis of rotation. Let thre be a particle of mass m at distance r from the axis of rotation. If ω is the angular velocity then the linear velocity of the particle of mass m is $r\omega$.

∴ Linear momentum = mass × velocity

$$P = m \times r\,\omega$$

The moment of this momentum is

$$P \times r = m \times r\,\omega \times r = m\,r^2\,\omega$$

i.e., angular momentum of a particle of mass $m = m\,r^2\,\omega$

Therefore angular momentum of the body

$$L = \Sigma m r^2 \omega$$

$$L = \omega \Sigma m r^2$$

$\therefore$ $$\boxed{L = I \omega}$$ $[\because I = \Sigma m r^2]$

Thus "angular momentum of the body is equal to the product of the moment of inertia and the angular velocity of the body about that axis."

Now we have $K = \frac{1}{2} I \omega^2$ and $L = I \omega$

$\therefore$ $$K = \frac{1}{2} \frac{I^2 \omega^2}{I} = \frac{L^2}{2I}$$

$$\boxed{K = \frac{L^2}{2I}}$$

This is relationship between angular momentum and kinetic energy of the body.

(c) Power and Work Done by a Torque : Let us consider a rigid body rotating about a fixed axis on which a torque acts.

This torque produces the angular acceleration and increases the kinetic energy of the body. We know that the rate of change of kinetic energy (work done) is equal to the power delivered by the torque *i.e.,*

Power = rate of change of kinetic energy or rate of change of work done

$$P = \frac{d\omega}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2} I \omega^2\right]$$

$$= I \omega \frac{d\omega}{dt}$$

$$= I \alpha \omega$$ [where α = angular acceleration]

$$\frac{d\omega}{dt} = P = \tau \omega$$

where τ is a torque acting on the body.

The work done due to small angular displacement $d\theta$ is

$$d\omega = \tau \omega \, dt$$

$$d\omega = \tau \, d\theta$$ $\left[\omega = \frac{d\theta}{dt}\right]$

Total work done

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau \, d\theta}$$

This is the work done by the torque.

(d) Relation between torque and angular acceleration : Let us consider a rigid body rotating about a given axis with a uniform angular acceleration α, and let a torque τ act on the body.

Consider that $m_1, m_2, m_3 \ldots$ are the masses of the particles of the body at perpendicular distances $r_1, r_2, r_3 \ldots$, respectively from the axis of rotation.

Since body is rigid so the angular acceleration of all particles of the body remains same while their linear acceleration is different due to different distances of the particles from the axis.

Let a_1, a_2, a_3 ... be the linear accelerations of the particles then

$$a_1 = r_1 \alpha, \ a_2 = r_2 \alpha, a_3 = r_3 \alpha, \dots$$

Force on particle of mass m_1

$$f_1 = m_1 a_1 = m_1 r_1 \alpha$$

Moment of this force about the axis of rotation

$$= f_1 \times r_1 = (m_1 r_1 \alpha) \times r_1 = m_1 r_1^2 \alpha$$

Similarly, moment of forces on other particles about the axis of rotation is $m_2 r_2^2 \alpha$, $m_3 r_3^2 \alpha$ ...

$\therefore$ Torque acting on the body

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \dots$$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2) \alpha$$

$$= (\Sigma m r^2) \alpha$$

$$\boxed{\tau = I \alpha} \qquad [\because I = \Sigma mr^2]$$

If $\alpha = 1$ then $\tau = I$

Hence "moment of inertia of a body about a given axis is numerically equal to torque acting on the body rotating with unit angular acceleration about it."

The above relation in vector form may be written as

$$\boxed{\vec{\tau} = I \vec{\alpha}}$$

This equation is called **fundamental equation of rotation or law of rotation.**

• 2.5. THEOREMS OF MOMENT OF INERTIA

There are two important theorems to determine the moment of inertia. They help in determining the moment of inertia about any axis if moment of inertia about one axis is known. They are :

(a) Theorem of parallel axes : According to this theorem, **"Moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus Mh^2, where M is the mass of the body and h the perpendicular distance between the two axes."** *i.e.,*

$$I = I_{cm} + Mh^2$$

This is the "theorem of parallel axes".

Proof : Let us consider a particle of the body of mass m at a distance r from the line AB. Here we have to calculate the moment of inertia of the body about the line GH which is parallel to the centre of mass axis AB. Let h be the perpendicular distance between AB and GH as shown in the fig. 4.

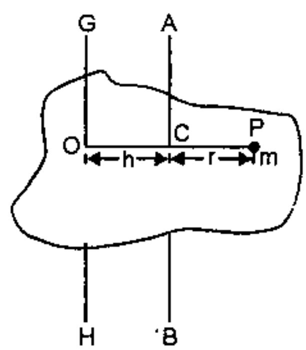


Fig. 4

Therefore the moment of inertia of the body about centre of mass axis AB is

$$I_{cm} = \Sigma m r^2$$

Now, moment of inertia of the body about the line GH is

$$I = \Sigma\, m\, (r + h)^2$$

$$= \Sigma\, m\, [r^2 + h^2 + 2rh]$$

$$= \Sigma\, m\, r^2 + \Sigma\, m\, h^2 + \Sigma\, 2\, m\, r\, h$$

$$= I_{cm} + h^2\, \Sigma\, m + 2h\, \Sigma\, m\, r$$

$$= I_{cm} + M\, h^2 + 0$$

where M = Total mass of the body and $\Sigma\, m\, r$ = sum of the moments of the masses of particles constituting the body about an axis through its mass must be zero.

$\therefore$ $$\boxed{I = I_{cm} + Mh^2}$$ **Hence proved.**

(b) Theorem of perpendicular axes : According to this theorem, **"The moment of inertia of a plane lamina (a two-dimensional body) about an axis perpendicular to its plane (OZ) is equal to sum of the moments of inertia about any two mutually perpendicular axes OX and OY in its plane intersecting on the first axis."**

i.e., $$I_z = I_x + I_y$$

where x, y, z axes are mutually perpendicular to each other.

Proof : According to this theorem, the sum of the moments of inertia of a plane lamina about any two mutually perpendicular axes in its plane is equal to its moment of inertia about an axis perpendicular to the plane of the lamina and passing through the point of intersection of the first two axes.

i.e., $$I_z = I_x + I_y$$

Now we have to prove it.

Let us consider a particle P of mass m at distance r from O and at distances x and y from OY and OX, respectively. Then the moment of inertia about OX is my^2 and hence I_x, the moment of inertia of the lamina about OX is

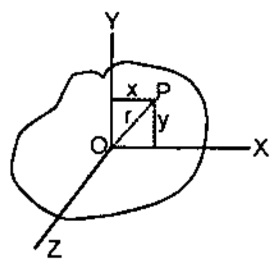


Fig. 5

$$I_x = \Sigma\, m\, .\, y^2 \qquad \ldots \text{(i)}$$

Similarly, the moment of inertia of the lamina about OY is

$$I_y = \Sigma\, m\, x^2 \qquad \ldots \text{(ii)}$$

and the moment of inertia of the lamina about OZ is

$$I_z = \Sigma\, m\, r^2$$

$$I_z = \Sigma\, m\, (x^2 + y^2) \qquad [\because r^2 = x^2 + y^2]$$

$$I_z = \Sigma\, m\, x^2 + \Sigma\, m\, y^2$$

$$I_z = I_y + I_x$$

Hence $$\boxed{I_z = I_x + I_y}$$ [from eqs. (i) and (ii)]

This proves the theorem of perpendicular axis.

• 2.6. MOMENT OF INERTIA OF A CIRCULAR DISC

(i) Moment of inertia of a circular disc about an axis through its centre and perpendicular to its plane : Let us consider a circular disc of mass M and radius R with

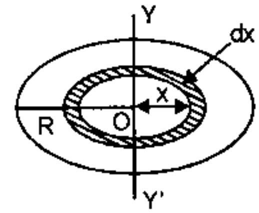


Fig. 6

centre 'O'. We have to calculate the moment of inertia of the disc about the line $y y'$, *i.e.*, the axis which passes through the centre O and is perpendicular to the plane of disc.

$\because$ Mass of the disc $= M$

and Area of the disc $= \pi R^2$

$\therefore$ Mass per unit area $= \dfrac{M}{\pi R^2}$

Consider a small element of the disc which is also circular in shape of radius x and width dx,

area of the element $= 2 \pi x \, dx$

$\therefore$ Mass of the element $= \dfrac{M}{\pi R^2} (2 \pi x \, dx) = \dfrac{2Mxdx}{R^2}$

M.I. of this element about $y y'$

$$= \frac{2 M x \, dx}{R^2} \cdot (x^2)$$

$$= \frac{2 M x^3 \, dx}{R^2}$$

$\therefore$ M.I. of the circular disc about $y y'$ is

$$I = \int_0^R \frac{2 M x^3}{R^2} dx = \frac{2M}{R^2} \int_0^R x^3 \, dx$$

$$\boxed{I = \frac{1}{2} M R^2}$$

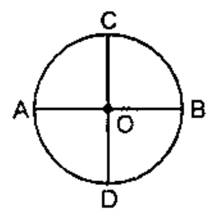


Fig. 7 (a)

(ii) About diameter : M.I. of the circular disc about any diameter can be obtained by using theorem of perpendicular axis, *i.e.*,

$$I = I_{AB} + I_{CD} = 2 I_{AB} \qquad [\because AB = CD]$$

$$\boxed{I_{AB} = \frac{1}{4} M R^2}$$

(iii) About tangent : M.I. of the disc about tangent EF can be obtained by using the theorem of parallel axis, *i.e.*,

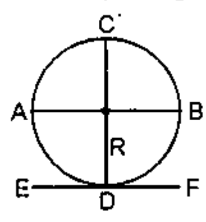


Fig. 7 (b)

$$I_{EF} = I_{AB} + M R^2 = \frac{1}{4} M R^2 + M R^2$$

$$= \frac{5}{4} M R^2$$

• 2.7. MOMENT OF INERTIA OF AN ANNULAR DISC

(i) Moment of inertia of an annular disc about an axis through its centre and perpendicular to its plane : Let us consider an annular disc of mass M and inner radius R_1 and outer radius be R_2 with centre 'O'. Consider a small strip of this disc, it will be a ring. Let x be the radius of this ring.

Mass of the disc $= M$

$\therefore$ Mass per unit area $= \dfrac{M}{\pi (R_2^2 - R_1^2)}$

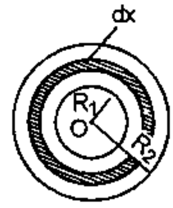


Fig. 8

$\because$ Area of small ring $= 2 \pi x \, dx$

$\therefore$ Mass of the ring $= \dfrac{M}{\pi (R_2^2 - R_1^2)} 2 \pi x^2 \, dx = \dfrac{2M}{(R_2^2 - R_1^2)} x \, dx$

M.I. of this ring about the axis passing through its centre and perpendicular to its plane

$$= \text{mass} \times (\text{distance})^2$$

$$= \frac{2M}{\pi (R_2^2 - R_1^2)} x \, dx \,.\, x^2 = \frac{2M}{(R_2^2 - R_1^2)} x^3 \, dx$$

M.I. of the disc about an axis which passes through the centre and is perpendicular to the plane is

$$I = \int_{R_1}^{R_2} \frac{2M}{(R_2^2 - R_1^2)} x^3 \, dx = \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{x^4}{4} \right]_{R_1}^{R_2}$$

$$= \frac{M}{2 (R_2^2 - R_1^2)} [R_2^4 - R_1^4] = \frac{M}{2} (R_2^2 + R_1^2)$$

$$\boxed{I = \frac{M}{2} (R_2^2 + R_1^2)}$$

(ii) About the diameter : Moment of inertia of the annular disc about any diameter can be obtained by using the theorem of perpendicular axis.

$$I = I_{AB} + I_{CD}$$

$$I = 2I_{AB} \qquad [\because AB = CD]$$

$$I_{AB} = \frac{1}{2} I$$

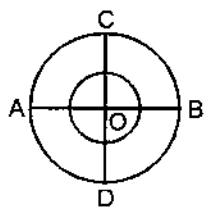


Fig. 9

$$I_{AB} = \frac{1}{2} \times \frac{M}{2} (R_2^2 + R_1^2)$$

$$\boxed{I_{AB} = \frac{M}{4} (R_2^2 + R_1^2)}$$

This is the desired result.

(iii) About the tangent : Moment of inertia of the annular disc can be obtained by using the theorem of parallel axis, *i.e.,*

$$I_{EF} = I_{cm} + M R_2^2$$

$$= I_{AB} + M R_2^2$$

$$= \frac{M}{4} (R_2^2 + R_1^2) + M R_2^2$$

$$I_{EF} = \frac{M}{4} (R_1^2 + 5 R_2^2)$$

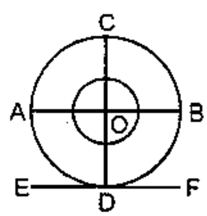


Fig. 10

• 2.8. MOMENT OF INERTIA OF A SOLID CYLINDER

Moment of inertia of a solid cylinder about an axis perpendicular to geometrical axis and passing through its centre.

Let us consider a solid cylinder of mass M and radius R with centre of mass 'O'. Let l be the length of the cylinder as shown in the fig. 11. Let XX' be the axis of the cylinder and YY' be the axis about which M.I. is to be determined.

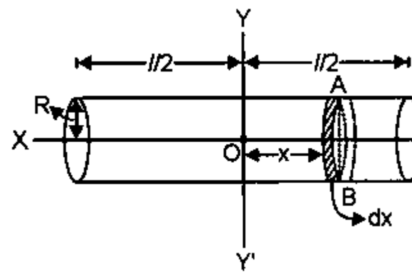


Fig. 11

Consider a small part of this cylinder. It will be a disc of the same radius R and width dx. Let x be the distance of this disc from YY'.

$\because$ Mass of the cylinder $= M$

and volume of the cylinder $= \pi R^2 l$

$\therefore$ Mass per unit volume $= \dfrac{M}{\pi R^2 l}$

$$\text{Volume of the disc} = \text{surface area} \times \text{width}$$

$$= \pi R^2\, dx$$

$$\therefore \quad \text{Mass of disc} = \frac{M}{\pi R^2 l} \,.\, \pi R^2\, dx = \frac{M}{l}\, dx$$

M.I. of this disc about its diameter, *i.e.*, about AB

$$= \frac{1}{4}\,\text{mass} \times \text{radius}^2$$

$$= \frac{1}{4}\frac{M}{l}\, dx \,.\, R^2 = \frac{1}{4}\frac{MR^2}{l}\, dx$$

M.I. of the disc about YY' can be obtained by using theorem of parallel axes

$$= \frac{1}{4}\frac{M}{l}\, dx \,.\, R^2 + \frac{M}{l}\, dx \,.\, x^2$$

$$= \frac{1}{4}\frac{M}{l}(R^2 + 4x^2)\, dx$$

M.I. of the solid cylinder about YY'

$$I = \int_{-l/2}^{l/2} \frac{1}{4}\frac{M}{l}(R^2 + 4x^2)\, dx$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 \int_{-l/2}^{l/2} dx + 4\int_{-l/2}^{l/2} x^2\, dx\right]$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 \,.\, \frac{2l}{2} + \frac{4}{24} \,.\, 2l^3\right]$$

$$= \frac{1}{4}\frac{M}{l}\left[R^2 l + \frac{8}{24} l^3\right]$$

$$\boxed{I = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]} \quad \ldots (1)$$

(b) If ρ is the density of its material then

$$M = \pi R^2 l \rho$$

$$\therefore \qquad R^2 = \frac{M}{\pi l \rho}$$

so by (1) we get

$$I = M\left(\frac{l^2}{12} + \frac{M}{4\pi l \rho}\right)$$

Now, I will be minimum if $\frac{dI}{dl} = 0$ *i.e.*,

$$\frac{d}{dl}\left[M\left(\frac{l^2}{12} + \frac{M}{4\pi l \rho}\right)\right] = 0$$

$$M\left[\frac{2l}{12} - \frac{M}{4\pi l^2 \rho}\right] = 0$$

$$\frac{l}{6} = \frac{M}{4\pi l^2 \rho} = \frac{\pi R^2 l \rho}{4\pi l^2 \rho}$$

$$\therefore \qquad \boxed{\frac{l}{R} = \sqrt{\frac{3}{2}}}$$

This is the required relation between l and R.

(c) If the rod is very thin so that its radius is negligible ($R = 0$) then from equation (1) M.I. of the rod about the axis in part (a) is

$$I = \frac{M l^2}{12}$$

If K is the radius of gyration of the rod about this axis then

$$I = \frac{M l^2}{12} = M K^2$$

$$K = \frac{l}{\sqrt{12}}$$

• 2.9. MOMENT OF INERTIA OF A CYLINDER ABOUT ITS OWN AXIS

(a) Moment of inertia of cylinder about its own axis : Let us consider a cylinder of mass M and radius R. Let l be the length of the cylinder as shown in the fig. 12. Let XX' be its geometrical axis about which its moment of inertia is to be determined. Consider a small part of this cylinder. It will be a disc. Let the mass of this disc be m.

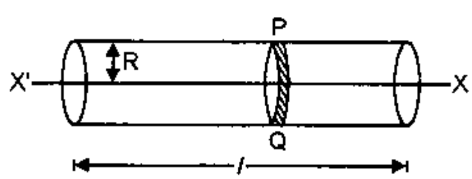


Fig. 12

M.I. of one disc PQ about

$$XX' = \frac{1}{2} \times (\text{mass}) \times (\text{radius})^2 = \frac{1}{2} m R^2$$

M.I. of the whole cylinder about XX' will be equal to the sum of the moment of inertia of these discs, *i.e.*,

$$I = \frac{1}{2} m R^2 + \frac{1}{2} m R^2 + \frac{1}{2} m R^2 + \ldots$$

$$I = \Sigma \frac{1}{2} m R^2$$

$$= \frac{1}{2} R^2 \Sigma m$$

$$I = \frac{1}{2} M R^2$$

where $\Sigma m = M$ = mass of the cylinder.

Now the M.I. of the cylinder about the line parallel to its axis and touching its surface can be obtained by using theorem of parallel axes, *i.e.*,

$$\frac{1}{2} M R^2 + M R^2 = \frac{3}{2} M R^2$$

(b) The M.I. of the cylinder about its own axis

$$= \frac{1}{2} M R^2$$

M.I. of the cylinder about the equatorial axis is

$$= M\left[\frac{l^2}{12} + \frac{R^2}{4}\right]$$

when these are equal, then

$$M\left[\frac{l^2}{12} + \frac{R^2}{4}\right] = \frac{1}{2} M R^2$$

$$\frac{l^2}{12} + \frac{R^2}{4} = \frac{R^2}{2}$$

$$\frac{l^2}{12} = \frac{R^2}{4}$$

$$l^2 = 3 R^2$$

$$\boxed{l = \sqrt{3} R}$$

This is the required relation between l and R.

• 2.10. MOMENT OF INERTIA OF A THIN SPHERICAL SHELL

(1) About a diameter : Let us consider a thin spherical shell of mass M and radius R with centre O.

Consider a small part of this shell. This lies between two parallel planes AB and CD and is perpendicular to XX'. This small part will be a ring. Let its thickness be dx at a distance x from the centre O as shown in fig. 13

From the fig.

Radius of ring $= R \cos \theta$

$$y = \sqrt{R^2 - x^2}$$

and $\quad x = R \sin \theta$

$\therefore \quad dx = R \cos \theta \, d\theta$

$\because$ Mass of the spherical shell $= M$

Surface area of the shell $= 4 \pi R^2$

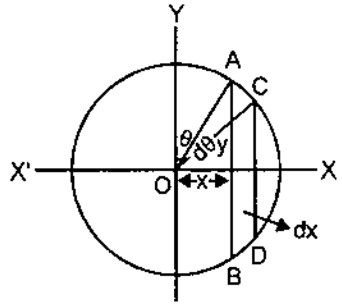


Fig. 13

$\therefore$ Mass per unit area $= \dfrac{M}{4\pi R^2}$

area of the ring = circumference × width

$$= 2\pi y\, AC$$
$$= 2\pi . R\cos\theta . R\, d\theta$$
$$= 2\pi R\, dx$$

$\therefore$ Mass of ring $= \dfrac{M}{4\pi R^2} . 2\pi R\, dx = \dfrac{M}{2R}\, dx$

M.I. of ring about XX' = mass × (radius)^2

$$= \frac{M}{2R} \times y^2 = \frac{M}{2R} \times (R^2 - x^2)\, dx$$

M.I. of the spherical shell is

$$I = \int_{-R}^{R} \left[\frac{MR^2}{2R} - \frac{Mx^2}{2R}\right] dx = \left[\frac{MR}{2} x - \frac{Mx^3}{6R}\right]_{-R}^{R}$$

$$\boxed{I = \frac{2}{3} MR^2}$$

(ii) About the tangent : In case when spherical shell is parallel to any diameter at a distance R from the centre. Now from theorem of parallel axis M.I. of the shell about tangent is

$$I_t = \frac{2}{3} MR^2 + MR^2 = \frac{5}{3} MR^2.$$

• 2.11. ABOUT DIAMETER

Let us consider a solid sphere of mass M and radius R with centre O. We have to calculate the moment of inertia of this sphere about its diameter, *i.e.*, about XX' as shown in the fig. 14

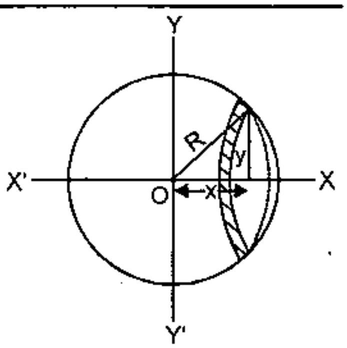


Fig. 14

Volume of the solid sphere $= \dfrac{4}{3}\pi R^3$

Mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Consider a small part of this sphere, it will be a disc. Let x be the radius of this disc and dx be the thickness of this disc with centre O.

From the figure the radius of this thin disc is

$$y = \sqrt{R^2 - x^2}$$

Volume of this disc $= \pi (R^2 - x^2)\, dx$

$\therefore$ Mass of disc $= \dfrac{3M}{4\pi R^3} . \pi (R^2 - x^2)\, dx$

$$= \left(\frac{3M}{4\pi R^3} . \pi R^2 - \frac{3M}{4\pi R^3} . \pi x^2\right) dx$$

$$= \left(\frac{3M}{4R} - \frac{3M}{4R^3} x^2\right) dx$$

$\therefore$ M.I. of this disc about $XX' = \frac{1}{2}$ mass $\times$ (radius)^2

$$= \frac{1}{2} \frac{3M}{4R^3} (R^2 - x^2)\, dx\, R^2 - x^2)$$

$$= \frac{1}{2} \frac{3M}{4R^3} (R^2 - x^2)^2\, dx$$

$$= \frac{1}{2} \frac{3M}{4R^3} (R^4 + x^4 - 2R^2 x^2)\, dx$$

$\therefore$ M.I. of the solid sphere about XX' is

$$I = \int_{-R}^{R} \frac{1}{2} \frac{3M}{4R^3} (R^4 + x^4 - 2R^2 x^2)\, dx$$

$$= \frac{1}{2} \frac{3M}{4R^3} \int_{-R}^{R} (R^4 + x^4 - 2R^2 x^2)\, dx$$

$$= \frac{1}{2} \frac{3M}{4R^3} \left[R^4 x - 2R^2 \cdot \frac{x^3}{3} + \frac{x^5}{5}\right]_{-R}^{R}$$

$$\boxed{I = \frac{2}{5} M R^2}$$

(ii) About a tangent : Any tangent to the sphere at any point is parallel to one of its diameters so by the theorem of parallel axes

$$I_t = \frac{2}{5} M R^2 + M R^2 = \frac{7}{5} M R^2.$$

• 2.12. CONSTRUCTION OF FLYWHEEL

(a) Construction of flywheel : A flywheel is a large heavy wheel with a long cylindrical axle supported on ball-bearings. The wheel is constructed such that whole of its mass is concentrated at its rim. Its centre of mass lies on its axis of rotation so that it remains at rest in any position. A peg is attached on its rim.

(b) Use of flywheel in stationary engines : The flywheel is used in stationary engines to make the motion of the engine uniform. In these engines, the linear motion of a piston is converted into the rotatory motion of a shaft. An arrangement for doing this is shown in fig. 15.

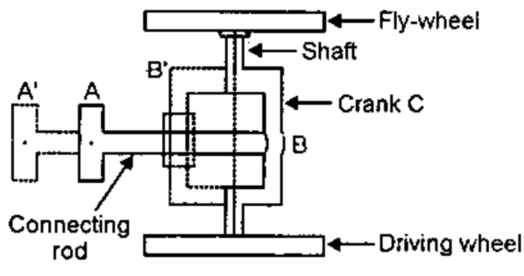


Fig. 15

A connecting rod is joined with the piston A and to a crank C and B. To one end of the crank-shaft is attached a large flywheel and at the other end there is the driving wheel which is connected to the machinery that is to be driven by the engine. As the piston moves from A' to A, the crank rotates from the position B' to the position B and as the piston returns back from the position B to B'. Thus the crank-shaft has made one complete rotation. When the connecting rod makes some angle with the crank, then a torque is developed which rotates the crank-shaft. But, in each rotation there are two points B and B' at which the connecting rod and the crank are in the same straight line.

(c) Determination of the moment of inertia of a flywheel : Now we have to calculate the moment of inertia of a fly-wheel. For this, its axle is mounted on ball-bearings in a horizontal position as shown in the fig. 16. A small peg P is attached to the axle. A loop made at one end of a fine cord is fastened to the peg. The cord is wrapped several times around the axle and mass m is attached to the other end of the cord.

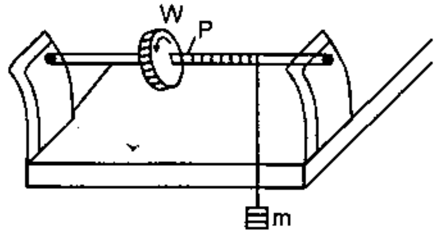


Fig. 16

When the mass m falls vertically down, then the cord is unwrapped and the wheel rotates. The gravitational potential energy lost by the mass in its vertical fall is converted partly into kinetic energy of translation of the falling mass itself, partly into the kinetic energy of rotation of the flywheel and partly used in doing work against the friction in the ball bearings.

Hence at B and B' the torque produced by the piston is zero. These points are called the 'dead centres'. At two other points the crank is at right angles to the connecting rod and then the torque is maximum. Thus the torque considerably varies. Therefore the machinery connected to the shaft can not run uniformly. But the flywheel makes it uniform. The flywheel rotates with the shaft and gains K.E. of rotation. As the torque becomes minimum at the dead centres, the flywheel continues to rotate on account of its large M.I. and carries the crank shaft with it. Hence the machine goes on running with practically the same speed.

Let h be the vertical distance through which the mass falls before the cord leaves the axle. Then the gravitational potential energy lost by the mass m is high.

Let I be the moment of inertia of the flywheel about the axle. Let v be the velocity gained by the mass and ω the angular velocity gained by the flywheel at the instant when the cord leaves the axle. Then, the kinetic energy of translation gained by the mass m is $\frac{1}{2}mv^2$ and the kinetic energy of rotation gained by the flywheel is $\frac{1}{2}I\omega^2$.

Let n_1 be the number of rotations of the flywheel before the cord leaves the axle and f the work done against the friction during each rotation. Then, the energy used up against the friction is $n_1 f$.

Now loss in potential energy = gain in K.E. by mass

+ gain in K.E. by flywheel + energy used against friction

$$\therefore \quad mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + n_1 f$$

But $v = r\omega$ where r is the radius of the axle.

$$\therefore \quad mgh = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}I\omega^2 + n_1 f \qquad \dots (1)$$

When the cord leaves the axle, the flywheel continues to rotate for some and finally comes into rest due to the friction force on its bearings. Let it make n_2 rotations more before finally coming to rest. This means that the kinetic energy $\frac{1}{2}I\omega^2$ of the flywheel is used up against the friction in n_2 rotations.

$$\therefore \quad \frac{1}{2}I\omega^2 = n_2 f$$

$$f = \frac{1}{2}\frac{I\omega^2}{n_2}$$

So by eqn. (1) we get

$$mgh = \frac{1}{2} mr^2\omega^2 + \frac{1}{2}\omega^2 \left(1 + \frac{n_1}{n_2}\right)$$

$$\therefore \qquad I = \frac{2mgh - mr^2\omega^2}{\omega^2\left(1 + \frac{n_1}{n_2}\right)}$$

$$I = \frac{m\left(\frac{2gh}{\omega^2} - r^2\right)}{\left(1 + \frac{n_1}{n_2}\right)} \qquad \ldots (2)$$

When the cord leaves the axle then the wheel has an angular velocity ω. After making n_2 rotations in time t, its velocity is reduced to zero owing to the friction force. Suppose the friction force remains constant during this time, the average angular velocity of the wheel is

$$\frac{\omega + 0}{2} = \frac{\omega}{2}$$

This should be equal to $\frac{2\pi n_2}{t}$, since the wheel makes n_2 rotations during time t, *i.e.*,

$$\frac{\omega}{2} = \frac{2\pi n_2}{t}$$

$$\therefore \qquad \omega = \frac{4\pi n_2}{t}$$

So by eqn. (2), we get

$$\boxed{I = \frac{m\left(\frac{ght^2}{8\pi^2 n_2^2} - r^2\right)}{1 + \frac{n_1}{n_2}}} \qquad \ldots (3)$$

This is the required expression for the moment of inertia of a flywheel.

Procedure : The cord is wrapped around the axle so that the mass m is at the same height as a fixed point marked on a wall. The length of the cord is so adjusted that when the mass m reaches the ground, the other end of the cord just leaves the axle. The distance h from the fixed point to the ground is measured by a scale. The cord is again wrapped around the axle and the mass is allowed to fall. The number of rotations, n_1, made by the wheel before the cord leaves the axle is counted. The cord is once again wrapped around the axle and the mass is allowed to fall. As soon as the mass strikes the ground, a stop watch is started. The time t and the number of rotations n_2 made by the wheel before coming to rest are noted. The radius r of the axle is obtained by measuring its diameter at several places by means of a vernier callipers, I is then calculated from the above relation (3).

• 2.13. BODY ROLLING DOWN AN INCLINED PLANE

(a) Body rolling down an inclined plane : Let us consider a body of mass M and radius R rolling down an inclined plane, which makes an angle θ with horizontal. When a body rolls without slipping then it rotates about a horizontal axis through its centre of mass and also its centre of mass moves. So, the rolling may be assumed as a rotational motion about an axis through its centre of mass plus a translational motion of the centre of mass.

In fig. 17, a body starts to roll down at certain height. When it rolls then it suffers loss in gravitational potential energy, but gains kinetic energy that of rotation. This loss in kinetic energy must be equal to the total gain in kinetic energy, provided no energy is lost due to friction between the body and the plane.

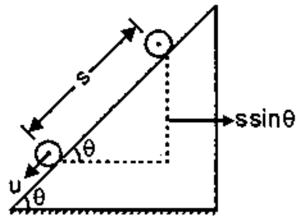


Fig. 17

Let v be the linear velocity of its centre of mass and ω be the angular velocity about the centre of mass after rolling down the plane a distance s.

The loss in gravitational potential energy

$$= \text{weight} \times \text{loss in vertical height} = Mgs \sin \theta.$$

Now, translational K.E. gained by the body $= \frac{1}{2} Mv^2$

and the rotational K.E. $= \frac{1}{2} I \omega^2$

where I is the moment of inertia about rotational axis.

Total energy gained by the body

$$= \frac{1}{2} Mv^2 + \frac{1}{2} I_{cm} \omega^2 \quad \ldots(1)$$

If K be the radius of gyration of the body about the axis of rotation, then

$$I = MK^2$$

also $$\omega = \frac{v}{R}$$

$\therefore$ by eqn. (1)

Total energy gained by the body

$$= \frac{1}{2} Mv^2 + \frac{1}{2} (MK^2) \frac{v^2}{R^2}$$

$$= \frac{1}{2} Mv^2 \left(1 + \frac{K^2}{R^2}\right)$$

Assuming that no energy is used up against friction, the loss in potential energy is equal to the total gain in kinetic energy, *i.e.*,

$$Mgs \sin \theta = \frac{1}{2} Mv^2 \left(1 + \frac{K^2}{R^2}\right)$$

$$v^2 = 2s = \frac{g \sin \theta}{1 + \frac{K^2}{R^2}}$$

$$v^2 = \frac{2gs \sin \theta}{1 + \frac{K^2}{R^2}}$$

$$\therefore \qquad v = \sqrt{\left(\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}\right)} \qquad \ldots (2)$$

This is the required expression for the final velocity of the body rolling on an inclined plane.

$$\because \qquad v^2 = 2as$$

$$a = \frac{v^2}{2s}$$

$$\boxed{a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}} \qquad \ldots (3)$$

This is the expression for the final acceleration.

(b) For different rolling bodies, acceleration is obtained as follows :

(i) Solid sphere : The moment of inertia of a solid sphere about its diameter is given by

$$I = MK^2 = \frac{2}{5}MR^2$$

$$\therefore \qquad \frac{K^2}{R^2} = \frac{2}{5}$$

so by eqn. (3)

$$a = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\sin\theta \qquad \ldots (4)$$

(ii) Disc : The M.I. of a disc about the axis passing through its centre and perpendicular to its plane is given by

$$I = MK^2 = \frac{1}{2}MR^2$$

$$\frac{K^2}{R^2} = \frac{1}{2}$$

so by eqn. (3) $$a = \frac{g\sin\theta}{1+\frac{1}{2}} = \frac{2}{3}g\sin\theta \qquad \ldots (5)$$

This is also the acceleration for cylinder.

(iii) Spherical shell : Its M.I. about the diameter is given by

$$I = MK^2 = \frac{2}{3}MR^2$$

$$\frac{K^2}{R^2} = \frac{2}{3}$$

so by eqn. (3) $$a = \frac{g\sin\theta}{1+\frac{2}{3}} = \frac{3}{5}g\sin\theta \qquad \ldots (6)$$

(iv) Ring : Its M.I. about the axis passing through its centre and perpendicular to its plane is given by

$$I = MK^2 = MR^2$$

$$\frac{K^2}{R^2} = 1$$

so by eqn. (3)

$$a = \frac{g \sin \theta}{1+1} = g \sin \theta \qquad \dots (7)$$

From eqns. (4), (5), (6) and (7) we get the ratio of acceleration of solid sphere, disc, spherical shell and ring as

$$= \frac{5}{7} : \frac{2}{3} : \frac{3}{5} : \frac{1}{2} = 150 : 140 : 126 : 105$$

Hence the acceleration of solid sphere, disc, shell and ring are in a decreasing order. Therefore, if all of them start rolling at the same instant then, the sphere will reach down the plane first, then the disc, then the shell and then the ring.

• STUDENT ACTIVITY

3. Find the relation between length and the radius of the cylinder so that M.I. is minimum.

4. If earth were to shrink suddenly, what would happen to the length of the day ?

• 2.14. PRECESSION

When a torque is exerted perpendicular to the axis of rotation of a rotating body then the rate of rotation of the body remains constant but the direction of the axis of rotation changes, *i.e.,* the axis of rotation itself rotates. The motion of the axis of rotation about a fixed axis due to an external torque is called **precession**. The axis about which the direction of rotation of the body precesses is called the axis of precession.

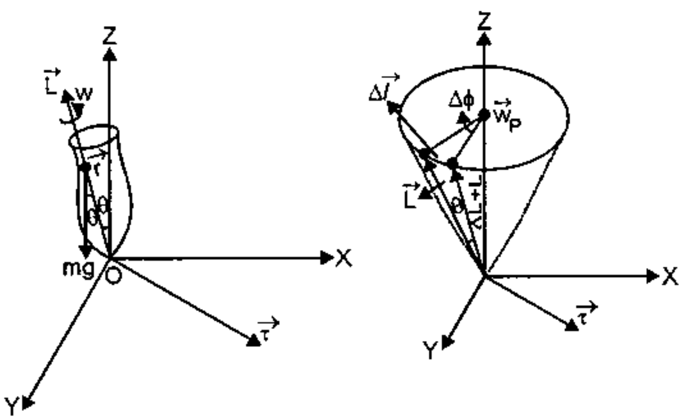
In other words we can say that **the turning of the axis of rotation is called precession.**

Gyroscope : In majority cases, the body, subjected to precessional motion, is supported at a point, away from the vertical line through its centre of gravity, where the axis of rotation is free to turn about the centre of gravity of the body.

Such a body, with its axis of spin supported at a point away from its centre of gravity and with the precessional rate of its spin axis maintained by the gravitational torque (due to its weight) about that point is called a **gyroscope or a top.**

Precession of a top spinning in earth's gravitational field : Top is a symmetrical body rotating about an axis, one point of which is fixed. In the fig. 18(a) top is spinning with angular velocity ω about its own axis of symmetry, O is the fixed point at the origin of an inertial reference frame. Its angular momentum is $\vec{L}$, pointing along the axis of rotation. This axis makes an angle θ with the vertical.

Let the position of centre of mass be $\vec{r}$ with respect to O.



(a) Fig. 18 (b)

The weight of the top is mg which exerts a torque about the fixed point O.

We have

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{g}$$

Its magnitude is

$$\tau = rmg\,(180^\circ - \theta) = rmg \sin\theta \qquad \dots (1)$$

According to right-hand-rule the direction of torque is perpendicular to the plane containing $\vec{r}$ and mg. This means that the torque $\vec{\tau}$ is perpendicular to $\vec{L}$ or perpendicular to the axis of rotation of the top.

The torque $\vec{\tau}$ changes the angular momentum $\vec{L}$ of the top. The change ΔL is also in the direction of the torque, *i.e.,* perpendicular to $\vec{L}$.

If this change takes place in a time Δt, the torque is given by

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta L} \qquad \dots (2)$$

The angular momentum $\vec{L} + \Delta\vec{L}$, after a time Δt is the vector sum of $\vec{L}$ and $\Delta\vec{L}$. When $\Delta\vec{L}$ the perpendicular to $\vec{L}$ and is very small so the new angular momentum vector

$\vec{L} + \Delta\vec{L}$ has the same magnitude as the initial angular momentum $\vec{L}$, but a different direction, *i.e.*, the angular momentum remains constant in magnitude but varies in direction. The top of the angular momentum vector L describes a circle around z-axis. In time Δt the radius of this circle moves through an angle $\Delta\phi$. The angular velocity of precession ω_P is defined as the rate at which the axis of rotation itself rotates about a fixed axis OL in the laboratory.

Now,
$$\Delta\phi = \frac{\Delta L}{L \sin\theta}$$
$$= \tau \frac{\Delta t}{L \sin\theta}$$
$$\omega_P = \frac{\Delta\phi}{\Delta t} = \frac{\tau \frac{\Delta t}{L \sin\theta}}{\Delta t}$$
$$\omega_P = \frac{\tau}{L \sin\theta} \qquad \ldots (3)$$

From eq. (1) putting the value of τ in eqn. (3) we get
$$\omega_P = \frac{rmg \sin\theta}{L \sin\theta}$$
$$\omega_P = \frac{mgr}{L}$$

Thus the angular velocity of precession is independent of θ and is inversely proportional to the magnitude of angular momentum. Larger the angular momentum smaller will be the precessional velocity. As the spinning to P slows down, its angular momentum $L\ (= I\omega)$ decreases and the angular velocity of precession increases. $\vec{\omega_P}$ is a vector pointing vertically upward as shown in fig. 18 (b).

From
$$\omega_P = \frac{\tau}{L \sin\theta}$$
$$\tau = \omega_P L \sin\theta$$

From the fig. 18 (b) it is clear that θ is the angle between $\vec{\omega_P}$ and $\vec{L}$, and $\vec{\tau}$ is a vector perpendicular to the plane formed by $\vec{\omega_P}$ and $\vec{L}$. So $\omega_P L \sin\theta$ is a vector product of $\vec{\omega_P}$ and $\vec{L}$, *i.e.*,

$$\boxed{\vec{\tau} = \vec{\omega_P} \times \vec{L}}$$

Hence Proved.

• 2.15. NEWTON'S LAW OF GRAVITATION

According to this law, **every object attracts every other object with a force which is directly proportional to the product of the two masses and inversely proportional to the square of distance between them.**

Let m_1 and m_2 be the masses separated by a distance r.

then
$$F \propto m_1 m_2$$
$$F \propto \frac{1}{r^2}$$

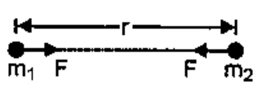


Fig. 15

On combining,
$$F \propto \frac{m_1 m_2}{r^2}$$

$$F = -\frac{G m_1 m_2}{r^2}$$

where G is called gravitational constant. G is also called universal constant because its numerical value does not depend on the nature of the medium. Negative sign shows attraction.

If $m_1 = m_2 = 1$ and $r = 1$ then

we get $$G = F.$$

Hence, the gravitational constant G is numerically equal to the force with which two particles, each of unit mass and placed at a unit distance apart, attract each other.

The value of G is $6.67 \times 10^{-11} \dfrac{\text{N m}^2}{\text{kg}^2}$

or $$G = 6.67 \times 10^{-8} \frac{\text{dynes cm}^2}{\text{gm}^2}$$

Since, $$G = F \frac{r^2}{m_1 m_2}$$

Dimension of G is $[G] = \dfrac{[\text{MLT}^{-2}]\,[\text{L}^2]}{[\text{M}]\,[\text{M}]} = [\text{M}^{-1}\,\text{L}^3\,\text{T}^{-2}]$

G is a scalar quantity and is different from the acceleration due to gravity $\vec{g}$ which is a vector quantity. $\vec{g}$ is neither universal nor constant.

Important Characteristics of Gravitational Force :

(1) This law is applicable for spherical bodies.

(2) It arises due to the masses of the bodies.

(3) They are attractive, central and form action-reaction pair.

(b) Gravity : Gravity is a special case of gravitation in which one of the objects must be earth, thus gravity represents forces of attraction between earth and any other object.

If m is the mass of a body placed on the surface of earth where acceleration due to gravity is g, then

$$\text{gravity pull} = \text{weight of body} = mg$$

The units and dimensions of gravity pull or weight are the same as those of force.

Difference between gravitation and gravity : The following points differentiate the gravitation and gravity.

1. Gravitation is the force of attraction acting between any two bodies of the universe, while gravity is the earth's gravitation pull on the body lying on or near the surface of earth.

2. The gravitational force on a body of mass m_1 due to an other body of mass m_2 placed at a distance r from each other, is

$$F = G \frac{m_1 m_2}{r^2}$$

while the force of gravity on a body of mass m is $F = mg$.

3. The force of gravitation between two bodies can be zero if the separation between two bodies becomes infinity while the force of gravity on a body is zero at the centre of the earth.

• 2.16. GRAVITATIONAL FIELD

We know that every particle of matter exerts a force of attraction on each other. **The gravitational field of a body at a point in a field is defined as the force experienced by the body of unit mass placed at that point provided the presence of unit mass does not disturb the original gravitational field.** *Or* the space surrounding the attracting particle within which its gravitational force of attraction can be experienced is called **gravitational field** of the particle.

It is always directed towards the centre of gravity of the body. It is a vector quantity.

Gravitational attraction : The intensity of gravitational field or the gravitational attraction, at a point in the field is the force experienced by a unit mass placed at that point, provided the unit mass itself does not produce any change in the field. Thus, if there is a gravitational field due to a particle of mass M, the attraction F at a point, whose distance r from the particle is

$$F = -G\frac{M \times 1}{r^2}$$

or

$$\boxed{F = -\frac{GM}{r^2}}$$

Gravitational attraction is a vector quantity.

Gravitational potential : When a mass moves in a gravitational field then work is done against the gravitational attraction but if it moves in the direction of the field then work is done by the field itself. Hence the gravitational potential at a point in a gravitational field is defined as **"The amount of work done in bringing a body of unit mass from infinity to that point without acceleration".**

Let us consider a body of mass M which is situated at O as shown in the fig. Let a unit mass be placed at point and at a distance r from O.

The gravitational attraction exerted on the unit mass is given by

$$F = -\frac{GM}{r^2} \text{ towards } O.$$

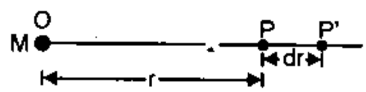


Fig. 20

Let the unit mass move against this attraction from P to P' through a distance dr.

The work done against the gravitational attraction $= -(F)\,dr$

$$= \frac{GM}{r^2}\,dr$$

Thus the work done as the unit mass moves from infinity will be

$$\int_r^\infty \frac{GM}{r^2}\,dr = GM\left[-\frac{1}{r}\right]_r^\infty = \frac{GM}{r}$$

Now work done as the unit mass moves from infinity to P will be

$$= -\frac{GM}{r}$$

This will be equal to the gravitational potential, *i.e.*,

$$\boxed{V = -\frac{GM}{r}}$$

Potential is a scalar quantity. Here the negative sign shows that the potential increases in the direction of r increasing. The potential at infinity is considered to be zero and becomes increasingly negative in a direction towards the attracting mass.

Relation between gravitational attraction and potential : Let F be the gravitational attraction at a point. The work done as a unit mass moves from this point to another point in the direction of the field through a small distance dr

$$= F(-dr)$$

By definition, this is equal to the difference of potential dV between two points.

$$\therefore \quad dV = F(-dr)$$

$$\boxed{F = -\frac{dV}{dr}}$$

This is the relationship between gravitational attraction and potential.

From above it is clear that **the rate of change of potential with respect to distance is equal to gravitational attraction.**

***Or* gravitational attraction at a point is the negative value of gradient of gravitational potential at that point.**

• 2.17. GRAVITATIONAL POTENTIAL OF SHELL AT AN EXTERNAL POINT

Let O be the centre of a thin spherical shell of radius R. Let P be the point at distance r from O at which the gravitational potential and attraction is to be determined. Let σ be the mass per unit area of the shell.

Imagine the shell to be divided into a number of circular rings, with centres on OP. Consider one such ring AB of radius $AT = R\sin\theta$ and thickness $R\,d\theta$.

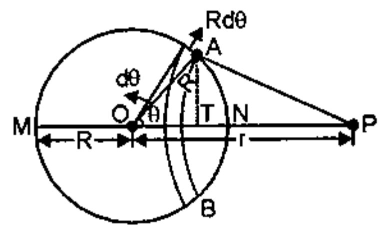


Fig. 21

The area of this ring = circumference × thickness

$$= 2\pi r \sin\theta \times (R\,d\theta)$$

Therefore its mass $= 2\pi R\sin\theta\,(R\,d\theta)\,\sigma$

$$= 2\pi R^2 \sigma \sin\theta\, d\theta$$

To first approximation every element of this ring at the same distance $AP = x$ from P. Therefore, the potential dV at P due to this ring is given by

$$dV = -G\,\frac{\text{Mass of the ring}}{x} = -G\,\frac{2\pi R^2 \sigma \sin\theta\, d\theta}{x} \quad \ldots (1)$$

From ΔAOP, we have

$$\therefore \quad \Delta AOP \cos\theta = \frac{R^2 + r^2 - x^2}{2Rr} \quad \ldots (2)$$

Differentiating eq. (2) we get

$$-\sin\theta\, d\theta = \frac{0 + 0 - 2x}{2Rr}$$

$$\sin\theta\, d\theta = \frac{x\,dx}{Rr}$$

Putting this value in eqn. (1) we get

$$dV = -G\,\frac{2\pi R^2 \sigma}{x}\left(\frac{xdx}{Rr}\right) = -G\,\frac{2\pi R\sigma}{r}\,dx$$

The potential V due to whole shell is obtained by integrating this expression between N and M, *i.e.*, $(r-R)$ to $(r+R)$.

$$\therefore \quad V = \int_{r-R}^{r+R} -G\frac{2\pi R\sigma}{r}\,dr$$

$$= -G\frac{2\pi R\sigma}{r}[x]_{r-R}^{r+R}$$

$$= -G\frac{2\pi R\sigma}{r}2R$$

$$= -G\frac{4\pi R^2\sigma}{r}$$

M is the mass of shell then $M = 4\pi R^2\sigma$

$$\therefore \quad \boxed{V = -G\frac{M}{r}}$$

This is the expression for the potential due to spherical shell at an external point.

Gravitational Attraction : As the shell is symmetrical with respect to the point P, the attraction F at P is along PO and is equal to negative gradient of potential, *i.e.*,

$$F = -\frac{dV}{dr}$$

$$F = -\frac{d}{dr}\left[\frac{-GM}{r}\right]$$

$$\boxed{F = -\frac{GM}{r^2}}$$

This is the required expression for the gravitational attraction at an external point due to spherical shell.

(b) Gravitational potential at an internal point : Consider a point P inside the shell at a distacne r from O as shown in the fig. 22.

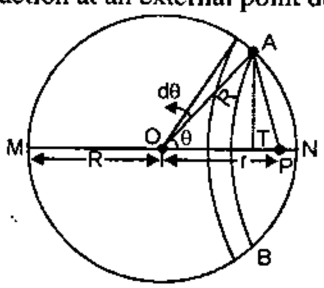


Fig. 22

The potential at P due to ring is

$$dV = -G\frac{2\pi R\sigma}{r}\,dx$$

The potential V due to the whole shell is obtained by integrating this expression between $N\,(n = R-r)$ and $M\,(x = R+r)$

$$V = \int_{R-r}^{R+r} -G\frac{2\pi R\sigma}{r}\,dx$$

$$V = -G\frac{2\pi R\sigma}{r}[x]_{R-r}^{R+r}$$

$$V = -G\frac{2\pi R\sigma}{r}[x]_{R-r}^{R+r}$$

$$V = -G\frac{2\pi R\sigma}{r}.\,2r = -G\,(4\pi r\sigma)$$

$$\boxed{V = -G\frac{M}{R}} \qquad [\because 4\pi R^2\sigma]$$

This is required expression for the potential due to spherical shell at an internal point.

From above it is clear that the potential is independent of r and is same as at a point on the surface of the shell. Thus the potential inside the shell is constant and equal to its value at the surface.

Gravitational attraction : We know

$$F = -\frac{dV}{dr}$$

$$= -\frac{d}{dr}\left[-G\frac{M}{R}\right]$$

$$\therefore \qquad F = 0$$

i.e., gravitational attraction inside the shell is zero.

Graphical representation :

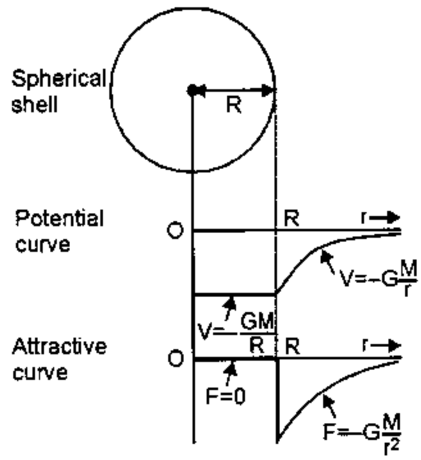


Fig. 23

• 2.18. GRAVITATIONAL POTENTIAL DUE TO A SOLID SPHERE AT AN EXTERNAL POINT

Let us consider a solid sphere of mass M and radius R with centre O. Let ρ be the density of the sphere.

Let P be the point outside the sphere at which the potential and attraction are to be determined. Let $OP = r$.

Imagine the sphere to be divided into a large number of thin concentric shells of masses m_1, m_2, m_3 ...etc. The potential at P due to these shells will be

$$-G\frac{M_1}{r}, -G\frac{M_2}{r}, -G\frac{M_3}{r} \ ....$$

where r is the distance of P from the centre O of the shells. As the potential is a scalar quantity, therefore the potential V due to whole sphere is equal to the sum of the potentials due to such shells, *i.e.*,

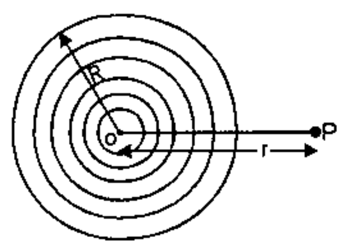


Fig. 24

$$V = -G\frac{M_1}{r} - G\frac{M_2}{r} - G\frac{M_3}{r} \ldots$$

$$= -\frac{G}{r}(m_1 + m_2 + m_3 + \ldots)$$

But $m_1 + m_2 + m_3 = \ldots = M$ is the mass of sphere

$$\boxed{V = -G\frac{M}{r}} \qquad \dots (1)$$

This is the expression for the potential due to solid sphere at an external point.

Gravitational Attraction : The gravitational attraction F at a point is equal to the negative gradient of potential at that point, *i.e.*,

$$F = -\frac{dV}{dr}$$

$$F = -\frac{d}{dr}\left[-G\frac{M}{r}\right]$$

$$\boxed{F = -G\frac{M}{r^2}}$$

This is the required expression for the attraction due to solid sphere at an external point.

Thus **the solid sphere exerts attraction at an external point as if its whole mass were concentrated at its centre.**

(ii) Potential due to solid sphere at an internal point : Let us consider a point P inside the sphere at a distance r from the centre O as shown in the fig. 25.

Imagine a concentric sphere, through P so that the point P will be external for the inner solid sphere of radius r and internal for the outer spherical shell of radius R.

The mass of the inner solid sphere is $\frac{4}{3}\pi r^3 \rho$. Therefore the potential at P due to this sphere is

$$V_1 = -G\frac{\frac{4}{3}\pi r^3 \rho}{r}$$

$$V_1 = -\frac{4}{3}\pi G \rho r^2 \qquad \dots (2)$$

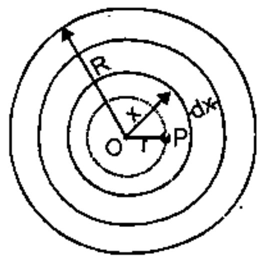


Fig. 25

The potential at point P due to outer spherical shell is obtained as follows :

Imagine this shell to be divided into a number of thin concentric shells. Consider one such shell of radius x and infinitesimaly small thickness dx. The volume of this shell is

$$= 4\pi x^2 (dx)$$

and its mass is $4\pi x^2 (dx)\rho$. As the potential at any point within a spherical shell is the same as on the surface, therefore the potential at P due to the thin shell under consideration will be

$$= -G\frac{\text{Mass of the shell}}{\text{radius of the shell}}$$

$$= -G\frac{4\pi x^2 (dx)\rho}{x}$$

$$= -G\,4\pi x\,(dx)\,\rho$$

The potential V_2 at P due to the whole shell of internal radius r and external radius R is obtained by integrating the expression between the limits $x = r$ and $x = R$.

Thus,

$$V_2 = \int_r^R -G\,4\pi x\,(dx)\rho$$

$$= -4\pi G \rho \left[\frac{x^2}{2}\right]_r^R$$

$$= -4\pi G \rho \left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$\therefore \quad V_2 = -\frac{4}{3}\pi G \rho \left[\frac{3R^2}{2} - \frac{3r^2}{2}\right] \quad \ldots (3)$$

as the potential is a scalar quantity the total potential V at P is obtained by adding the potentials given by eqns. (2) and (3).

$$\therefore \quad V = V_1 + V_2$$

$$V = -\frac{4}{3}\pi G \rho r^2 - \frac{4}{3}\pi G \rho \left[\frac{3R^2}{2} - \frac{3r^2}{2}\right]$$

$$= -\frac{4}{3}\pi G \rho \left(r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi G \rho \left(\frac{3R^2}{2} - \frac{r^2}{2}\right)$$

$$= -\frac{4}{3}\pi G \rho R^3 \left(\frac{3R^2 - r^2}{2R^3}\right)$$

If M is the mass of the spherre then $M = \frac{4}{3}\pi R^3 \rho$

$$\therefore \quad \boxed{V = -GM\left(\frac{3R^2 - r^2}{2R^3}\right)}$$

This is the expression for the potential due to solid sphere at an internal point.

Gravitational Attraction :

We know

$$F = -\frac{dV}{dr}$$

$$F = -\frac{d}{dr}\left[-GM\left(\frac{3R^2 - r^2}{3R^3}\right)\right]$$

$$= \frac{GM}{2R^3}(0 - 2r)$$

$$\boxed{F = -G\frac{Mr}{R^3}}$$

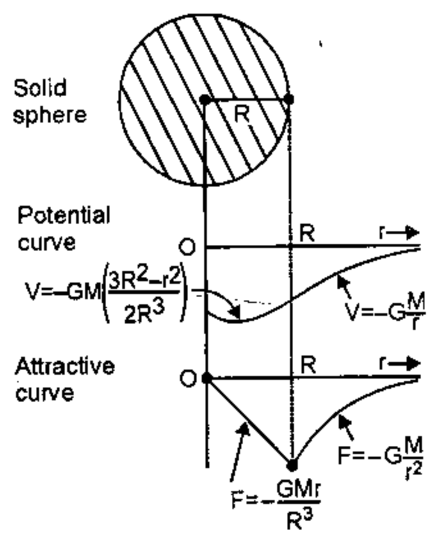


Fig. 26

This is the expression for the attraction.

Thus, **the gravitational attraction at a point inside a solid sphere is proportional to its distance from the centre.**

Graphical representation :

At the centre $(r = 0)$

$$\therefore \quad V_C = -\frac{3}{2}\frac{GM}{R}$$

and at the surface ($r = R$)

$$\therefore \quad V_S = -\frac{GM}{R}$$

Thus $$\boxed{V_S = \frac{2}{3} V_C}$$

• 2.19. KEPLER'S LAWS OF PLANETARY MOTION

Kepler gave three laws which describe the motion of the planets around the sun. These are :

1. Law of Orbit : The path of each planet about the sun is an ellipse with the sun at one focus.

$(F_1P + F_2P)$ is same for all points in the orbit.

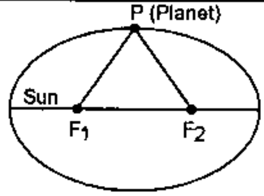


Fig. 27

2. The Law of Areas : Each planet moves in such a way that an imaginary line drawn from the sun to the planet sweeps out equal areas in equal time, *i.e.*, the areal velocity of the radius vector is constant.

From fig. 28

$$\boxed{\text{Area } SAB = \text{Area } SCD}$$

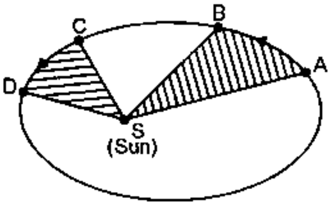


Fig. 28

The planet moves faster when it is nearer to the sun.

3. Law of Period : The square of the period of revolution of any planet around the sun is proportional to the cube of the semi-major axis of its elliptical orbit.

i.e., $$\boxed{T^2 \propto a^3}$$

T = period a = semi-major axis

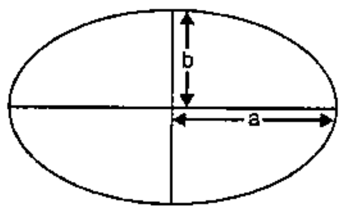


Fig. 29

Derivation of Kepler's Laws from Newton's Law of Gravitation : Let us consider a planet of mass m revolving around the sun in an elliptical orbit under gravitational field. According to Newton's Law of Gravitation, the attractive force acting on the planet due to the sun is given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \quad \ldots (1)$$

Since the gravitational force is a central force. Hence the angular momentum $\vec{L}$ is conserved in magnitude and direction. As a result of this, the motion of the planet must take place in a fixed plane and the areal veloctiy of its radius vector should be constant. **This is Kepler's second Law.**

The magnitude of areal velocity is

$$\frac{dA}{dt} = \frac{L}{2m}$$

Let $\frac{dA}{dt} = \frac{h}{2}$ where h is constant

and $$L = I\omega = mr^2 \frac{d\theta}{dt}$$

so $$\frac{h}{2} = \frac{L}{2m} = \frac{1}{2m}\left[mr^2 \frac{d\theta}{dt}\right]$$

$$\therefore \quad h = r^2 \frac{d\theta}{dt} \quad \ldots (2)$$

The radial force on the planet is

$$F = \text{mass} \times \text{radial acceleration}$$

$$F = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \quad \ldots (3)$$

$\because$
$$F = -G\frac{Mm}{r^2}$$

$\therefore$ From eq. (3)

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{GMm}{r^2}$$

$\therefore$
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \quad \ldots (4)$$

From eqn. (2), we get

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

$\therefore$ By eqn. (4)

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2} \quad \ldots (5)$$

Let
$$r = \frac{1}{u}$$

$\therefore$
$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) \quad \text{[from eqn. (2)]}$$

$$= -h\frac{du}{d\theta}$$

Again, differentiating
$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Putting this value of $\frac{d^2r}{dt^2}$ in eqn. (5) we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^3u^3 = -GMu^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

$\therefore \qquad \dfrac{d^2u}{d\theta^2} + \left(u - \dfrac{GM}{h^2}\right) = 0$

$\because \qquad \dfrac{GM}{h^2} = \text{constant}$

$\therefore \qquad \dfrac{d^2}{d\theta^2}\left(u - \dfrac{GM}{h^2}\right) + \left(u - \dfrac{GM}{h^2}\right) = 0$

This is the differential equation of second order.
Let the solution of this equation be

$$u - \frac{GM}{h^2} = -C\cos\theta$$

$$\therefore \qquad u = \frac{GM}{h^2} - C\cos\theta$$

$$\therefore \qquad \frac{h^2/GM}{r} = 1 - \frac{h^2C}{GM}\cos\theta \qquad \ldots(6)$$

or

$$\boxed{\frac{l}{r} = 1 - e\cos\theta}$$

where eccentricity, $e = \dfrac{h^2C}{GM}$ and $l = \dfrac{h^2}{GM}$ = semi-latus rectum.

Since the orbit of the planet around the sun must be a 'closed' one, hence the total energy of the planet, *i.e.*, $E = \dfrac{1}{2}mv^2 - \dfrac{GMm}{r}$ should be 'negative'. We find that E is negative only when $e < 1$.

Hence the orbit of the planet around the sun is an ellipse. **This is Kepler's First Law.**

Let 'a' and 'b' are be semi-major and semi-minor axes of the ellipse, then

$$l = \frac{b^2}{a} = \frac{h^2}{GM}$$

If T be the time period of the planet around the sun then

$$T = \frac{\text{area of ellipse}}{\text{areal velocity}}$$

$$\therefore \qquad T = \frac{\pi ab}{h/2}$$

$$\therefore \qquad T^2 = \frac{4\pi^2 . b^2 a^2}{h^2}$$

$$T^2 = \frac{4\pi^2 b^2 a^2}{GMb^2/a}$$

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

$$\therefore \qquad \boxed{T^2 \propto a^3}$$

This is **Kepler's Third Law.**

• 2.20. PERIOD OF MOTION OF A PLANET AROUND THE SUN

We know that all the planet in universe revolve around the sun in an elliptical orbit. Except for Mercury and Pluto, the orbits of all the planets may be considered approximately circular.

Let us consider a planet of mass m, revolving around the sun of mass M, in an orbit of radius r. R is the radius of the orbit of sun. C is the centre of system, then we have

$$MR = mr$$

Since $\quad M >> m$

and $\quad R << r$

hence we may assume that the sun is situated at C. In this position the force of attraction between Sun and Planet is given by

$$F = \frac{G\,Mm}{r^2}$$

and centripetal force is

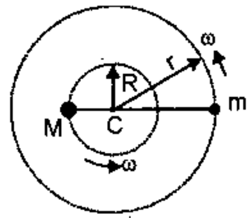


Fig. 30

$$F = mr\,\omega^2$$

$$\therefore \quad G\frac{Mm}{r^2} = mr\,\omega^2$$

$$GM = \omega^2 r^3$$

Since $\omega = \frac{2\pi}{T}$ where T is the period of revolution of the planet.

$$\therefore \quad GM = \frac{4\pi^2}{T^2} r^3$$

$$\therefore \quad \frac{T^2}{r^3} = \frac{4\pi^2}{GM} \quad \ldots (1)$$

$$\boxed{\frac{T^2}{r^3} = \text{constant}} \quad \left[\because \frac{4\pi^2}{GM} = \text{constant}\right]$$

This is **Kepler's Third Law**

From eqm. (1)

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}}}$$

This is the expression for the period of revolution.

• 2.21. WEIGHTLESSNESS INSIDE A SATELLITE

We know that the only force acting on satellite revolving around the earth in an orbit of radius r is the gravitational attraction of the earth which is directed towards the centre of the earth.

In this position the acceleration is

$$a = \frac{GM}{r^2} \quad \ldots (1)$$

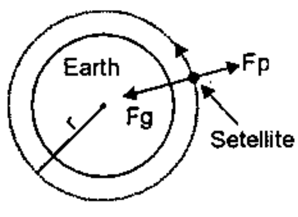


Fig. 31

where M is the mass of the earth. The gravitational force F_g is given by

$$F_g = G\frac{Mm}{r^2}$$

and fictitious or pseudo force is given by $F_p = -ma$.

This is directed away from the centre of the orbit.

For system the force is given by

$$F = F_g + F_p = \frac{GMm}{r^2} - ma$$

$$= \frac{GMm}{r^2} - m\frac{GM}{r^2} = 0 \qquad \text{[by eqn. (1)]}$$

Thus, the net force on all bodies inside the satellite is zero and they will be in the state of 'weightlessness'. Hence they will appear to move without acceleration inside the satellite.

From the above we conclude that the value of acceleration due to gravity, i.e., g for a pendulum inside a satellite would be zero and hence the simple pendulum experiment can not be conducted inside the satellite.

• 2.22. ENERGY CONSIDERATION IN THE MOTION OF PLANETS AND SATELLITES

Let us consider a satellite of mass m in a circular orbit of radius r about earth of mass M which is assumed to be at rest in an inertial reference frame. The potential energy of the system is

$$U(r) = -\omega_{\infty r} + U(\infty)$$

where $\omega_{\infty r}$ is the work done by the gravitational force of the earth on the satellite as the satellite moves from infinity to a distance r. The potential energy at infinity is assumed to be zero, *i.e.*,

$$U(\infty) = 0 \text{ then}$$

$$U(r) = -\omega_{\infty r} = -\int_{\omega}^{r} F(r)\,dr$$

$$\because \quad F(r) = -\frac{GMm}{r^2}$$

$$\therefore \quad U(r) = -\int_{\infty}^{r} -\frac{GMm}{r^2}\,dr$$

$$U(r) = \left[\frac{-GMm}{r}\right]_{\infty}^{r}$$

$$\boxed{U(r) = -\frac{GMm}{r}}$$

Thus the potential energy is negative. The kinetic energy of the system is given by

$$K = \frac{1}{2}mv^2$$

We know that the gravitational force supplies the centripetal force, *i.e.*,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\therefore \qquad mv^2 = \frac{GMm}{r}$$

so kinetic energy is

$$K = \frac{1}{2}\frac{GMm}{r}$$

Kinetic energy is always positive. This shows that the moving body can do work in coming to zero energy.

Total energy is

$$E = U + K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

$$\boxed{E = -\frac{1}{2}\frac{GMm}{r}}$$

This energy is constant and negative. Kinetic energy can never be negative but it would be zero at $r = \infty$. The potential energy is always negative, being zero at $r = \infty$.

The meaning of negative total energy is that the orbit of satellite is a 'closed' one, *i.e.*, the satellite is always bound under the gravitational force of the earth.

(b) Variation in kinetic energy and potential energy of a planet moving around the Sun in an Ellipitical orbit : The total energy of the system, *i.e.*, sun + satellite is given by

$$E = \text{kinetic energy} + \text{gravitational potential energy}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Since the force on the planet due to sun is conservative, the total energy E is constant in time. Hence as the planet max moves then both r and v vary and hence the kinetic energy and potential energy individually vary, but the total energy remains unchanged.

Hence the kinetic is minimum when r is smallest, *i.e.,* when the planet is at the point of closet approach to the sun and minimum when r is largest. Potential energy is maximum negative when r is smallest and minimum negative when r is largest.

(c) Satellite moving down : When the satellite moves in a lower orbit then its energy dissipates due to the atmospheric friction. Therefore, in a particular orbit the gravitational attraction on the satellite exceeds the force required to keep the satellite in that orbit. As a result of which the satellite moves down towards the earth into lower orbit. In the lower orbit the potential energy decreases, *i.e.*, becomes more negative so that the kinetic energy increases because the total energy is conserved. Hence the satellite describes a smaller orbit with increased speed. In fact, due to atmospheric friction, the satellite spirals down towards the earth with increasing speed and ultimately burns out in the denser lower atmosphere.

• 2.23. TWO-BODY PROBLEM AND REDUCED MASS

The two body problem with central forces can always be reduced to the form of one body problem.

Let us consider two particles of mass m_1 and m_2 whose position vectors with respect to an origin O in an inertial reference frame are $\vec{r_1}$ and $\vec{r_2}$ as shown in fig. 32.

From fig. 32 $\qquad \vec{r} = \vec{r_1} - \vec{r_2}$

The particles exert gravitational forces of attraction on each other which act along the vector $\vec{r}$ and are thus "central forces". Let $\vec{F_{12}}$ and $\vec{F_{21}}$ be the forces acting on particles m_1 due to m_2 and on m_2 due to m_1. Then the equations of motion for m_1 and m_2 with respect O are

$$m_1 \frac{d^2\vec{r_1}}{dt^2} = \vec{F_{12}}$$

and
$$m_2 \frac{d^2\vec{r_2}}{dt^2} = \vec{F_{21}}$$

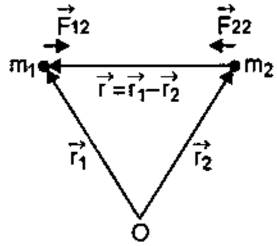


Fig. 32

By Newton's Third Law

$$\vec{F_{21}} = -\vec{F_{12}} = \vec{F} \text{ then}$$

$$\frac{d^2\vec{r_1}}{dt^2} = \frac{\vec{F}}{m_1} \text{ and } \frac{d^2\vec{r_2}}{dt^2} = \frac{\vec{F}}{m_2}$$

On substituting these equations we get

$$\frac{d^2(\vec{r_1} - \vec{r_2})}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}$$

or
$$\frac{d^2\vec{r}}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}$$

The central force $\vec{F}$ may be written as

$$\vec{F} = F\,\hat{r}$$

where F is the magnitude of the force and is any function of r and $\hat{r}$ is the unit vector along $\vec{r}$. Then

$$\frac{d^2\vec{r}}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)F\,\hat{r}$$

$$\therefore \quad \frac{d^2\vec{r}}{dt^2} = -\frac{1}{\mu}F\,\hat{r} \qquad \ldots (1)$$

where
$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}}$$

or
$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

From eqn. (1), $\mu \dfrac{d^2\vec{r}}{dt^2} = -F\,\hat{r}$

This is exactly the same as the equation of motion of a single particle of mass μ at a position vector $\vec{r}$ from a fixed centre which exerts on it a central force $F\,\hat{r}$.

Here the negative sign shows that the force on the particle and its vector distance from the centre of force are opposite in direction.

Thus, the original two-body problem involving two vectors $\vec{r_1}$ and $\vec{r_2}$ has been reduced to a one-body problem involving a single vector $\vec{r}$. Here $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is known as the **'reduced mass'** of the system of two particles and has a value less than both m_1 and m_2.

• SUMMARY

- Angular displacement is the angle described by the position vector $\vec{r}$ about the axis of rotation.
- The rate of change of angular displacement is known as angular velocity.

- The rate of change of angular velocity of a body about the axis of rotation is known as angular acceleration.
- Torque is defined as the external force acting on the body which rotates the body about fixed axis.
- The rate of change of angular momentum of a particle is equal to the torque acting on the particle.
- The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of the masses of the various particles and squares of their perpendicular distance from the axis of rotation. It given by $I = \Sigma mr^2$.
- Theorem of parallel axes states that Moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus Mh^2, where M is the mass of the body and h the perpendicular distance between the two axes.
- The equation $\vec{\tau} = I\vec{\alpha}$ is called fundamental equation of relation or law of rotation, where τ = Torque, I = Moment of Inertia, α = Angular acceleration.
- Theorem of perpendicular axes states that "The moment of inertia of a plane lamina (a two-dimensional body) about an axis perpendicular to its plane (***OZ***) is equal to sum of the moments of inertia about any two mutually perpendicular axes ***OX*** and ***OY*** in its plane intersecting on the first axis."
- A flywheel is a large heavy wheel with a long cylindrical axle supported on ball-bearings. The wheel is constructed such that whole of its mass is concentrated at its rim.
- The motion of the axis of rotation about a fixed axis due to an external torque is called precession.
- The axis about which the direction of rotation of the body precesses is called the axis of precession.
- According to this law, every object attracts every other object with a force which is directly proportional to the product of the two masses and inversely proportional to the square of distance between them.
- Gravity is a special case of gravitation in which one of the objects must be earth, thus gravity represents forces of attraction between earth and any other object.
- The space surrounding the attracting particle within which its gravitational force of attraction can be experienced is called gravitational field of the particle.
- The gravitational potential at a point in a gravitational field is defined as **"The amount of work done in bringing a body of unit mass from infinity to that point without acceleration".**
- Kepler gave three laws which describe the motion of the planets around the sun. These are:

  1. Law of Orbit : The path of each planet about the sun is an ellipse with the sun at one focus.

  2. The Law of Areas : Each planet moves in such a way that an imaginary line drawn from the sun to the planet sweeps out equal areas in equal time, *i.e.*, the areal velocity of the radius vector is constant.

  3. Law of Period : The square of the period of revolution of any planet around the sun is proportional to the cube of the semi-major axis of its elliptical orbit.
- $T = 2\pi\sqrt{\frac{r^3}{GM}}$ is the expression for the period of revolution and is derived using Kepler's laws.

• STUDENT ACTIVITY

5. State and explain Newton's law of universal gravitation.

6. What do you mean by gravity ?

7. Differentiate between gravitation and gravity.

8. Define gravitational field.

9. Define gravitational attraction.

10. Define gravitational potential.

• TEST YOURSELF

1. Define torque $\vec{\tau}$ acting on a particle about an axis. Obtain its angular momentum. Find out the relationship between torque and angular momentum.
2. Define moment of inertia of a body. Give its physical significance.
3. Derive an expression for the kinetic energy of a body rotating about an axis. hence define the M.I. of the body.
4. Prove that for a rigid body the angular momentum about an axis of rotation is equal to the product of M.I. and the angular velocity about that axis. Hence show that the K.E. of rotation is $L^2/2I$.
5. Obtain a relation between the torque applied and the angular acceleration produced in the body.
6. State and prove the theorem of parallel axes.
7. State and prove the theorem of perpendicular axis.
8. Obtain an expression for M.I. of a thin circular disc (i) about an axis passing through its centre and perpendicular to its plane (ii) about a diameter and (iii) about a tangent in its plane.
9. Find M.I. of annular disc of mass M, inner radius R_1 and outer radius R_2 : (i) about an axis passing through its center and perpendicular to its plane (ii) about a diameter and (iii) about the tangent in its plane.
10. A solid sphere rolls on a table. What fraction of its total K.E. is rotational ?

    [Ans. 2/7]
11. Deduce expressions for the gravitational potential and attraction due to thin uniform spherical shell at a point (a) outside and (b) inside the shell.
12. What are Kepler's laws of planetary motion ?
13. Show how by introducing the idea of reduced mass, a two body problem under central forces can be reduced to a one body problem.
14. If the rotational motion of a body about an axis is to be changed then we must apply about the axis :

    (a) Torque (b) Torque and Force
    (c) Force (d) None of these
15. The moment of inertia of a thin uniform circular disc of mass M and radius R about any tangent is :

    (a) $\frac{MR^2}{4}$ (b) $\frac{MR^2}{2}$ (c) $\frac{5\,MR^2}{2}$ (d) None of these

16. The moment of inertia of uniform solid sphere of mass M and radius R about diameter is :

(a) $\frac{2}{5}MR^2$ (b) $\frac{2}{3}MR^2$ (c) $\frac{2}{7}MR^2$ (d) None of these

17. The moment of inertia of uniform solid cylinder of mass M, radius R and length l about long axis of symmetry is :

(a) $\frac{1}{4}MR^2$ (b) $\frac{M\rho^2}{12}+\frac{MR^2}{4}$ (c) MR^2 (d) None of these

18. The rate of change of angular momentum is equal to :

(a) Force (b) Angular Acceleration
(c) Torque (d) Moment of Inertia

19. A gymnast is sitting on a rotating stool with her arms outstretched. Suddenly she folds her arms near the body. Which of the following is correct :

(a) Angular speed decreases
(b) Moment of inertia decreases
(c) Angular momentum decreases
(d) Angular speed remains constant

20. Moment of momentum is called :

(a) Torque (b) Weight
(c) Moment of inertia (d) Angular momentum

21. When the torque acting on a system is zero, which of the following will be constant?

(a) Force (b) Linear momentum
(c) Angular momentum (d) Linear impulse

22. A solid sphere of mass M rolls down an inclined plane without slipping from rest at the top of the inclined plane. The linear speed of the sphere at the bottom of the inclined plane is V. The K.E. of the sphere is :

(a) $\frac{1}{2}MV^2$ (b) $\frac{5}{3}MV^2$ (c) $\frac{2}{5}MV^2$ (d) $\frac{7}{10}MV^2$

23. Two circular discs A and B have equal mass and thickness but are made of metals with densities d_A and d_B ($d_A > d_B$). If their moments of inertia about an axis passing through the centre and normal to circular faces be I_A and I_B then :

(a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A \geq I_B$

24. What must be the relation between l and R if the moment of inertia of the cylinder about its axis is be the same as the moment of inertia about the equatorial axis :

(a) $l=\sqrt{3}\,R$ (b) $R=\sqrt{3}\,l$ (c) $l=\frac{\sqrt{3}}{2}R$ (d) $R=\frac{3}{\sqrt{2}}l$

25. The total kinetic energy of a rolling uniform disc is equal to :

(a) $\frac{2}{3}$ translational kinetic energy (b) $\frac{3}{2}$ translational kinetic energy
(c) 2 translational kinetic energy (d) None of these

26. The moment of inertia of the spherical shell of mass M and radius R about a tangent is :

(a) $\frac{5}{3}MR^2$ (b) $\frac{2}{3}MR^2$ (c) $\frac{7}{5}MR^2$ (d) $\frac{2}{5}MR^2$

27. Fly-wheels have wide applications in :

(a) Stationary engines (b) Mobile engines
(c) Transport (d) None of these

28. A body of mass M and radius R rolls down a plane inclined at an angle θ to the horizontal without slipping, the acceleration of a body will depend upon :

(a) Mass (b) Angle θ (c) Height (d) None of these

29. A sphere, a spherical shell, a ring and a cylinder are allowed to roll down simultaneously an inclined plane from the same height without slipping which will reach earlier :

(a) Shell (b) Cylinder (c) Sphere (d) Ring

20. The relation between angular momentum and angular velocity is :

(a) $\vec{J} = \vec{r} \times \vec{\omega}$ (b) $\vec{J} = \vec{\omega} \times \vec{r}$ (c) $\vec{J} = \frac{1}{\vec{\omega}}$ (d) $\vec{J} = I\,\vec{\omega}$

ANSWERS

14. (a) **15.** (c) **16.** (a) **17.** (d) **18.** (c) **19.** (b) **20.** (d) **21.** (c) **22.** (d) **23.** (c) **24.** (a) **25.** (b) **26.** (a) **27.** (a) **28.** (b) **29.** (c) **30.** (d)

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UNIT 3

PROPERTIES OF MATTER

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- Various properties of matter such as elasticity, surface tension, surface energy, etc. in detail.
- Determination of young's modulus and its application.
- The flow of liquid in detail along with the various principal's applicable.
- Standard notations such as Reynold's number and Stoke's law.

• 3.1. ELASTICITY

A body undergoes a change in its shape or size or both, when external force is applied. In this position it is called deformed. When external forces are removed, and if the body returns to its original form completely then the body is said to be perfectly elastic body and **this property of the body to regain its original form, on the removal of the deforming force, is called elasticity of the body.**

(a) Perfectly Elastic Body : A body which regains its original configuration immediately and completely after the removal of deforming force from it, is called perfectly elastic body. For example, quartz is perfectly elastic body.

(b) Perfectly Plastic Body : It may be defined as the body which does not regain its original configuration at all on the removal of deforming force. Putty and Paraffin wax are the example, of perfectly plastic body.

(c) Stress : When a deforming force is applied on the body then it changes the configuration of the body by changing the normal positions of the molecules or atoms of the body. Due to this an internal restoring force comes into play which tends to bring the body back to its original form.

The stress of the body may be defined as, "the internal restoring force per unit area of a deformed body is called stress", *i.e.*,

$$\text{Stress} = \frac{\text{Restoring force}}{\text{Area}}$$

The unit of stress in S.I. system is Nm^{-2} and in CGS system it is dyne/cm^2 and dimension is $[ML^{-1}T^{-2}]$.

Stress is of three types :

Tensile stress, compressional stress and tangential stress.

(d) Strain : When a deforming force is applied on a body then there is a change in the configuration of the body. In this position the body is said to be strained or deformed.

The ratio of the change in configuration to the original configuration is known as strain, *i.e.*,

$$\text{Strain} = \frac{\text{Change in configuration}}{\text{Original configuration}}$$

It has no unit and no dimensions. Strain is of 3 types : Longitudinal strain, volumetric strain and shearing strain.

Shearing strain = 2 × Longitudinal strain

Volumetric strain = 3 × Longitudinal strain

(e) Shear : When a body is acted upon by deforming forces tangential to its surface then it suffers a change in shape and in this position the body is said to be **sheared.**

(f) Elastic Limit : Elastic limit is the upper limit of deforming force upto which if deforming force is removed, the body regains its original form completely and beyond which if deforming force is increased the body loses its property of elasticity and gets permanently deformed.

Elastic limit is the property of the body while elasticity is the property of material of the body.

• 3.2. MODULUS OF ELASTICITY

Modulus of elasticity or coefficient of elasticity of a body is defined as the ratio of the stress to the corresponding strain produced, within the elastic limit.

It is denoted by E

$$E = \frac{\text{Stress}}{\text{Strain}}$$

It does not depend upon the magnitude of stress and strain but depends upon the nature of the material of the body.

Modulus of elasticity is of the following types :

(a) Young's Modulus of Elasticity (Y) : It is defined as "within the limit of elasticity the ratio of normal stress to longitudinal strain is known as Young's modulus of elasticity." It is denoted by Y.

$$Y = \frac{\text{Normal stress}}{\text{Longitudinal strain}}$$

Let us consider a wire of length l, radius r and uniform area of cross-section A. Let it be suspended from a rigid support. When normal force F is applied at its free end then it increases in its lenght by Δl (say).

So longitudinal strain $= \frac{\Delta l}{l}$

and normal stress $= \frac{F}{A} = \frac{F}{\pi r^2}$

$$\therefore \quad Y = \frac{f/\pi r^2}{\Delta l / l}$$

$$\boxed{Y = \frac{F\, l}{\pi\, r^2 \Delta l}}$$

This is the expression for Young's modulus.

Its unit is Nm^{-2} in S.I. system and dyne/cm^2 in C.G.S. system. **'Y' for liquid is zero.**

(b) Bulk Modulus of Elasticity (K) : It is defined as "the ratio of normal stress to the volumetric strain within the limit of elasticity, *i.e.*,

$$K = \frac{\text{Normal stress}}{\text{Volumetric strain}}$$

Let us consider a spherical body of volume V and surface area A. Let a force F be applied normally on the entire surface of the body. Due to this let its volume decrease by ΔV.

So volumetric strain $= -\frac{\Delta V}{V}$

Negative sign shows that the volume is decreasing when force is applied and normal stress $= \frac{F}{A}$.

$$\therefore \quad K = \frac{F/A}{-\Delta V/V} = \frac{-FV}{A\Delta V}$$

$$\boxed{K = -\frac{FV}{A\Delta V}}$$

This is the required expression for the bulk modulus of elasticity.

It has unit of Nm^{-2} in S.I. system and dyne/cm^2 in C.G.S. system. Gases have two types of volume elasticity, they are

(i) Isothermal elasticity $E_T = -V\left(\frac{dP}{dV}\right)_T$ = Pressure, P

(ii) Adiabatic elasticity $E_S = -V\left(\frac{dP}{dV}\right)_S$

$$= \gamma_P$$

where $\gamma = \frac{C_P}{C_V}$

The reciprocal of bulk modulus is known as compressibility, *i.e.*,

$$\text{Compressibility} = \frac{1}{\text{Bulk modulus}}$$

(c) Modulus of Rigidity (η) **:** It is defined as the ratio of tangential stress to the shearing strain, *i.e.*,

$$\eta = \frac{\text{Tangential stress}}{\text{Shearing strain}}$$

Let the lower face of the cube be fixed and let a tangential force be applied at the upper face of area A as shown in the fig. 1

$$\text{Tangential stress} = \frac{F}{A} \text{ and shearing strain} = \theta$$

so $$\eta = \frac{-F/A}{\theta}$$

$$\boxed{\eta = \frac{F}{A\theta}}$$

This is the expression for modulus of rigidity.

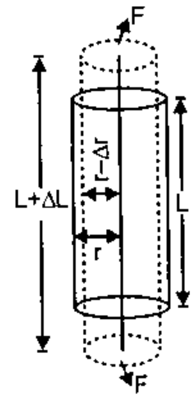

Fig. 1

(d) Poisson Ratio (σ) **:** When two equal and opposite forces are applied along the length of a wire then the wire extends along its length while contracts radially. Then

$$\text{Longitudinal strain} = \frac{\Delta L}{L}$$

$$\text{Lateral strain} = \frac{\Delta r}{r}$$

where L is the initial length, ΔL is extension is initial radius. Δr is contraction in radius.

The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio (σ).

$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$\sigma = -\frac{\Delta r/r}{\Delta L/L}$$

(–) sign shows that the change in length and radius are opposite signs, *i.e.*, if one decreases, the other increases.

σ lies between -1 and $\frac{1}{2}$ but in practice σ lies between 0 and $\frac{1}{2}$.

If there is no change in volume of a wire on loading then its σ will be $\frac{1}{2}$.

(e) Elastic After-effect : It is found that some solids come back to their position when the deforming force is removed while some solids take some time to do so. For example, quartz, silver and gold return to their original position as soon as the deforming force is removed but glass takes some time to do so. Hence, "This delay in recovering back to the original condition on the removal of deforming force is called elastic after-effect."

(f) Elastic Fatigue : When a wire vibrates continuously for a long time, for example, for some days, the rate of decay of the vibration is much greater than when the wire is pressrd. The wire gets tired due to continuous vibrations. This is called **elastic fatigue**.

(g) Elastic Hysteresis : Due to elastic after-effect the strain in the body tends to lag behind the stress. This lagging of strain behind the stress is called **elastic hysteresis.**

(h) Hooke's Law : This law is the fundamental law of elasticity. According to this law **"within the limits of elasticity the stress is proportional to the strain."**

That is, Stress ∝ Strain

$$\frac{\text{Stress}}{\text{Strain}} = \text{constant} = E$$

This is Hooke's law. E is known as **Modulus of Elasticity**. This law is applicable only when **Strain is small.**

• 3.3. RELATION AMONG ELASTIC CONSTANTS.

Let us consider a cube of side L each. Let the sides be parallel to x, y and z axis. (Fig. 2)

When extensional stress (P) is applied in any direction then there will be extension in that direction and at the same time there will be contraction in the remaining two perpendicular directions. This means that extensional stress applied in x direction will produce the extension along the x-axis and contraction along the y and z-axis.

Now extension (longitudinal) and contraction (lateral) may be calculated as follows:

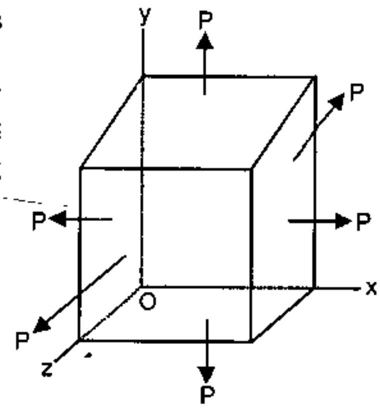

Fig. 2

$$\text{Longitudinal strain} = \frac{\Delta L}{L}$$

Change in length (ΔL) = Longitudinal Strain × L ... (1) (original length)

Again, when extensional stress is applied in x direction :

extension along the x-axis $= \frac{PL}{Y}$

contraction along the y-axis $= \sigma \frac{PL}{Y}$

and contraction along the z-axis $= \sigma \frac{PL}{Y}$

When extensional stress is applied along y-direction :

extension along the y-axis $= \frac{PL}{Y}$

contraction along the x-axis $= \frac{\sigma PL}{Y}$

and contraction along the z-axis $= \frac{\sigma PL}{Y}$

When extensional stress is applied along z-axis :

extension along the z-axis $= \frac{PL}{Y}$

contraction along the x-axis $= \frac{\sigma PL}{Y}$

and contraction along the y-axis $= \frac{\sigma PL}{Y}$

and $$Y = \frac{\text{Long. Stress}}{\text{Long.Strain}}$$

or $\quad$ Long. Strain $= \frac{P}{Y}$ ... (2)

Putting in eq. (1)

Change in length $= \frac{P}{Y} \times L$ ... (3)

Similarly, contraction may be calculated as follows :

Lateral Strain $= \frac{\Delta L}{L}$

Change in length (ΔL) compression = Lateral strain × original length (L) ... (4)

and $\quad \sigma = \frac{\text{Lateral Strain}}{\text{Long. Strain}}$

Lateral Strain $= \sigma \times$ Long. Strain

Lateral Strain $= \sigma \times \frac{P}{Y}$ [From (2)]

Putting in (4)

Change in Length (ΔL) (contraction) $= \sigma \times \frac{P}{Y} \times L$ ... (5)

Hence, Net extension along x-axis

$$= \frac{PL}{Y} - \sigma \frac{PL}{Y} - \sigma \frac{PL}{Y}$$

$$= \frac{PL}{Y}(1 - 2\sigma)$$

Similarly, the same extension will be obtained along y and z-axis.
New side of the cube = original length + extension

$$= L + \frac{PL}{Y}(1 - 2\sigma)$$

The new volume $= \left[L + \frac{PL}{Y}(1 - 2\sigma)\right]^3$

$$= L^3\left[1 + \frac{P}{Y}(1 - 2\sigma)\right]^3$$

$$= L^3\left[1 + \frac{3P}{Y}(1 - 2\sigma)\right] \quad \text{(approx.)}$$

(By Binomial theorem)

Original volume $= L^3$

Change in volume $= L^3\left[1 + \frac{3P}{Y}(1 - 2\sigma)\right] - L^3$

$$= \frac{3PL^3}{Y}(1 - 2\sigma)$$

Volume Strain $= \frac{\text{Change in volume}}{\text{Initial volume}}$

$$= \frac{\frac{3PL^3}{Y}(1 - 2\sigma)}{L^3} = \frac{3P}{Y}(1 - 2\sigma)$$

Therefore, Bulk Modulus $(K) = \frac{\text{Normal Stress}}{\text{Volume Strain}}$

$$= \frac{P}{\frac{3P}{Y}(1-2\sigma)}$$

$$K = \frac{Y}{3(1-2\sigma)}$$

$$\boxed{Y = 3K(1-2\sigma)} \qquad \ldots \text{(A)}$$

Now, again let us consider a cube of side L. Now extensioal stress is applied along x-axis and compression stress is applied along y and z-axis (Fig. 3).

When extensional stress is applied along x-direction

extension along the x-axis $= \frac{PL}{Y}$

contraction along the y-axis $= \frac{\sigma PL}{Y}$

and contraction along the z-axis $= \frac{\sigma PL}{y}$

When compressional stress is applied along y-direction :

contraction along the y-axis $= \frac{PL}{Y}$

extension along the x-axis $= \frac{\sigma PL}{Y}$

and extension along the z-axis $= \frac{\sigma PL}{Y}$

Net extension along x-axis $= \frac{PL}{Y} + \sigma\frac{PL}{Y}$

$$= \frac{PL}{Y}(1+\sigma)$$

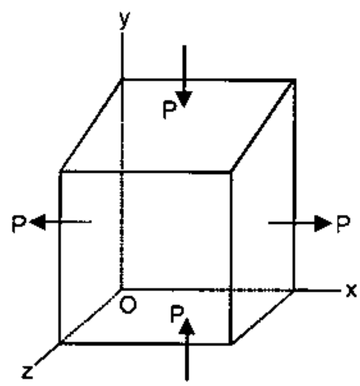

Fig. 3

Net compression along the y-axis $= \sigma\frac{PL}{Y} + \frac{PL}{Y}$

$$= \frac{PL}{Y}(1+\sigma)$$

Since there is equal extension and compression, hence there will be no change along the z-axis.

Again, extensional strain $= \frac{\text{Change in length}}{\text{Original length}}$

$$= \frac{\frac{PL}{Y}(1+\sigma)}{L} = \frac{P}{Y}(1+\sigma)$$

Similarly, compressional strain

$$= \frac{\frac{PL}{Y}(1+\sigma)}{L} = \frac{P}{Y}(1+\sigma)$$

We know

extensional strain + compressional strain = Shear (θ)

$$\frac{P}{Y}(1+\sigma)+\frac{P}{Y}(1+\sigma)=\theta$$

$$\frac{2P}{Y}(1+\sigma)=\theta$$

$$\frac{P}{\theta}=\frac{Y}{2(1+\sigma)}$$

$$\eta=\frac{Y}{2(1+\sigma)}$$

$$\boxed{Y=2n(1+\sigma)} \quad \ldots (B)$$

Again, from (B)

$$\frac{Y}{2n}=(1+\sigma)$$

$$\sigma=\frac{Y}{2\eta}-1$$

Put in (A)

$$Y=3K(1-2\sigma)$$

$$=3K\left[1-2\left(\frac{Y}{2n}-1\right)\right]$$

$$=3K\left(3-\frac{Y}{\eta}\right)$$

$$\boxed{Y=\frac{9K\eta}{3K+\eta}}$$

This can also be written as

$$\boxed{\frac{9}{Y}=\frac{1}{K}+\frac{3}{\eta}} \quad \ldots (C)$$

Theoretical limiting values of σ :

From (A) and (B)

$$Y=2\eta(1+\sigma)$$

or $$Y=3K(1-2\sigma)$$

or $$3K(1-2\sigma)=2\eta(1+\sigma) \quad \ldots (6)$$

Since K and η are positive quantities, therefore σ may either be a positive or negative quantity. If σ is positive, right hand side of equation (6) is positive and for the left hand, side to be positive

$$1-2\sigma>0$$

$$1>2\sigma$$

$$\sigma<\frac{1}{2}$$

If σ is a negative quantity, left hand side of (6) is positive and for the right hand side to be positive.

$$1+\sigma>0$$

$$\sigma>-1$$

Thus theoretically σ must lie between $\frac{1}{2}$ and -1.

But σ can not be negative. Since negative value of σ would mean that on being extended, a body should also expand laterally. Since no substance behaves in this way. Hence, in practice σ lies between 0 and $\frac{1}{2}$.

• 3.4. ANGLE OF TWIST AND ANGLE OF SHEAR

Consider a cylindrical rod of length l and radius r, clamped at the upper end. This is twisted through an angle θ at the lower end in the direction of arrow. As a result, the radius of each circular cross-section of the cylinder rod is turned about its axis through an angle θ. This angle θ is called the angle of twist and is proportional to the distance of the cross- section from the clamped end, *i.e.*, decreases towards the clamped end. This is an example of pure shear because the twist produces a charge neither in length nor in the radius of the cylindrical rod.

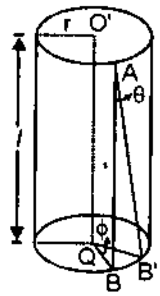

Fig. 4

Due to elasticity of the material a restoring couple is set up inside the cylindrical rod which is equal and opposite to external twisting couple. A generating line AB parallel to the axis of the cylindrical rod is turned, through an angle ϕ, to a new position AB'. The angle ϕ is called the angle of shear.

It is clear from fig. (4) that the angle of shear is maximum for the outermost layer and reduces to zero for the innermost larger, *i.e.*, at the axis.

(a) Twisting couple on a cylindrical rod : Imagine the cylindrical rod to be divided into a large number of infinitesimally thin coaxial cylindrical shells and consider one such shell of radius x, thickness dx and length l (Fig. 5). Let a generating line AB on the surface of this thin cylindrical shell be displaced into the position AB' when the cylindrical rod is twisted. If ϕ be the angle through which this thin spherical shell is sheared and θ, the corresponding angle of twist then

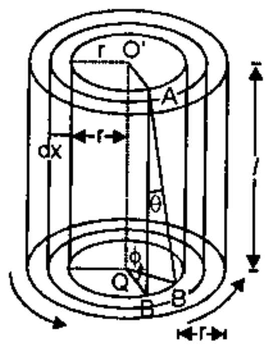

Fig. 5

$$BB' = l \,.\, \phi \quad \text{(arc = angle × radius)}$$

So angle of shear $\phi = \frac{BB'}{l}$

also $BB' = x\,\theta$

Thus, $$\phi = \frac{x.\,\theta}{l} \qquad \text{... (i)}$$

If F be the tangential force acting on the base of this thin elementary cylindrical shell of area $2\pi x \,.\, dx$, producing a shear ϕ in it, then

$$\text{Tangential stress} = \frac{\text{Tangential force}}{\text{area}}$$

$$= \frac{F}{2\pi r \,.\, dx} \qquad \text{... (ii)}$$

If η be the modulus of rigidity of the material of the rod, then

$$\eta = \frac{\text{Tangential stress}}{\text{shear } \phi}$$

$$= \frac{F/2\pi x \,.\, dx}{x\,\theta/l} \qquad \text{[From equations (i) and (ii)]}$$

$$= \frac{F}{2rx \cdot dx} \cdot \frac{l}{x\theta}$$

or $$F = \frac{2\pi n\theta}{l} \cdot x^2\, dx$$

The moment of this force about the axis OO' of the rod is

$$F \cdot x = \frac{2\pi n\theta}{l} \cdot x^3\, dx$$

This is equal to the couple required to twist the elementary shell through an angle θ. Therefore, the couple τ required to twist total rod of radius r is obtained by integrating the last expression between the limits $x = 0$ to $x = r$.

Thus, $$\tau = \int_0^r \frac{2\pi n\theta}{l} \cdot x^3 \cdot dx$$

$$= \frac{2\pi n\theta}{l}\left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{2\pi n\theta}{l} \cdot \frac{r^4}{4}$$

$$\tau = \frac{\pi\eta r^4}{2l}\,\theta$$

This is the required expression, where θ is in radians. This shows that the couple required is proportional to the twist θ.

If C (say) be the couple to produce a twist of one radian, then

$$C = \frac{\pi\eta r^4}{2l} \qquad (\theta = 1)$$

C is called the torsional rigidity or torsional constant of the wire. Since twisting couple is numerically equal to the restoring couple, C is also called the 'restoring couple per unit twist'. Also, the couple required to twist the rod through 90° (*i.e.*, $\frac{\pi}{2}$ radian) will be

$$\tau = C \cdot \frac{\pi}{2} = \frac{\pi\eta r^4}{2l} \cdot \frac{\pi}{2}$$

$$\tau' = \frac{\pi^2\eta r^4}{4l}$$

(b) Couple required to twist a hollow cylinder : We shall proceed exactly as in earlier, only the limits of integration will be changed. Now for hollow cylinder limits are $x = r_1$ to $x = r_2$. Thus the couple required to twist the hollow cylinder through an angle ϕ

$$\tau = \int_{r_1}^{r_2} \frac{2\pi\eta\theta}{l} x^3\, dx = \frac{2\pi\eta\theta}{l}\left[\frac{x^4}{4}\right]_{r_1}^{r_2}$$

$$= \frac{2\pi\eta\theta}{l}\left[\frac{r_2^4 - r_1^4}{4}\right] = \frac{\pi\eta\,(r_2^4 - r_1^4)}{2l} \cdot \theta$$

Therefore, the couple required to twist the hollow cylinder through one radian is

$$C = \frac{\pi\eta\,(r_2^4 - r_1^4)}{2l}$$

where C is called the torsional constant or torsional rigidity of the given hollow cylinder.

3.5. STATISTICAL METHOD OF DETERMINING MODULUS OF RIGIDITY

The method is due to Barton. The given rod is rigidly clamped at one end A_1 and hung vertically. A brass cylinder is attached to the lower free end. The flexible threads are wound around the cylinder such that they leave it tangentially at the opposite ends of a diameter and then pass over to two frictionless pulleys P_1 and P_2. The pans are attached to the ends of these threads, on which weights may be placed. At a known distance from the upper end, a pointer is fixed. When the wire is twisted, the pointer moves over a circular scale S graduated in degrees [Fig. (6)].

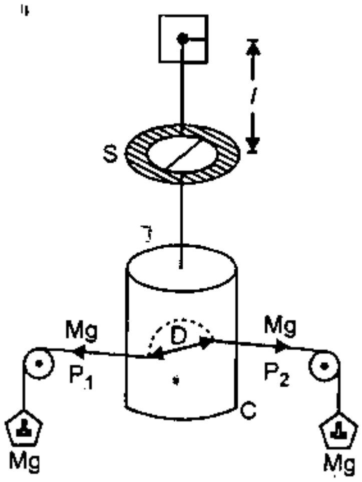

Fig. 6

Theory : Let a mass M be placed on each pan then the threads will experience equal and opposite parallel forces each being Mg. Thus, if D be the diameter of the cylinder, the moment of the couple twisting the cylinder $= MgD$.

Twisting of the rod will develop a restoring couple, equal and opposite to twisting couple in the rod. If C be restoring couple per unit twist and θ be the twist in the rod at a distance l from the fixed end then restoring couple for twist $\theta = C\,\theta$.

At the equilibrium, twisting couple = restoring couple

$$Mg\,.\,D = C\,\theta$$

$$= \frac{\pi\eta r^4}{2l}\,.\,\theta$$

where r is the radius of the rod. If θ be expressed in degrees, then

$$Mg\,.\,D = \frac{\pi r^4}{2l}\,.\,\theta \times \frac{\pi}{180}$$

Thus, $$\eta = \frac{360\,.\,Mg\,.\,D\,.\,l}{\pi^2 r^4 \theta}$$

Method : (i) A series of weights is used and the twists are read on the scale, both for load increasing and for load decreasing. To avoid error due to eccentricity of the rod with respect to the scale, both ends of the pointer are read and mean value of twist is calculated.

(ii) A graph (fig. 7) is plotted in Mg and corresponding mean twist θ. This is a straight line whose slope gives Mg / θ.

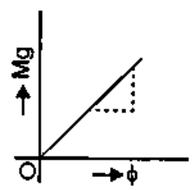

Fig. 7

(iii) The length l is measured by a meter scale.

(iv) The radius r is measured by screw guage at several points in two perpendicular directions and mean value is calculated.

(v) The diameter D of cylinder is measured by vernier callipers. Then by using the previous deduced relation, η for the material of the rod can be calculated.

3.6. TORSIONAL OSCILLATIONS

Let a body, say a heavy disc, be hung by a long and thin vertical wire whose upper end is rigidly clamped (fig. 8).

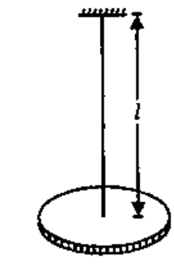
Fig. 8

Let the body be given a slight rotation in the horizontal plane by applying a couple. The wire is twisted and a restoring couple is developed in it due to its elasticity. This restoring couple, if twisting couple be removed, being unbalanced will produce an angular acceleration in the wire in a direction opposite to that of the twist.

The body, therefore, returns to its mean position but due to inertia it does not stop at mean position and now moves in the opposite direction, twisting the wire. Due to this twist, again restoring couple is set up in the wire which again arrests its motion and makes it return. The whole phenomenon is then repeated. Thus the body oscillates in the horizontal plane about the wire as axis. Such oscillations are called torsional oscillations and the system is called a torsional pendulum.

Theory : If θ be the angular twist or displacement any time t, then angular acceleration produced $= \frac{d^2\theta}{dt^2}$.

Let I be the M.I. of the disc about the wire as axis, then torque applied to the wire

$$= I\frac{d^2\theta}{dt^2}.$$

If C be the restoring couple per unit twist produced in the wire, then restoring couple for twist $\theta = C\theta$.

The equation of motion of the disc

$$I\frac{d^2\theta}{dt^2} + C\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \mu^2\theta = 0,$$

which represents a simple harmonic motion of time period.

$$T = \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{I}{C}} \qquad \text{... (1)}$$

To determine the modulus of rigidity : A regular body of known M.I., I_1 is placed at the disc such that its centre of mass coincides with that of disc. The whole system is then set into torsional vibrations. Suppose time period comes out to be T_1, then

$$T_1 = 2\pi\sqrt{\left(\frac{I+I_1}{C}\right)} \qquad \text{... (2)}$$

From equations (1) and (2), we get

$$T_1^2 - T^2 = \frac{4\pi^2(I+I_1)}{C} - \frac{4\pi^2 I}{C} = \frac{4\pi^2 I_1}{C}$$

$$C = \frac{4\pi^2 I_1}{(T_1^2 - T^2)}$$

$$\frac{n\pi r^4}{2l} = \frac{4\pi^2 I_1}{(T_1^2 - T^2)} \qquad \text{(Putting value of } C\text{)}$$

$$\therefore \qquad \eta = \frac{8\pi l I_1}{r^4 [T_1^2 - T^2]} \qquad \ldots (3)$$

Thus, measuring the radius of wire r by a screw gauge, length l by metre scale, I_1 from the mass and dimensions of the regular body, and then substituting the values of l, I_1, r, T_1 and T, the value of η, the modulus of rigidity, can be calculated.

• 3.7. MAXWELL'S NEEDLE

Maxwell's needle is a modification of torsional pendulum. It consists of a hollow cylindrical brass tube of length L, suspended by a steel wire, under experiment, fastened at its middle. The other end of the wire is attached to a fixed support. The tube is open at both ends. The hollow tube is just fitted with four brass cylinders, two solids S, S and two hollows H, H ; each having a length $L/4$.

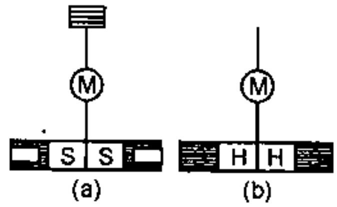

Fig. 9

All these cylinders are also of small radii. These cylinders are inserted in the hollow cylinder symmertically so that either solid cylinders S, S are inside and hollow H, H outside or hollow ones inside and solid cylinders outside. A mirror M is attached to the wire for counting of vibrations with lamp and scale arrangement.

Theory and Procedure : To start the experiment, the brass tube is set into oscillations in horizontal plane. Let first, solid cylinders be inside and hollow cylinders outside as in fig. (9a) If the period of torsional vibration in the case be T_1, then

$$T_1 = 2\pi \sqrt{\frac{I_1}{C}} \qquad \ldots \text{(i)}$$

where $I_1 \rightarrow$ M.I. of the loaded tube with solid cylinders inside and hollow cylinders outside, about

the suspension wire as axis.

$C \rightarrow$ restoring couple per unit twist of the wire.

The solid and hollow cylinders are now interchanged in position as shown in fig. (9b) and if T_2 be the time period of torsional vibration, then

$$T_2 = 2\pi \sqrt{\frac{I_2}{C}} \qquad \ldots \text{(ii)}$$

where $I_2 \rightarrow$ M.I. of the loaded tube with solid cylinders outside and hollow inside, about the suspension as axis.

From eq. (i) and (ii) ; we get

$$T_1^2 - T_2^2 = \frac{4\pi^2 (I_2 - I_1)}{C} \qquad \ldots \text{(iii)}$$

But
$$C = \frac{\pi \eta r^4}{2l}$$

where l and r are length and the radius of the wire under experiment, respectively. Thus, relation (iii) becomes

$$T_1^2 - T_2^2 = \frac{4\pi^2 (I_1 - I_2)}{\dfrac{\pi \eta r^4}{2l}}$$

$$\eta = \frac{8\pi l (I_1 - I_2)}{(T_2^2 - T_2^2) r^4}$$

To calculate $(I_1 - I_2)$.

Let $m_S \rightarrow$ mass of each of the solid cylinders

$m_H \rightarrow$ mass of each of the hollow cylinders.

If I_0 be the M.I. of the brass tube about the suspension wire as axis and I_s, I_h be the M.I. of solid and hollow cylinders respectively about vertical axes passing through their respective centre of mass, then by the theorem of parallel axes :

(a) M.I. of the solid cylinder about the axis of wire

$$= I_s + m_s\left(\frac{L}{8}\right)^2$$

(b) M.I. of hollow cylinder about the axis of wire

$$= I_h + m_H\left(\frac{3L}{8}\right)^2$$

(c) M.I. of hollow brass tube about the axis of wire $= I_0$

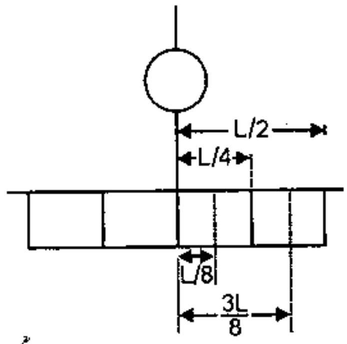

Fig. 10

Hence, the M.I. of the oscillating system in the first case will be

$$I_1 = I_0 + 2\left[I_s + m_s\left(\frac{L}{8}\right)^2\right] + 2\left[I_h + m_H\left(\frac{3L}{8}\right)\right] \quad \ldots \text{(v)}$$

Similarly, with the reference to fig. 10, the M.I. of the oscillating system in the second case is

$$I_2 = I_0 + 2\left[I_h + m_H\left(\frac{L}{8}\right)^2\right] + 2\left[I_s + m_s\left(\frac{3L}{8}\right)^2\right] \quad \ldots \text{(vi)}$$

From eq. (v) and (vi), we have

$$(I_1 - I_2) = (m_H - m_s)\frac{L^2}{4} \quad \ldots \text{(vii)}$$

Substituting the value of $(I_1 - I_2)$ in (vi), we get

$$\eta = \frac{8\pi l}{(T_1^2 - T_2^2)\, r^4} \cdot \frac{(m_H - m_s)\, L^2}{4}$$

$$= \frac{2\pi\,(m_s - m_H)\, L^2 l}{(T_1^2 - T_2^2)\, r^4}$$

Thus, knowing all the factors on right hand side of the above equation, modulus of rigidity can be calculated.

Superiority over Torsional Pendulum :

(i) The total weight of the suspended system remains unchanged throughout the experiment. Hence the torsional constant C of the wire remains constant, as assumed in the theory.

(ii) Instead of M.I., we require masses which can be known with more accuracy.

• 3.8. TERMS

(a) Beam : A bar of uniform cross-section whose length is much greater as compared to the thickness is called a beam.

(b) Longitudinal Filaments : A rectangular beam may be supposed as made up of a number of thin layers plane placed in contact parallel to one another.

Further, each layer may be considered to be a collection of thin fibres lying parallel to the length of the beam. These fibres are called longitudinal filaments.

(c) **Neutral Surface :** When equal and opposite couples are applied at the ends of a beam in a plane parallel to its length, the beam bends into circular arc. Fig. 11 shows the vertical section of such a beam. Due to bending, the filaments lying on concave side are compressed while those lying on the convex side are extended. There is, however, a plane in the beam in which the filaments remain unchanged in length. This is called the neutral plane or neutral surface. It passes through the centres of area of the cross-sections of the rod. In fig. 11 the middle line represents the inter-section of the neutral surface. It passes through the centres of area of the inter-section of the neutral surface by the plane of the diagram.

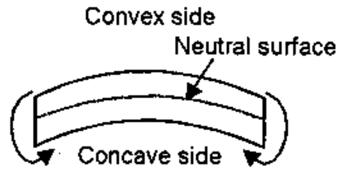

Fig. 11

(d) **Plane of Bending :** The plane in which the beam bends is called the plane of bending. Obviously, it is the vertical plane when the beam is placed horizontally. It is the plane parallel to the long axis of symmetry of the beam and passing through it and its centre of curvature.

(e) **Neutral Axis :** The line obtained by the inter-section of neutral surface and plane of bending is called neutral axis.

(f) **Bending Moment :** When a beam is bent by external applied couple, an internal restoring couple is developed at each cross-section of the beam due to its elasticity. In the equilibrium state, the restoring couple is equal and opposite to the external couple. The magnitude of this restoring couple is called the bending moment, and is obviously equal to the external couple.

• 3.9. COUPLE REQUIRED TO BEND A BEAM (BENDING MOMENT)

Fig. 12 represents the vertical section of a beam *AB* bent under the action of equal and opposite couples *T*, *T*, at its ends. The arc *NN* represents the intersection of the neutral surface by the plane of diagram. The other arcs represent the filaments in this place.

Let the beam be divided into two parts by a plane *C* and consider the equilibrium of *CB* part. Because of bending, the filaments of the beam above the neutral surface are extended while those below are compressed. The change in length of any filament is proportional to its distance from the neutral surface. There are stresses corresponding to these strains on the cross-sections of these filaments.

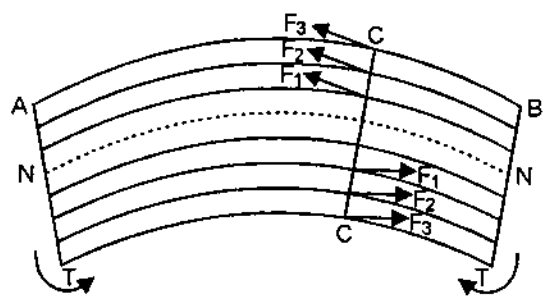

Fig. 12

The portion of a filament in the part *AC* above the neutral surface exerts an extensional force on its portion in the part *CB*. Similarly, the portion of a filament in the part *AC* below the neutral surface exerts a compressional force on its portion in the part *CB*. The magnitudes of these internal forces increase above and below the neutral surface as indicated by arrows. These forces form a system of anticlockwise couples. Their resultant is the restoring couple acting at the section *C*. The magnitude of this couple is called the bending moment. Obviously, it is exerted by the part *AC* over the part *CB*. As the part *CB* is in equilibrium, the antioclockwise restoring couple at *C* must be equal to the clockwise external couple *T* at *B*. We shall now, calculate the magnitude of restoring couple.

Figure 13 represents the part CB of the bent beam and force acting over the section C. Let it subtend an angle θ at the curvature of O. Let R be the radius of curvature of the neutral centre surface NN. Let us consider a filament PQ at a distance Z above the neutral surface. From the fig., it follows that $PQ = (R + Z)\,\theta$ and $NN = R\,\theta$. Now, before bending, the length of PQ was $R\,\theta$, *i.e.*, same as that of NN. Therefore the extension in the filament PQ is $(R + Z)\,\theta - R\,\theta = Z\,\theta$. Hence the extensional strain in this filament is

$$= \frac{\text{Extension}}{\text{Original length}}$$

$$= \frac{Z\,\theta}{R\,\theta} = \frac{Z}{R}$$

If f be the force acting on the cross-section of this filament and a be the area of cross-section, then

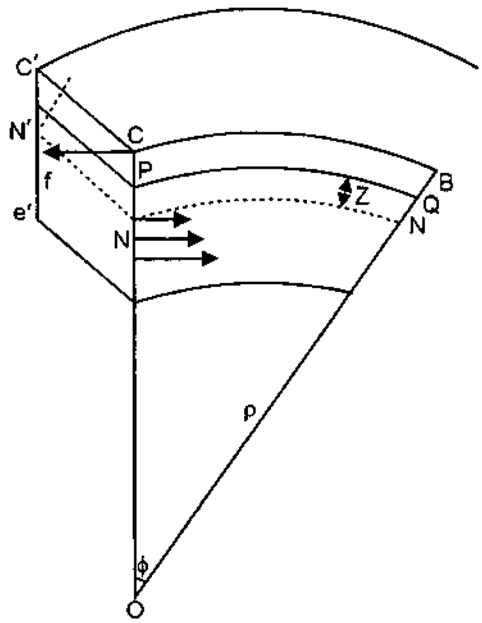

Fig. 13

$$\text{Stress} = \frac{f}{a}$$

Hence Young's modulus is given by

$$Y = \frac{\text{Stress}}{\text{Strain}} = \frac{f/a}{Z/l}$$

$$f = \frac{Ya}{l}\,.\,Z$$

The moment of this force about the neutral surface (Fig. 13) is

$$f\,Z = \frac{Ya}{l}\,.\,Z^2$$

The sum of the moments of this force about the neutral surface (Fig. 13) is

$$f\,Z = \frac{Ya}{l}\,.\,Z^2$$

The sum of the moments of all the forces acting over the whole cross-section C is the magnitude of the restoring couple or the bending moment. Therefore,

$$\text{bending moment} = \Sigma\, f\,Z$$

$$= \Sigma\,\frac{Ya}{l}\,.\,Z^2$$

$$= \frac{Y}{l}\,\Sigma\,.\,Z^2 = \frac{Y}{l}\,I$$

where $I = \Sigma\, aZ^2$, a quantity analogous to M.I. with the difference that mass is replaced by area and is called geometrical M.I. of the cross-section about the natural surface.

Now we shall consider two types of cross-section :

(i) For a rectangular cross-section

$$I = aK^2$$

where a is area of cross-section

and K the radius of gyration $= (b \times d)\,\dfrac{d^2}{12} = \dfrac{b\,d^2}{12}$

where b is breadth and d the width of rectangular beam.

Therefore bending moment for this type of cross-section

$$= \frac{Ybd^3}{12R}$$

(ii) For a circular cross-section

$$I = aK^2 = \pi r^2 \,.\, \frac{r^2}{4} = \frac{\pi r^4}{4}$$

where r is the radius of circular cross-section.

Therefore bending moment $= \dfrac{Y\,.\,\pi r^4}{4R}$.

• 3.10. BEAM LOADED AT FREE END

Let us consider a thin, uniform and light beam of lenght l, clamped horizontally at one end A [Fig. 14] and loaded with a weight ω at the free end B. When loaded end B is depressed downward compared to A, the beam undergoes bending.

The position of beam before bending is shown by dotted boundary. Such a system is called cantilever. Since the beam is light, the whole depression may be taken as due to the load W.

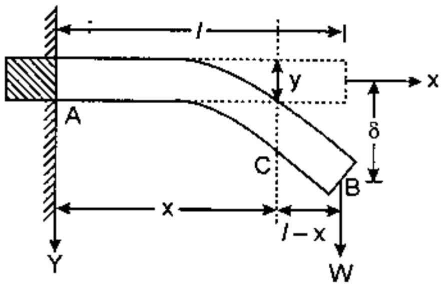

Fig. 14

Let us take a section of the beam at C, distant x from A. Consider the equilibrium of part CB. Since the beam is fixed at A, the load W at B exerts an external torque tending to rotate it clockwise. Its magnitude is obviously, $W(l-x)$. This torque is balanced by an anticlock-wise restoring couple supplied by the internal forces exerted by the part AC over the section C. These forces are caused due to the elastic reaction against the extension of filaments above the neutral surface and compression below the neutral surface. The magnitude of the restoring torque is YI/ρ where Y is Young's modulus of the beam, I the geometrical M.I. of the section C about the neutral surface and ρ the radius of curvature of the bent beam at C. At equilibrium, therefore,

$$\omega\,(l-x) = \frac{YI}{\rho} \qquad \ldots (1)$$

Let y be the depression at the point C. Taking the end A as origin, let us draw X, Y-axes as shown in the fig. Then $(x\,,y)$ are the co-ordinates of the point C, and the radius of curvature at this point is given by

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dr}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

If the depression be within elastic limit, the slope $\frac{dy}{dx}$ of the tangent at the point (x, y) will be very small and thus $\left(\frac{dy}{dx}\right)^2$ can be neglected in comparison to unity, Then

$$\rho = \frac{1}{\left(\frac{d^2y}{dx^2}\right)}$$

Putting this value of R in (1), we get

$$\omega\,(l - x) = Y\,I\,\frac{d^2y}{dx^2}$$

or
$$\frac{d^2y}{dx^2} = \frac{\omega}{Y\,I}\,(l - x)$$

Integrating this, we get

$$\frac{dy}{dx} = \frac{\omega}{Y\,I}\left(lx - \frac{x^2}{2}\right) + C_1 \qquad \ldots (2)$$

where C_1 is the constant of integration. At the fixed end A of the beam, the tangent is horizontal, *i.e.*, at $x = 0$, we have $\frac{dy}{dx} = 0$. Therefore $C_1 = 0$. Hence

$$\frac{dy}{dx} = \frac{\omega}{Y\,I}\left(lx - \frac{x^2}{2}\right) \ldots (3)$$

Integrating again, we get

$$y = \frac{\omega}{Y\,I}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

where C_2 is again a constant of integration.

Again, at $x = 0$, we have $y = 0$, so that $C_2 = 0$. Hence

$$y = \frac{\omega}{Y\,I}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

At B (where $x = l$), the depression y is maximum. Let it be equal to δ. Then, substituting l and δ for x and y, respectively in the last equation, we get

$$\delta = \frac{\omega}{Y\,I}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{\omega\, l^3}{3\,Y\,I} \qquad \ldots (5)$$

If the beam is of rectangular cross-section

$$I = \frac{bd^3}{12}$$

where b is breadth and d is the thickness of the beam. Thus

$$\delta = \frac{4\,\omega\, l^3}{Y\,b\,d^3} \qquad \ldots (6)$$

If the beam is of circular cross-section of radius r, then

$$I = \frac{\pi\, r^4}{4}$$

Thus, $$\delta = \frac{4\,\omega\, l^3}{3\,Y \pi r^4} \qquad \text{... (7)}$$

Equations (3) and (4) are the required expressions.

• 3.11. BEAM LOADED AT THE MIDDLE POINT

Let AB be a beam of rectangular cross-section resting symmetrically on two horizontal knife-edges K_1 and K_2 and loaded at its middle point D by a weight W. let l be the distance between K_1 and K_2. The reaction at each knife-edges is $\frac{W}{2}$ and acts vertically upwards. The beam is bent as shown. The depression is maximum at the loaded point.

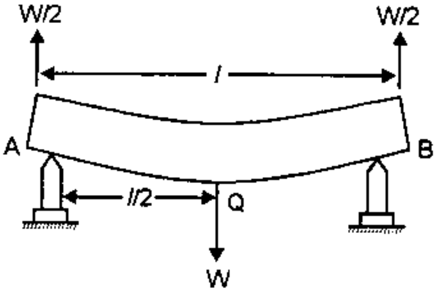

Fig. 15

It is clear from fig. 15 that the tangent at D will be horizontal. Hence each part DB_1 and DB_2, may be considered as an inverted cantilever of length $l/2$, fixed at one end D and loaded by an upward load $\frac{W}{2}$ at the other. The elevation of the loaded end of any cantilever will be the same as the depression in the middle in the actual case (fig. 16). Thus the problem is reduced to determine the elevation of B above D. Let us take a section at C distant x from O, and consider the equilibrium of the part CB.

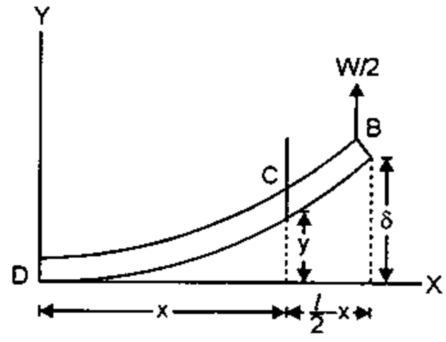

Fig. 16

Since the beam is fixed at D, the load $\frac{W}{2}$ at B exerts a torque on CB tending to rotate in anticlockwise. Its magnitude is clearly, $\frac{W}{2}\left(\frac{l}{2} - x\right)$. This torque is balanced by a clockwise restoring torque supplied by the internal forces exerted by part DC over the section C. These internal forces arise due to elastic reaction against the extension of filaments on one side of the neutral surface and compression on the other. The magnitude of the restoring torque is

$$= \frac{YI}{\rho}$$

where $Y \rightarrow$ Young's modulus of the material of the beam.

$I \rightarrow$ geometrical moment of inertia of the section C about the neutral surface.

$\rho \rightarrow$ radius of curvature at C.

Therefore, at equilibrium

$$\frac{\omega}{2}\left(\frac{l}{2} - x\right) = \frac{YI}{\rho} \qquad \text{... (1)}$$

Let us choose the co-ordinate axes X and Y as shown the origin being at D. If y be the elevation at C with co-ordinates $(x\,, y)$, the radius of curvature at C is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

If elevation y be small then $\left(\frac{dy}{dx}\right)^2$ can be neglected in comparison to one. Therefore

$$\rho = \frac{1}{\frac{d^2y}{dx^2}}$$

Substituting this value of R in (1), we get

$$\frac{W}{2}\left(\frac{l}{2} - x\right) = YI\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dx^2} = \frac{W}{2YI}\left(\frac{l}{2} - x\right)$$

Integrating this, we get

$$\frac{dy}{dx} = \frac{W}{2YI}\left(\frac{l}{2}x - \frac{x^2}{2}\right) + C_1,$$

where C_1 is the constant of integration.

At fixed point D, where $x = 0$ and $\frac{dy}{dx} = 0$. Therefore $C_1 = 0$. Hence

$$\frac{dy}{dx} = \frac{W}{2YI}\left(\frac{l}{2}x - \frac{x^2}{2}\right)$$

Integrating again, we get

$$y = \frac{W}{2YI}\left(\frac{l}{2}\cdot\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

where C_2 is another constant of integration.

Again at D, we have $x = 0$ and $y = 0$. Therefore $C_2 = 0$. Hence

$$y = \frac{\omega}{2YI}\left(\frac{l}{4}\cdot x^2 - \frac{x^3}{6}\right) \quad \ldots(2)$$

Now at the end B $\;x = \frac{l}{2}$ and $y = \delta$ (say)

The elevation δ at B is thus obtained by substituting δ for y and $l/2$ for x in eqn. (2)

$$\delta = \frac{W}{2YI}\left(\frac{l}{4}\cdot\frac{l^2}{4} - \frac{l^3}{48}\right)$$

If b and d be the width and thickness of the beam respectively, we have $I = \frac{b\,d^3}{12}$. Therefore

$$\delta = \frac{W\,l^3}{48Y\cdot\frac{bd^3}{12}} = \frac{W\,l^3}{4\,Ybd^3} \quad \ldots(3)$$

This is the same as the depression at the middle point D of the beam in the actual case.

• 3.12. DETERMINATION OF YOUNG'S MODULUS

We have

$$y = \frac{W\,l^3}{4bd^3\,\delta} \quad \ldots(4)$$

Beam is placed symmetrically on two knife edges K_1 and K_2 fixed at a distance l apart as shown in fig. 17. A hanger with a hook is placed in the middle of the beam. The load may be applied by suspending weight from the hook. The depression produced is measured by a spherometer placed in series with a battery and a galvanometer G. The galvanometer shows a deflection just as the central screw of spherometer touches the hook. At this stage, the reading of the spherometer is noted. Now the beam is loaded in equal steps and each time the screw is adjusted to touch the hanger and the reading is noted. The observation is repeated with decreasing load. The distance l, between the knife edges is measured by metre scale. The width b and the thickness d of the beam are measured by vernier callipers and screw gauge respectively. Since d is a small quantity occurring in third power, it should be measured accurately at several places and then mean value should be taken.

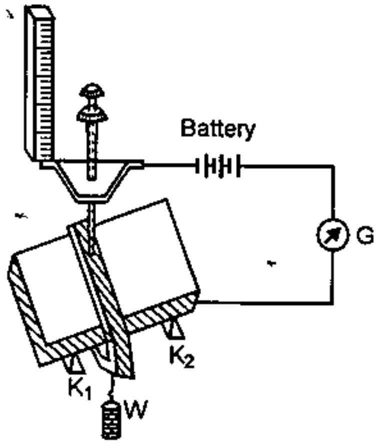

Fig. 17

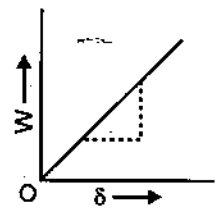

Fig. 18

A graph is plotted between the mean depression δ and the corresponding load W. The slope of the straight line so obtained gives $\frac{W}{S}$ (Fig. 18). Putting the value of $\frac{W}{S}$, l, b and d we can calculate the value of Young's modulus Y from above expression.

• STUDENT ACTIVITY

1. Define elasticity, perfectly elastic body and perfectly plastic body.

2. Define stress, strain and shear.

3. Define modulus of elasticity.

4. What is Hooke's law ?

5. Which of the two-glass and rubber is more elastic and why ?

6. What is meant by "plane of bending" ?

7. What is neutral axis ?

8. What is bending moment ?

• 3.13. SURFACE TENSION

"Surface tension is the property of the liquid by virtue of which the free surface of liquid at rest tends to have minimum area and as such it behaves as if covered with a stretched membrance.

Measurement of Surface Tension : Imagine a line AB drawn tangentially anywhere on the liquid surface. In this position the force of surface tension acts at right angles to this line on both its sides and also along the tangent to the liquid surface as shown in the fig. The surface tension may also be defined as **"the force acting per unit length is known as surface tension."**

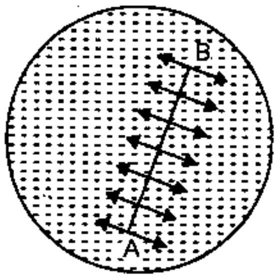

Fig. 19

Let F be the force acting on imaginary line of length l drawn tangentially on the liquid surface at rest, hence surface tension T by definition is given by

$$T = \frac{F}{l}$$

unit of surface tension is dyne/cm in C.G.S. system and Nm^{-1} in S.I. system.

The dimension formula of surface tension is $[MT^{-2}]$. Surface tension is a scalar quantity because it has no particular direction for a given liquid.

Examples of Surface Tension :

(1) Rain drops are spherical in shape because each drop tends to acquire minimum surface area due to surface tension and for a given volume the surface area of sphere is minimum.

(2) When mercury is split on a clean glass plate then it forms globules. Tiny globules are spherical due to surface tension because force of gravity is negligible while the bigger globules get flattened from the middle but have round shape near the edges.

Hence the globule takes spherical shape due to surface tension but in case of big mercury globule the force of gravity is large while centre of gravity of the big globule is lowered due to its heavy weight. Due to this, heavier big globule gets flattened from the middle.

(3) When a light iron needle is placed gently on the surface of water at rest then it does not prick the water surface and the needle floats on the surface of water. A slight depression on the surface of water is observed just below the needle which shows that the water surface behaves like a stretched membrance.

3.14. SURFACE ENERGY

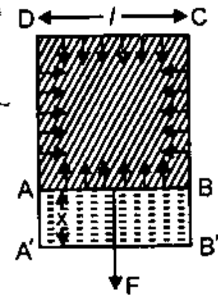

Fig. 20

We know that the liquid behaves as though its surface were covered with a skin under tension. Hence free surface of the liquid is always under tension and tends to get the least surface. If the area of the liquid surface is increased then work is done against surface tension. This work is stored in the surface in the form of potential energy.

For determination of surface energy consider a wire frame *ABCD* with two parallel sides *AD* and *BC* at distance l as shown in the fig. 20, in which the wire *AB* is movable. Now dip the frame in soap solution. Due to this a film is formed across it. Since this film has a tendency to contract, so it pulls the wire *AB* inwards by a force

$$F = T \times 2l$$

where T is surface tension.

Here l is taken twice because the film has two surfaces.

Now the side *AB* is moved to new position *A′B′* by a small distance x by the force F. In this position the work done by the external force is

$$W = \text{force} \times \text{distance} = F \times x$$
$$= T \times 2l \times x = T \times A$$

where $A = 2lx$. This is the total increase in area of the soap film on both front and back side.

so $$T = \frac{W}{A}$$

Hence the surface tension is equal to work done per unit area. Thus surface tension can also be defined in joule per square meter.

This work done is stored in the surface molecule in the form of potential energy. Hence we can say that **the potential energy per unit area of the molecules in the surface is called surface energy.** This potential energy of the film is transformed into kinetic energy of the scattered particle.

Here it is assumed that the temperature of the film remains constant when it is stretched. But in fact the temperature falls slightly in the process. Therefore to keep the temperature constant some heat must be supplied per unit change of area. Hence the total surface energy is given by

$$\boxed{E = T + H}$$

where H is heat supplied.

From above it is clear that the surface tension is equal to the mechanical part $(E–H)$ of the surface energy which is also called free surface energy.

3.15. PRESSURE DIFFERENCE ACROSS A LIQUID SURFACE

(i) From the fig. 21, we conclude that when the free surface of the liquid is plane then a molecule in the surface is attracted by other molecules equally in all directions. Hence resultant force due to surface tension on a molecule in the surface is zero as shown in the fig. 21(a).

(ii) When the liquid surface is concave as shown in fig. 21 (b) then the resultant force due to surface tension on the surface will act outwards and hence it diminishes the cohesion pressure. Due to this the curved liquid surface tends to contract. In order to keep the liquid

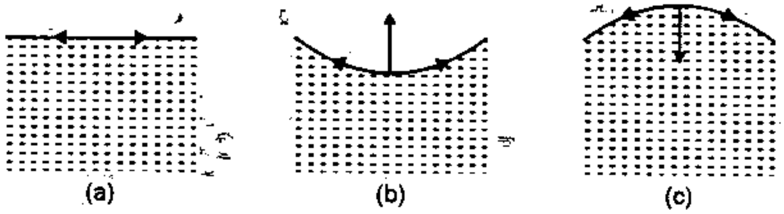

Fig. 21

surface in equilibrium, the pressure on the vapour side of the surface must be greater than pressure on the liquid side.

(iii) In the case when liquid surface is convex fig. 21(c) then the resultant force on it due to surface tension acts inwards of the liquid and hence it increases the cohesion pressure. In order to keep this surface in equilibrium the pressure on liquid side should be greater than the pressure on the vapour side.

The excess pressure can be obtained by the following formula

$$P = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad \ldots (1)$$

where T = surface tension, R_1 and R_2 = Principal radii of curvature.

Now following cases may arise :

(i) **For liquid drop :** Since liquid drop is spherical so $R_1 = R_2 = R$, *i.e.*, radius of sphere, so equation (1) reduces to

$$\boxed{P = T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{2T}{R}}$$

(ii) Air bubble in the liquid : In this case, there is only one free surface, so excess pressure is

$$P = T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{2T}{R} \qquad [\because R_1 = R_2 = R]$$

(iii) Soap bubble : A soap has two surfaces, so excess pressure is

$$P = 2T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{4T}{R}$$

(iv) Cylindrical film : For a cylindrical film $R_1 = R$ (radius of the cylinder) and $R_2 = \infty$ therefore

so $\quad P = \frac{T}{R}$ for a single surface

$P = \frac{2T}{R}$ for two surfaces.

• 3.16. CAPILLARITY

A tube with a fine and uniform bore is called a capillary tube and the phenomenon of rise or fall of liquid in a capillary tube is called capillarity.

Examples of Capillarity :

(i) The fine pores of a blotting paper act like a capillary tube. Ink rises in them leaving the paper dry.

(ii) A towel soaks water on account of capillary action.

(iii) The oil rises in the long narrow spaces between the threads of a wick because they act as fine capillaries.

(iv) Ploughing of fields is essential for preserving moisture in the soil. By ploughing, the fine capillaries in the soil are broken due which the water from within the soil does not rise and evaporate off.

• 3.17. METHOD FOR RISE OF LIQUID IN A CAPILLARY TUBE

Let us consider a glass capillary tube of uniform bore which is dipped vertically in a liquid fig. 22. This liquid rises in the tube and forms a concave meniscus as shown in the fig. Since the surface tension tends to make the area of the free-surface minimum. It acts downwards along the tangents to the liquid meniscus at the point of contact. This force (action) of surface tension acts along all the circle of contacts which pulls the tube downwards.

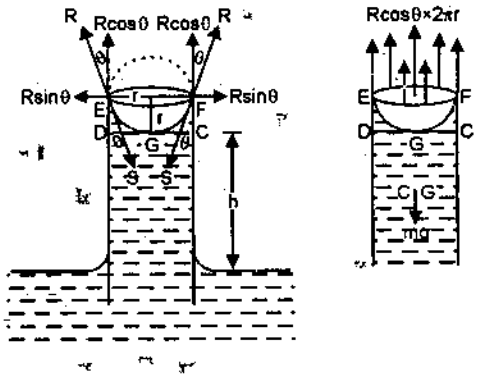

Fig. 22

At this instant another force (reaction) begins to act on the liquid mensicus in upward direction. The action is unable to pull the tube downward while reaction pulls the liquid up and the liquid goes on rising until the upward reaction is balanced by the weight of the liquid.

Let r be the radius of capillary tube, h the height of liquid rise, ρ density of liquid, θ be the angle of contact, T be the surface tension and R be the reaction, then

The reaction R has two components :

(i) $R\cos\theta$ acting tangentially upward

(ii) $R\sin\theta$ acting perpendicular to the wall of the tube.

Here only $R\cos\theta$ is responsible for the rising of the liquid in the tube.

Now, from the fig. the circumference of the circle $= 2\pi r$

$\therefore$ Total force (upward) $= R\cos\theta \times 2\pi r$

$$= T\cos\theta \times 2\pi r \qquad [\because R = T]$$

Volume of the liquid in capillary tube $= \pi r^2 h$

$\therefore$ Volume of liquid under the meniscus = volume of the cylinder *CDEF* – volume of semisphere *EFGE*

$$= \pi r^2 . r - \frac{1}{2}\frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

$\therefore$ total volume of liquid in capillary $= \pi r^2 . h + \frac{1}{3}\pi r^3$

$$= \pi r^2 \left(h + \frac{r}{3}\right)$$

Now, mass of the liquid that rises in tube = volume × density

$$\therefore \qquad m = \pi r^2 \left(h + \frac{r}{3}\right)\rho$$

weight of this liquid $= mg = \pi r^2 \left(h + \frac{r}{3}\right)\rho . g$

In equilibrium position we have

Total upward force = weight of liquid that rises in the tube

$$\therefore \qquad T\cos\theta \times 2\pi r = \pi r^2 \left(h + \frac{r}{3}\right)\rho g$$

$$\therefore \quad h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3}$$

If the tube is very narrow then $\frac{r}{3}$ can be neglected hence

$$h = \frac{2T\cos\theta}{r\rho g}$$

This is the required expression. This is known as **ascent formula.**

- **STUDENT ACTIVITY**

9. Explain why a small iron needle sinks in water while a large iron ship floats ?

10. Water on a clean glass surface tends to spread out while mercury on the same surface tends to form drops. Why ?

11. Why does oil spread over the surface of water ?

• 3.18. STREAM-LINED FLOW (STEADY FLOW)

"When a liquid flows steadily such that each particle passing through a certain point follows exactly the same path and has the same velocity as its preceding particle then this flow of liquid is known as stream-lined or steady flow".

In such a flow the velocity at every point within the liquid remains constant both in magnitude and direction.

Let us consider a liquid passing through the glass tube as shown in figure 23.

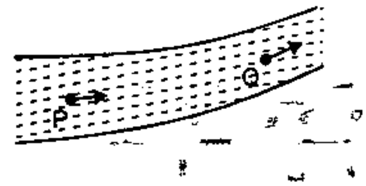

Fig. 23

If the velocity of the liquid is small then all the particles which come to P will have same speed and will move in same direction. As a particle goes from P to another point Q then its speed and direction may be changed but all the particles reaching P will have the same speed at P and all particles reaching at Q will have the same speed at Q and also if one particle passing through P has gone through Q then all the particles passing through P go through Q. Such a flow of fluid is called a **steady flow.**

In steady flow the velocity of fluid particles reaching a point is same at all times. Thus each particle follows the same path as taken by a previous particle passing through that point.

The path taken by a particle in flowing fluid is called **its path of flow.** If we draw a tangent at any point on the path of flow then it gives the direction of motion of that particle at that point. In the case of steady flow, all the particles passing through a given point follow the same path and hence we have a unique line of flow passing through a given point. In this case the line of flow is also called **a stream line.** Thus the tangent to the streamline at any point gives the direction of all the particles passing through that point. From above it is clear that the two streamlines can not intersect each others if they cut then they give two different directions of motion and this is against the streamlined flow.

• 3.19. PRINCIPLE OF CONTINUITY

According to this principle, **"When an incompressible, non-viscous fluid flows steadily through a tube of non-uniform cross-section, then the product of area of cross-section and the velocity of flow is same at every point in the tube."**

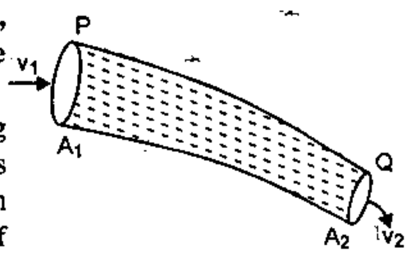

Fig. 24

Proof : Let us consider a tube of varying cross-section through which a non-viscous incompressible fluid of density ρ flows as shown in figure 24. Let A_1 and A_2 be the cross sectional areas of the tube at the points P and Q and let v_1 and v_2 be the velocities of the liquid at P and Q, respectively.

Now mass of fluid entering the area A_1 per second

= Area of cross-section × distance travelled × density

$$m_1 = A_1 \times \upsilon_1 \rho \qquad \text{... (1)}$$

Similarly, the mass of fluid leaving the area.

$$A_2 \text{ per second} = A_2 \upsilon_2 \rho$$

$$m_2 = A_2 \upsilon_2 \rho \qquad \text{... (2)}$$

Since $m_1 = m_2$ so $A_1 \times v_1 \times \rho = A_2 \times v_2 \times \rho$

$$A_1 v_1 = A_2 v_2,$$

$$\boxed{Av = \text{constant}}$$

This is the "equation of continuity". It states that the speed of flow through a tube is inversely proportional to the cross-sectional area of the tube.

3.20. VISCOSITY

The property of a liquid by virtue of which a liquid resists the relative motion between its different layers is called viscosity.

Consider a liquid flowing in a stream line over a fixed horizontal surface. The layer in contact with fixed surface will be at rest while the velocity of other layers increases uniformly with their distance from the fixed surface, *i.e.*, greater the distance of a liquid layer from the fixed surface, greater will be its velocity.

In the fig. 25 (a) a portion of the liquid which at some instant has the shape $pqrs$ becomes changed to the shape $p'q'r's'$ after a moment and this change continuously increases. Two layers are shown in the fig. 25 (b) in which the lower layer exerts a force F_1 on the upper layer which tends to drag it in backward direction while the upper layer which is faster exerts equal force F_2 on the lower layer which tends to drag it in forward direction. Thus the viscous forces tend to destroy the relative motion between the layers and so stopping the flow of liquid. Hence if the liquid is to be kept flowing then some external force must be applied to overcome the effect of viscous forces. This property of a fluid to oppose relative motion between its layers is called **viscosity** and the forces between its layers which opposes the relative motion between them are known as the **forces of viscosity**. Thus viscosity may be considered as the internal friction of a fluid in a motion.

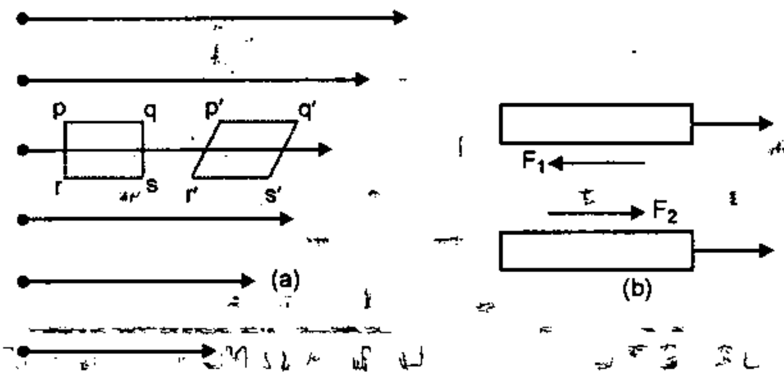

Fig. 25

Coefficient of Viscosity : Consider two layers P and Q. The layer Q moving faster than P which tries to accelerate P and the layer P which is slower tries to retard Q. Thus two layers tend to destroy their relative motion as if there is the backward dragging force acting tangentially on the layers. Therefore to maintain the flow of layer an external force equal and opposite to backward dragging force must be applied.

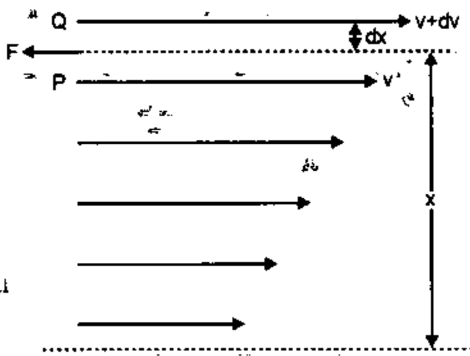

Fig. 26

According to Newton the force F depends upon the following factor

(i) $F \propto A$ (area of each layer)

(ii) $F \propto \frac{dv}{dx}$ (velocity gradient between the layers)

so $F \propto A \frac{dv}{dx}$

$$\boxed{F = -\eta A \frac{dv}{dx}}$$

where η is constant and this constant is known as **coefficient of viscosity** of the liquid. In this the negative sign shows that the direction F is opposite to the direction of velocity.

If $A = 1, \frac{dv}{dx} = 1$ then $F = -\eta$ or $\boxed{\eta = -F}$

Thus the coefficient of viscosity of a liquid is defined as **"the coefficient of viscosity is equal to the viscous force with negative sign when the velocity gradient between two layers of unit area is unity."**

Dimension of η is as follows :

$$\therefore \quad \eta = \frac{F}{A\, dv/dx} = \frac{[\mathrm{MLT^{-2}}]}{[\mathrm{L^2}]\left[\frac{\mathrm{LT^{-1}}}{\mathrm{L}}\right]} = [\mathrm{ML^{-1}T^{-1}}]$$

and unit of η is **poise** in C.G.S. system or dyne cm^{-2} sec.

In S.I. system unit of η is **Pascal-second** or **decapoise.**

and the relation between decapoise and poise is

$$\boxed{1 \text{ decapoise} = 1\ \mathrm{N\,s\,m^{-2}} = 10 \text{ poise}}$$

• 3.21. STREAMLINED, LAMINAR AND TURBULENT FLOW

The streamline flow of a liquid is that flow in which every particle of the liquid follows exactly the same path as its preceding particle and has the same velocity in magnitude and direction as that of its preceding particle while crossing through that point. Flow of liquid will be stream- lined if the velocity of liquid flow is less than the critical velocity of the liquid.

A flow of liquid in which it moves in the form of layers of different velocities which do not mix with each other, is known as **laminar flow**. In this flow the velocity of liquid is always less than the critical velocity of the liquid. In general, laminar flow is a streamline flow.

Flow of liquid will be **turbulent** if the velocity of liquid flow is greater than the critical velocity. In this flow the motion of the particles of the liquid is irregular.

• 3.22. CRITICAL VELOCITY

The critical velocity is that velocity of liquid flow upto which its flow is streamlined and above which its flow becomes turbulent. It is defined by v_c and is given by

$$v_c = \frac{k\eta}{\rho r}$$

where k is Reynold's number. η is the coefficient of viscosity, ρ is the density of the liquid and r is radius of the tube.

• 3.23. REYNOLD'S NUMBER

"It is a pure number which determines the nature of flow of liquid through a tube."

According to Reynold the critical velocity is given by

$$v_c = \frac{k\eta}{\rho r}$$

k is constant and it is known as Reynold number.

If the value of Reynold number k is less than 2000 than the flow of liquid is **streamline** and if the value of k is greater than 3000 then the flow of liquid is **turbulent.**

• 3.24. EFFECTS OF TEMPERATURE AND PRESSURE

Viscosity of liquids affected by temperature and pressure as follows :

(i) Effects of Temperature : On increasing in temperature, the viscosity of liquid decreases. The viscosity of liquid varies with temperature by the expression

$$\eta_t = \eta_0 /(1 + \alpha t + \beta t^2)$$

where η_t and η_0 are the coefficient of viscosity at t°C and 0°C respectively and α and β are constants.

But viscosity of all gases increases with increase in temperature by the relation $\eta \propto \sqrt{T}$. This result is obtained from kinetic theory of gases.

(ii) Effects of Pressure : With increase in pressure, the viscosity of liquids increases but viscosity of water decreases while the viscosity of gases remains unchanged.

• 3.25. PRACTICAL USES OF THE KNOWLEDGE OF VISCOSITY

There are some important uses of knowledge of viscosity, they are:

(i) The knowledge of viscosity and its variation with temperature helps us to select a suitable lubricant for a given machine.

(ii) The knowledge of viscosity of some organic liquids such as proteins, cellulose etc. helps us in determining their shape and molecular weight.

(iii) At railway terminals, the liquids of high viscosity are used as buffers.

(iv) Motion of some instruments is damped by using the viscosity of air or liquid.

(v) The knowledge of viscosity helped Millikan in determining charge on an electron.

(vi) The phenomenon of viscosity plays an important role in the circulation of blood through arteries and veins of human body.

• 3.26. FLOW OF LIQUID THROUGH A CAPILLARY TUBE (POISEUILLE'S FORMULA)

Poiseuille obtain the expression for the rate of flow of a liquid through a horizontal capillary tube and concluded that the volume V of the liquid flowing per second through a capillary tube depends upon following factors :

(i) $V \propto P$ (Pressure difference), (ii) $V \propto r^4$ (radius), (iii) $V \propto \frac{1}{l}$ (length), so on combining these factors we get

$$V \propto \frac{P r^4}{\eta l} \quad \text{or} \quad V = \frac{\pi}{8}\frac{P r^4}{\eta l}$$

where $\frac{\pi}{8}$ is a constant of proportionality.

This equation is known as **Poiseuille's equation** and it is true for the steady flow of liquid through a horizontal capillary tube.

Proof : Let us consider a horizontal tube of length l and radius 'r' as shown in the fig. 27. Let a constant pressure difference (P) is applied between its ends

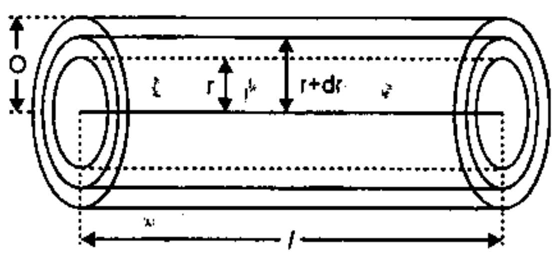

Fig. 27

on a liquid to maintain the flow of liquid. The velocity of the liquid flowing in the tube is maximum along the axis and zero at the walls of the cube. Let v be the velocity of the layer at a distance r from the axis of the tube. At each point of this layer the velocity gradient perpendicular to the direction of flow is $-dv/dr$. Here $-$ ve sign shows that v decreases as r increases.

According to Newton's Hypothesis the viscous force due to the layer at a distance r on the cylinder = coefficient of viscosity × surface area of the layer × velocity gradient

$$F_1 = \eta\,(2\,\pi r\,l)\left(-\frac{dv}{dr}\right) \quad \ldots (1)$$

and the force which tends to accelerate this liquid cylinder

= pressure difference × area of the cross-section

$$F_2 = P\,(\pi\,r^2) \quad \ldots (2)$$

From (1) and (2)

$$F_1 = F_2 \quad \text{(since no acceleration of the liquid)}$$

$$P\,\pi r^2 = -\eta\,(2\,\pi r\,l)\,\frac{dv}{dr}$$

$$\frac{dv}{dr} = -\frac{P}{2\,\eta l}\,r$$

On integrating we get

$$v = -\frac{P}{2\,\eta\,l}\left(\frac{r^2}{2}\right) + A \quad \ldots (3)$$

where A is the constant of integration whose value can be obtainted by boundary conditions, *i.e.*,

at $r = a$, $v = 0$ so by (3)

$$0 = -\frac{P}{2\,\eta l}\left(\frac{a^2}{2}\right) + A$$

$$A = \frac{P}{2\,\eta l}\left(\frac{a^2}{2}\right)$$

Putting this value in (3)

$$\boxed{v = \frac{P}{4\,\eta l}\,(a^2 - r^2)} \quad \ldots (4)$$

This is the equation of parabola. From this equation it is clear that it is independent of the length l of the tube because $\frac{P}{l}$ is the pressure drop per unit length and has the same value at each point along the length of the tube.

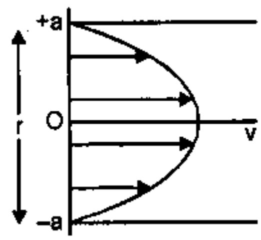

Fig. 28

If a graph is plotted between v and r then a curve of parabolic shape is obtained. In this position we say that the flow has a **parabolic velocity profile.**

Now the volume of the liquid flowing per second is

$$dV = \text{velocity} \times \text{cross–sectional area of the cylinder of radius } r$$

$$dV = v\,(2\,\pi r\,dr) = \frac{P}{4\,\eta l}\,(a^2 - r^2)\,2\,\pi r\,dr \quad \text{[by (4)]}$$

The volume of the liquid flowing per second through whole tube is

$$V = \int_0^a \frac{P}{4\eta l}(a^2 - r^2)\, 2\pi r\, dr = P\frac{2\pi}{4\eta l}\int_0^a (a^2 r - r^3)\, dr$$

$$= \frac{\pi P}{2\eta l}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a = \frac{\pi P a^4}{8\eta l}$$

$$\boxed{V = \frac{\pi P a^4}{8\eta l}}$$

This is required expression. This is known as Poiseuille's formula.

(a) Limitations and Corrections to Poiseuille's Formula : The limitations of Poiseuille's formula are as follows :

(i) This formula applies only to streamline flow through the tube. The flow is said to be streamline when the velocity of flow is less than the critical velocity. Since the critical velocity of the liquid is inversely proportional to the radius of the tube, so this flow will tend to become turbulent if tubes are of wide bore. Thus, **Poiseuille's formula is valid for narrow tubes only.**

(ii) The formula breaks down if the liquid layers in contact with the walls are not stationary. For this the pressure difference across the capillary should be kept low so that liquid flows very slowly through the tube.

(iii) The Poiseuille's formula holds good only so long as the tube is horizontal and escaping fluid has negligible kinetic energy.

(iv) Poiseuille's formula is not valid for gases.

(b) Corrections :

(i) The flow has been assumed to be steady and uniform throughout the length of tube. But in practice we see that the flow of liquid is slightly accelerated at the entrance and becomes uniform after travelling a small length of the tube. This difficulty is removed by taking the effective length of the tube which is $(l + 1.64\, a)$ in place of l where 'a' is the radius of the tube.

(ii) It is further assumed that the applied pressure difference between the ends of the flow tube is entirely used in over-coming viscous resistance but in fact a part of it is used in imparting kinetic energy to the fluid leaving the tube. For removing this difficulty the effective pressure difference between the ends of the tube should be taken as

$$\left(P - \frac{V\rho}{\pi a^2}\right)$$

where ρ is the density of the liquid.

• 3.27. ROTATION VISCOMETER

This is a device which is used to determine the viscosity of highly viscous liquids and of gases also.

Construction : The actual form of apparatus is shown in the figure 29. It consists of two coaxial metal cylinders. The inner cylinder is suspended inside the outer cylinder by a phosphor-bronze suspension wire and the outer cylinder is clamped. The space between two cylinders is filled with experimental liquid. A small mirror M is connected with phosphor-bronze wire. The outer cylinder is rotated with a constant angular velocity by an electric motor and number of rotations is recorded on an automatic counter.

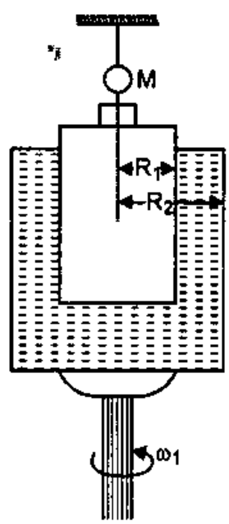

Fig. 29

Theory : When the outer cylinder is set in rotation with a certain angular velocity then the layers of liquid which is in contact with outer cylinder also rotates with the same velocity whereas the layer in contact with the inner cylinder is at rest. Due to this a relative motion is set up between the different layers of the liquid and the viscous forces are produced.

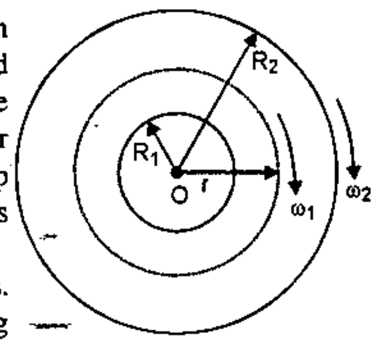

Fig. 30

Let us consider the cross-sections of two cylinders. Let R_1 and R_2 be their radii and let outer cylinder be moving with the angular velocity ω_2 (fig. 30). Let us consider an imaginary coaxial cylinder of radius r and length l. Let ω be the angular velocity of the surface of this cylinder. In this position the linear velocity gradient at distance r from the centre will be

$$\frac{d}{dr}(r\,\omega_1) = \omega_1 + r\,\frac{d\,\omega_1}{dr} \quad \ldots (1)$$

If the liquid had no viscosity then it would have rotated like a rigid body with a constant angular velocity ω_1 so

$$\frac{d}{dr}(r\,\omega_1) = \omega_1$$

the term $r\,\dfrac{d\,\omega_1}{dr}$ is the velocity gradient due to viscosity effects.

Now, from Newton's hypothesis, the tangential viscous force F at the surface of imaginary cylinder exerted by the liquid outside it is given by

$$F = \text{viscosity} \times \text{surface area} \times \text{velocity gradient}$$

$$F = \eta\,(2\,\pi r\,l)\left(r\,\frac{d\,\omega_1}{dr}\right)$$

and torque is $\tau = Fr$

$$\tau = \eta\,(2\,\pi r\,l)\left(r\,\frac{d\,\omega_1}{dr}\right)r = 2\,\pi\,\eta\,lr^3\,\frac{d\omega_1}{dr} \quad \ldots (2)$$

This torque is constant throughout the liquid.

by (2) $$2\,\pi\,\eta l\,d\omega_1 = \tau\,\frac{dr}{r^3}$$

On integrating

$$2\,\pi\,\eta\,l\,\omega_1 = -\frac{\tau}{2}\left[\frac{1}{r^2}\right] + A_1$$

where A_1 is the constant of integration.

Since at $r = R_1$ $\omega_1 = 0$ then we get

$$A_1 = \frac{\tau}{2}\,\frac{1}{{R_1}^2}$$

$$\therefore \quad 2\,\pi\,\eta\,l\,\omega_1 = -\frac{\tau}{2}\cdot\frac{1}{r^2} + \frac{\tau}{2}\,\frac{1}{{R_1}^2}$$

For outer cylinder put $r = R_2$ and $\omega_1 = \omega_2$

$$2\pi\eta l\omega_2 = -\frac{\tau}{2}\frac{1}{R_2^2} + \frac{\tau}{2}\frac{1}{R_1^2}$$

$$= \frac{\tau}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right]$$

$$\therefore \quad \tau = 4\pi\eta l\omega_2 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2)} \quad \ldots (3)$$

This torque is balanced by the restoring torque in the suspension. Let ϕ be the steady angular deflection of the inner cylinder and C be the restoring torque per angle twist in the suspension.

The restoring torque for the twist $= C\phi$

Hence by (3)

$$\tau = 4\pi\eta l\omega_2 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2)} = C\phi$$

$$\therefore \quad \boxed{\eta = \frac{C\phi(R_2^2 - R_1^2)}{4\pi l\omega_2 R_1^2 R_2^2}} \quad \ldots (4)$$

This is expression for viscosity.

End correction : It is found that a torque is also exerted on the bottom of the inner cylinder by the liquid between the bottom and the base of the rotating cylinder. This effect can be eliminated by using two different lengths l_1 and l_2 of the liquid covering the inner cylinder. In this position τ_1 and τ_2 are the torques exerted on the inner cylinder in the two cases. Then

So,
$$\tau_1 = 4\pi\eta l_1 \omega_2 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2)} + \tau(B) = C\phi_1 \quad \ldots (5)$$

$$\tau_2 = 4\pi\eta l_2 \omega_2 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2)} + \tau(B) = C\phi_2 \quad \ldots (6)$$

where $\tau(B)$ is the torque of bottom. On solving these eqns. (5) and (6) we get

$$\eta = \frac{C(\phi_1 - \phi_2)(R_2^2 - R_1^2)}{4\pi(l_1 - l_2)\omega_2 R_1^2 R_2^2} \quad \ldots (7)$$

The value of C is obtained by the following formulae

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{and} \quad T' = 2\pi\sqrt{\frac{I+I'}{C}}$$

where T is time period when the inner cylinder **alone** is set into torsional oscillations about the wire ; T' is the time period when an angular metallic disc of known moment of inertia is placed centrally on the cylinder and I and I' are the moment of inertia of the inner cylinder and angular disc about the wire respectively.

From above $C = \dfrac{4\pi^2 I'}{(T'^2 - T^2)}$

on putting this value in (7), η can be calculated.

• 3.28. STOKE'S LAW

When a small spherical body moves through a viscous medium at rest then the layers of the body touching the medium are dragged along with it. But the layer of the medium (liquid) away from the body are at rest. Due to this, a relative motion is produced between the different layers of the medium.

Hence, a viscous force begins to act, which opposes the motion of the body. This viscous force increases with the velocity of the body and finally becomes equal to force driving the body. The body then falls with a constant velocity which is known as **terminal velocity.**

Stoke found that the viscous force F acting on a small spherical body is given by

$$F = 6\pi\eta r v$$

This is known as "Stoke's Law of Viscosity".
where r is the radius of body, v is the terminal velocity and η is the coefficient of viscosity.

This law can be derived dimensionally in the following way.

Let $$F = K\eta^a v^b r^c \qquad \text{... (A)}$$

where K is dimensionless constant.

On writing the dimensions

$$[MLT^{-2}] = [ML^{-1}T^{-1}]^a [LT^{-1}]^b [L]^c$$

$$[MLT^{-2}] = [M^a L^{-a+b+c} T^{-a-b}]$$

On comparing $\boxed{a = 1}$

and $$-a + b + c = 1$$

$$-1 + b + c = 1 \qquad [\because a = 1]$$

$$b + c = 2 \qquad \text{... (1)}$$

and $$-a - b = -2$$

$$-1 - b = -2$$

$$\boxed{b = 1}$$

Put this value in (1) we get

$$\boxed{c = 1}$$

On putting the value of a, b and c in (A) we get

$$F = K\eta r v$$

Experimentally the value of K was found to be 6π *i.e.*,

$$F = 6\pi\eta r v$$

This law is valid only when the sphere is small in size, it moves with terminal velocity and the fluid is perfectly homogenous.

Terminal velocity : Let us consider a small sphere of radius r and density ρ, falling freely from rest under gravity through a liquid of density σ and coefficient of viscosity η (fig. 31). When it moves with the terminal velocity v then the following forces act on it.

(i) downward force due to gravity = weight of the body

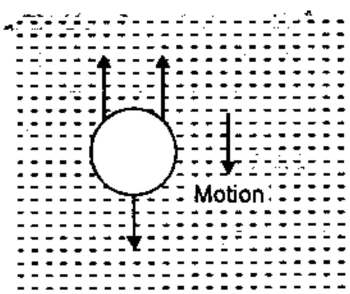

Fig. 31

$$= \frac{4}{3} \pi r^3 \rho g$$

(ii) upward thrust due to buoyancy

$$= \frac{4}{3} \pi r^3 \sigma g$$

(iii) viscous force = $6 \pi \eta r v$

The resultant downward driving force

$$= \frac{4}{3} \pi r^3 \rho g - \frac{4}{3} \pi r^3 \sigma g = \frac{4}{3} \pi r^3 (\rho - \sigma) g$$

when the sphere attains constant velocity then this driving force is equal to viscous force, *i.e.*,

$$\frac{4}{3} \pi r^3 (\rho - \sigma) g = 6 \pi \eta r v$$

$$\boxed{v = \frac{2}{9} \frac{r^2 (\rho - \sigma) g}{\eta}}$$

This is the required expression for terminal velocity.

Importance of Stoke's Law :

Some important applications of Stoke's law are as follows :

(i) This law is used in the determination of elecrtric charge with the help of Millikan's experiment.

(ii) This law accounts for the formation of clouds.

(iii) This law gives, why the speed of rain drops is less than the speed of body falling freely with a constant velocity from the height of clouds.

• SUMMARY

- Property of the body to regain its original form, on the removal of the deforming force, is called elasticity of the body.
- Modulus of elasticity or coefficient of elasticity of a body is defined as the ratio of the stress to the corresponding strain produced, within the elastic limit.
- Young's Modulus of Elasticity is defined as "within the limit of elasticity the ratio of normal stress to longitudinal strain.
- Bulk Modulus of Elasticity is defined as "the ratio of normal stress to the volumetric strain within the limit of elasticity
- Modulus of Rigidity is defined as the ratio of tangential stress to the shearing strain.
- The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio (σ).
- Hooke's Law states that "within the limits of elasticity the stress is proportional to the strain."
- Angle of Twist (θ) is the angle by which the radius of each circular cross section of the cylinder rod is turned about its axis.
- The angle of shear (ϕ) is the angle formed by the line generated parallel to he axis of the cylindrical root as a result of restoring couple.
- Surface tension is the property of the liquid by virtue of which the free surface of liquid at rest tends to have minimum area and as such it behaves as if covered with a stretched membrance.
- The potential energy per unit area of the molecules in the surface is called surface energy.
- A tube with a fine and uniform bore is called a capillary tube and the phenomenon of rise or fall of liquid in a capillary tube is called capillarity.

- The ascent formula is used for determining the rise of liquid in a capillary tube. It is given by $h = \frac{2T\cos\theta}{r\rho g}$.
- Principle of Continuity states that "When an incompressible, non-viscous fluid flows steadily through a tube of non-uniform cross-section, then the product of area of cross-section and the velocity of flow is same at every point in the tube."
- The property of a liquid by virtue of which a liquid resists the relative motion between its different layers is called viscosity.
- Reynold's number "It is a pure number which determines the nature of flow of liquid through a tube."
- The equation $V = \frac{\pi}{8}\frac{P\,r^4}{\eta l}$ is known as **Poiseuille's equation** and it is true for the steady flow of liquid through a horizontal capillary tube.
- "Stoke's Law of Viscosity" $F = 6\,\pi\,\eta\,r\,v$, where r is the radius of body, v is the terminal velocity and η is the coefficient of viscosity.

• STUDENT ACTIVITY

12. Water flows faster than honey. Why ?

13. A bigger rain drop falls faster than smaller one. Why ?

14. What is an ideal liquid ?

• TEST YOURSELF

1. Deduce the relaion among the elastic constants γ, κ, η and σ.
2. Define angle of twist and angle of shear. Dedue the expression for the couple required to twist a uniform cylinder.
3. What is Maxwell's needle ? Describe and explain how Maxwell's needle can be used to determine the modulus of rigidity of the material of wire.
4. A uniform beam is clamped horizontally at one end and loaded at the other. Calculate the depression at the free end.
5. Define surface tension. Write its SI units and dimensions. Give some important examples.
6. What is surface energy of a liquid ? Obtain the relation between surface tension and surface energy.
7. What is capillarity ? Give practical examples of caillarity.
8. Describe the method of finding the surface tnsion of water, using capillary rise method.
9. Explain streamlined flow. State and prove the principle of continuity in the flow of liquids.
10. Derive Poisenseuille's formula for the viscosity of a liquid flowing through a narrow tube.
11. Describe with necessary theory, the rotation viscometer method of determining the coefficient of viscosity of a fluid.
12. Derive Stoke's formula for the velocity of a small sphere falling through a viscous liquid, using dimension method. Hence obtain the expression for the terminal velocity.
13. Dimensional formula for modulus of elasticity is :
    (a) $[MLT^{-2}]$ (b) $[ML^{-1}T^{-2}]$ (c) $[ML^{-2}T^{-1}]$ (d) $[ML^{-2}T^{-2}]$
14. Energy per unit volume in a stretched wire is :
    (a) $\frac{1}{2} \times$ load $\times$ strain (b) load $\times$ strain
    (c) stress $\times$ strain (d) $\frac{1}{2} \times$ stress $\times$ strain
15. The Young's modulus of a wire is numerically equal to the stress which will :
    (a) not change the length of wire
    (b) double the length of wire
    (c) increase the length by 50%
    (d) change the area of cross-section of wire to half
16. The Poisson ratio can not have the value :
    (a) 0.7 (b) 0.2 (c) 0.1 (d) 0.5
17. What is the relation between Y, B and η for isotopic material ?
    (a) $\eta = \frac{3BY}{9B+Y}$ (b) $\eta = \frac{9BY}{4B+Y}$
    (c) $\eta = \frac{9BY}{9B-Y}$ (d) $Y = \frac{9B\eta}{3B+\eta}$
18. Poisson's ratio is equal to :
    (a) $\frac{\text{Lateral Strain}}{\text{Longitudinal Strain}}$ (b) $\frac{\text{Longitudinal Strain}}{\text{Lateral Strain}}$
    (c) Longitudinal Strain $\times$ Lateral Strain
    (d) None of these
19. On increasing temperature, the value of Young's modulus :
    (a) decreases (b) increases
    (c) remains constant (d) has no effect

20. If T is the surface tension of soap solution the amount of work done in blowing a soap bubble form a diameter D to diameter $2D$ is :

(a) $2\pi D^2 T$ (b) $4\pi D^2 T$ (c) $6\pi D^2 T$ (d) $8\pi D^2 T$

21. A capillary tube of radius R is immersed in water and water rises in it to a height H. Mass of water in capillary tube is M. If the radius of the tube is doubled, mass of water that will rise in capillary will be :

(a) $2M$ (b) M (c) $\frac{M}{2}$ (d) $4M$

22. The terminal velocity v of a spherical ball of radius r falling through a viscous liquid varies with r such that :

(a) $\frac{v}{r}$ = constant (b) vr = a constant

(c) vr^2 = constant (d) $\frac{v}{r^2}$ = constant

23. An ice cube containing a glass ball is floating on the surface of water contained in a trough. The whole of the ice melts, the level of water in the trough :

(a) rises (b) falls

(c) remains unchanged (d) first falls and then rises

24. A spherical liquid drop of radius R is divided into eight equal droplets. If surface tension is T, then work done in the process will be :

(a) $2\pi R^2 T$ (b) $3\pi R^2 T$ (c) $4\pi R^2 T$ (d) $2\pi RT^2$

25. The cause of viscosity of liquid is :

(a) diffusion (b) adhesive force

(c) gravitational force (d) cohesive force

ANSWERS

13. (b) **14.** (d) **15.** (b) **16.** (a) **17.** (d) **18.** (a) **19.** (a) **20.** (c) **21.** (a) **22.** (d) **23.** (b) **24.** (c) **25.** (d)

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UNIT 4

RELATIVITY

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- Michelson-Morley experiment to detect th relative motion between earth and ether.
- Special theory of relativity, its postulates and the equations of relativity.
- Various relations such as those of time, energy, velocity, etc. as per the theory of relativity.

• 4.1. FRAME OF REFERENCE

A co-ordinate system relative to which the position and motion of an object are specified is called a frame of reference, for example, aeroplane, train, car, earth etc.

There are two types of frames of reference.

(a) Inertial frame : The non-accelerated frames are known as inertial frames or we can say that the frames in which Newton's laws hold good are called inertial frames. All inertial frames are equally valid. There is no universal frame of reference that can be used everywhere and no absolute motion is possible.

(b) Non-inertial frames : The accelerated frames are called non-inertial frames.

• 4.2. MICHELSON–MORLEY EXPERIMENT

According to Huygen's theory, light motion is a wave motion. But for the propagation of wave, a medium is required. For this he imagined an imaginary (hypothetical) medium, called ether and this ether is transparent, highly elastic and filled in all space. Now the question arises whether the ether remains stationary in space or it is in motion. If it is stationary then there **must be relative motion between a body (Earth) and Ether. Thus, to detect the relative motion between earth and ether many experiments were performed and one of famous experiments is Michelson–Morley experiment.**

They used a device known as interferometer which is based on the interference of light. The experiment is shown in figure 1. S is the monochromatic source of light. P is a semi-silvered plate held at 45°. M_1 and M_2 are two fully silvered mirrors; placed at perpendicularly each other and at equal distance l from the plate P. T is the telescope in which the fringes are to be seen.

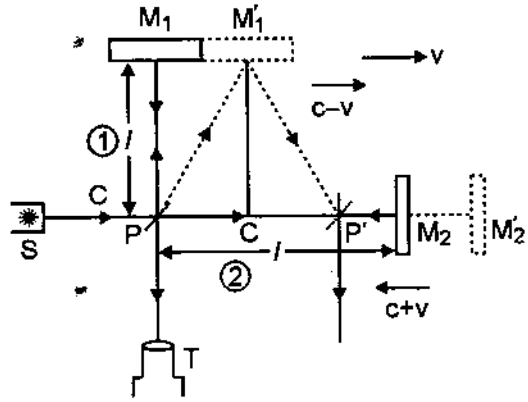

Fig. 1

Working : Light from the source S falls on the plate P. It is divided into two beams (1) and (2). The beam (1) travels to the mirror M_1 and is reflected back. The beam (2) travels towards M_2 and is reflected back. These two beams recombine at P and then enter into the telescope T in which interference fringes are observed.

If the interferometer (earth) is at rest in ether then the two beams would take the same time to return to P. But in fact the earth, *i.e.*, interferometer is moving in ether (space) with a velocity v (speed of earth in its orbit). The interferometer is so adjusted that it is always moving in the direction of incident beam of light and ether is assumed to be stationary. In this position the time taken by the beams (1) and (2) on their journeys is not equal.

Let c be the speed of light then velocity of light beam (2) towards the mirror M_2 is

$$= c - v$$

Time taken in this journey $= \dfrac{l}{c - v}$

Velocity of light beam (2) after reflection from the mirror M_2

$$= c + v$$

Time taken in this journey $= \dfrac{l}{c + v}$

Total time taken $\quad T_1 = \dfrac{l}{c - v} + \dfrac{l}{c + v}$

$$T_1 = \frac{2lc}{(c^2 - v^2)}$$

$$T_1 = \frac{2lc}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

$$T_1 = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$\boxed{T_1 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right)}$$

Since the interferometer is moving with velocity v so it will cover a distance vt in time t. Hence the new position of the mirrors M_1 and M_2 will be M_1' and M_2' and the new position of the plate P will be P', where $M_1 M_1' = vt$. In this position the beam (1) will follow the path $P M_1'$ instead of PM_1 where $PM_1 = ct$ and is shown by dotted line.

$\therefore$ In triangle $P M_1' C$

$$(P M_1')^2 = (PC)^2 + (C M_1')^2$$

$$P M_1' = ct,\ PC = M_1, M_1' = vt,\ C M_1' = P M_1 = l$$

$$(ct)^2 = (vt)^2 + (l)^2$$

$$t = \frac{l}{\sqrt{(c^2 - v^2)}} = \frac{l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

total time (T_2) taken by the beam (1) to travel the whole path $P M_1' P'$ then

$$T_2 = 2t = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\boxed{T_2 = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)}$$ neglecting the higher terms.

Time difference between the two beams

$$\Delta T = T_1 - T_2$$

$$\Delta T = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

$$\Delta T = \frac{2l}{c} \cdot \frac{v^2}{2c^2}$$

$$\Delta T = \frac{lv^2}{c^3}$$

Path difference = speed × time difference

$$= c \times \frac{l v^2}{c^3}$$

$$= \frac{l v^2}{c^2}$$

If the apparatus is rotated through 90° then the
New path difference

$$= -\frac{l v^2}{c^2}$$

$$\text{Total path difference} = \frac{l v^2}{c^2} - \left(-\frac{l v^2}{c^2}\right)$$

$$= \frac{2 l v^2}{c^2}$$

No. of fringes shifted due to this path difference

$$\Delta N = \frac{\text{path difference}}{\text{wavelength of light used}}$$

$$= \frac{2 l v^2}{c^2 \lambda}$$

Michelson and Morley used $l = 11$ meter, $c = 3 \times 10^8$ m/sec $v = 3 \times 10^4$ meter/sec and $\lambda = 6 \times 10^{-7}$ meter (approx.)

$$\text{Expected fringe shift} = \frac{2 l v^2}{c^2 \lambda}$$

$$\Delta N = \frac{2 \times 11 \times (3 \times 10^4)^2}{(3 \times 10^8)^2 \times (6 \times 10^{-7})}$$

$$= 0.4$$

Thus a shift of 0.4 of a fringe width was expected. But no fringe shift was observed. The experiment was repeated at different times and various seasons of the year, at different places but no shifting was observed. This shows that this experiment gives a negative result. **The negative result suggests that the velocity of light is constant in all directions.**

Thus, **the relative motion between earth and ether could not be detected.**

Explanation of the Negative Results : Many theories were given to explain the negative or null results of the Michelson-Morley experiment. The important ones are :

(i) Ether drag hypothesis : According to it, the moving earth drags the ether with it, hence there is no relative motion between the two so that no shift is observed. This explanation was not accepted for two reasons :

(a) It is against the observed phenomenon of aberration of light from stars.

(b) Fizeau experimental observation of particle dragging of light waves by a moving body, was explained on the basis of electromagnetic theory, without considering the ether-drag hypothesis.

(ii) Fitzgerald–Lorentz contraction hypothesis : According to this hypothesis, all material bodies moving through the ether are contracted in length along the direction of motion by the factor $\sqrt{1-\frac{v^2}{c^2}}$. This contraction in the interferometer arm prevents the shift of the fringes. Rayleigh had shown that such a contraction is expected to provide double refraction, which was never observed. This explanation was also rejected.

(iii) Light velocity hypothesis : According to this hypothesis, the velocity of light from a moving source is the vector sum of its natural velocity and the velocity of source. This explanation was also rejected because it is against the velocity of light and some astronomical evidences.

• 4.3. BASIC POSTULATES OF SPECIAL THEORY OF RELATIVITY

In 1905 Einstein presented his famous theory of relativity. It is based upon two postulates :

Postulate I : The laws of physics are the same in all inertial frames of reference. It is also known as **principle of relativity :** This means that if any physical quantity like kinetic energy, angular momentum etc., is conserved in one inertial frame then it would be conserved in all the inertial frames. Similarly, if it is not conserved in one inertial frame then it would not be conserved in any other inertial frame.

"According to this postulate, there is only relative motion and no absolute motion, *i.e.*, it is impossible to find absolute motion. In other words, we can say that there is no universal frame of reference relative to which motion of other frames could be detected.

Postulate II : The speed of light in free space (c) has the same value in all inertial frames of reference, *i.e.*, the speed of light is constant.

• 4.4. GALILEAN OR NEWTONIAN TRANSFORMATIONS

Galilean transformations (equations) can be applied only for smaller velocities, not for high velocities, *i.e.*, comparable to the velocity of light (c).

Let us consider two frames S and S' moving in x direction, Let S' be moving with velocity v relative to S.

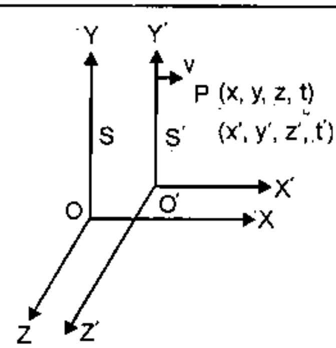

Fig. 2

To obtain transformation equations, let us consider two inertial frames S and S' as shown in the fig. 2. Let S' be moving with velocity v relative to S along the direction of x–axis. Let us consider that at an instant the origin O and O' of two frames. Suppose a pulse of light is generated at $t=0$ at the origin O. It will spread into the space and consider the situation when pulse reaches at P. Observers at O and O' measure co-ordinates of P as (x, y, z, t) and (x', y', z', t') respectively. When pulse is observed by an observer in S then

$$\text{Velocity of light} = \frac{\text{distance}}{\text{time}}$$

$$c = \frac{(x^2+y^2+z^2)^{1/2}}{t}$$

$$x^2+y^2+z^2 = c^2 t^2 \quad \ldots (1)$$

and when the pulse is observed in S' we have

$$x'^2+y'^2+z'^2 = c^2 t'^2 \quad \ldots (2)$$

According to Galilean transformation

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

These equations are known as Galilean transformation (equations).

Putting these values in (2)

$$x'^2+y'^2+z'^2 = c^2 t'^2$$

$$(x-vt)^2+(y)^2+(z)^2 = c^2 (t)^2$$

$$x^2 - 2xvt + v^2 t^2 + y^2 + z^2 = c^2 t^2$$

which is not same as equation (1).

This shows clearly that Galilean transformation equations are not valid for very high velocities, *i.e.*, velocities comparable to the velocity of light and this is the violation of I postulate.

Again, if we measure the speed of light in x direction in the S frame to be c, then from the frame S' it will be $c' = c - v$. This is the violation of II postulate.

• 4.5. LORENTZ TRANSFORMATION EQUATIONS OF RELATIVITY

These equations were obtained by Lorentz and these equations relate the co-ordinates of any system into another, when we deal with particles like electrons, protons, mesons etc., which move with velocities comparable with c. These equations are known as "Lorentz Transformation Equations".

From the frame S and S' it is clear that

$$x' \propto (x - vt)$$

or

$$x' = \gamma (x - vt) \quad \ldots (1)$$

where γ is a factor which does not depend upon either x or t (it may depend upon v)

Similarly, $$x = \gamma (x' + vt') \quad \ldots (2)$$

Putting the value of (1) in (2)

$$x = \gamma [\gamma (x - vt) + vt']$$

$$\frac{x}{\gamma} = \gamma x - \gamma vt + vt'$$

$$vt' = \frac{x}{\gamma} + \gamma vt - \gamma x$$

$$= \gamma\left[\frac{x}{\gamma^2} + vt - v\right]$$

$$t' = \frac{x}{\gamma v} + \gamma t - \frac{\gamma x}{v}$$

$$t' = \gamma t - \frac{\gamma x}{v}\left(1 - \frac{1}{\gamma^2}\right) \qquad \ldots (5)$$

Similarly, $$t = \gamma t' + \frac{\gamma x}{v}\left(1 - \frac{1}{\gamma^2}\right) \qquad \ldots (6)$$

The value of γ can be calculated from II postulate. Let a light pulse be sent at the origin O at time $t = t' = 0$ and $x = x' = 0$, *i.e.*, when O and O' coincide then light pulse in frame S, $x = ct$ and light pulse in frame S' $x' = ct'$

Putting these values in (1) and (2)

$$c\,t' = \gamma\, t\,(c - v)$$

$$c\,t = \gamma\, t'\,(c + v)$$

On multiplying

$$(ct)\,.\,(ct') = r^2\, t\, t'\,(c - v)\,(c + v)$$

$$c^2 t\, t' = r^2\, t\, t'\,(c^2 - v^2)$$

$$c^2 = r^2\,.\,c^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\boxed{r = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Putting in (1) and (2)

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ x &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}} \qquad \ldots \text{(a)}$$

Again from $$r = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$r^2 = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$\frac{1}{r^2} = 1 - \frac{v^2}{c^2}$$

$$1-\frac{1}{\gamma^2}=1-\left(1-\frac{v^2}{c^2}\right)$$

$$1-\frac{1}{\gamma^2}=\frac{v^2}{c^2}$$

Put in (3) and (4)

$$t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

and $$t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{...(b)}$$

Since the motion is in x direction therefore y and z components will be unaffected. That is

$$y'=y \quad \text{... (c)}$$

$$z'=z \quad \text{... (d)}$$

Equations (a), (b), (c) and (d) are known as Lorentz transformation equations.

If $v <<< c$ then Lorentz transformation equations reduce to Galilean (Newtonion or Classical) equations, *i.e.*, Lorentz trnasformation are applicable only when velocities are very large.

• 4.6. LENGTH CONTRACTION

When a body moves with very high speed v relative to an observer then it appears to be contracted by a factor $\sqrt{1-\frac{v^2}{c^2}}$ in the direction of motion only. (not in perpendicular direction). This concept is known as length contraction and this shows clearly that length is not absolute but relative.

If L_0 is the proper length (actual length) and L is apparent length (improper length) then

$$L=L_0\sqrt{1-\frac{v^2}{c^2}}$$

i.e., $$L<L_0$$

This means that L is decreased to L_0 by a factor $\sqrt{1-\frac{v^2}{c^2}}$.

Proof : Let us consider two frames of reference moving in X direction, let S' be moving with velocity v relative to S.

Let a rod be placed in S' and let it be in rest. Let x_1' and x_2' be the co-ordinates of the ends of the rod, the length L_0 of the rod in S' is given by

$$L_0=x_2'-x_1' \quad \text{... (1)}$$

Now let an observer in S measure the length of the rod when S' is in motion. If x_1 and x_2 are the co-ordinates of the ends of the rod, then length of the rod L for him is given by

$$L = x_2 - x_1 \qquad \ldots (2)$$

Using Lorentz transformation

From

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \qquad x_1' = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } x_2' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting in (1)

$$L_0 = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_0 = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_0 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{L = L_0 \sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{i.e., } L < L_0$$

This shows clearly the L appears to be contracted by the factor $\sqrt{1 - \frac{v^2}{c^2}}$. This contraction is known as **Lorentz-Fitzgerald contraction.**

when $v = c$ then $L = 0$. This means that a rod moving with the speed of light will appear as reduced to a point to a stationary observer.

If the rod is placed perpendicular to the motion of reference frame, *i.e.*, along y' or z' direction, then no contraction in its length along these directions is observed. This is due to invariance in Lorentz transformations for these co-ordinates, *i.e.*, $y = y'$ and $z = z'$.

Thus a moving square appears as rectangle with a shorter side along its motion (x–axis) and a circle appears as ellipse with minor axis along its motion as shown in the following figure.

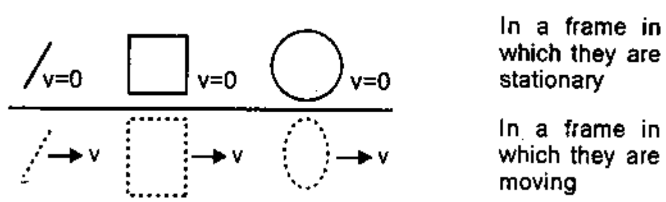

Fig. 3

• 4.7. TIME DILATION

The word **dilate** means "enlarge beyond normal size". Time is also not absolute but relative. It is found that a moving clock in the spacecraft appears to tick at a slower rate than the stationary one on the ground, as seen by an observer on the ground.

Thus, if a clock moves with very high speed v relative to an observer then it appears to be slowed down or time interval is increased (dilated) by a factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

In other words we can say that the tick-tick of a moving clock are slower than a clock at rest. This concept is known as 'time dilation'.

Consider a clock placed at point x' in the moving frame S' and an observer in S' feels that the clock gives two ticks at time t_1' and t_2' then the time interval between the ticks is

$$t_0 = t_2' - t_1'$$

t_0 is the time interval calculated in the frame in which the clock is at rest. Now let t_1 and t_2 be the time of two ticks measured in frame S' which is moving with velocity v relative to S.

The time interval appears to him as

$$t = t_2 - t_1$$

Now we have Lorentz transformations

$$t = \frac{t' + \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \qquad t_2 = \frac{t_2' + \frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } t_1 = \frac{t_1' + \frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

so

$$t_2 - t_1 = \frac{t_2' + \frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{t_2' + \frac{x'v}{c^2} - t_1' - \frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

i.e., $\qquad t > t_0$

From this equation it is clear that the stationary observer in S feels the time interval to be **lengthened** by a factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. This phenomenon is known as **Time Dilation.**

Thus in a moving frame the stationary clock measures longer time than in the case of stationary frame. In other words we can say that a moving clock appears to be slowed down to a stationary observer.

If $v = c$ then we get $t = \infty$. This means that when a clock moves with the speed of light then it will appear to be completely stopped to stationary observer.

• 4.8. SIMULTANEITY (TIME IS RELATIVE)

According to classical mechanics or Newtonian mechanics if two events are simultaneous in one inertial frame then they are also simultaneous in all inertial frame. But according to relativity or Einstein simultaneity is only a relative concept, *i.e.,* two simultaneous events occurring at two different places for an observer in S will not be simultaneous for another observer in S'. In other words we can say that **simultaneity is not absolute, it is relative**.

The simultaneity of two events means their occurrence at exactly the same time. Let us see whether two events occurring simultaneously in a stationary frame S also appear to be so in a reference frame S', moving relative to it with velocity v in the direction of x.

Let two clocks located at $x = 0$ and $x = l$. A flash bulb is located in between as shown in fig. 4. Clock starts on receiving flash of light. Flash takes $\frac{l}{2c}$ time, equal for both clocks so that clocks are exactly synchronised.

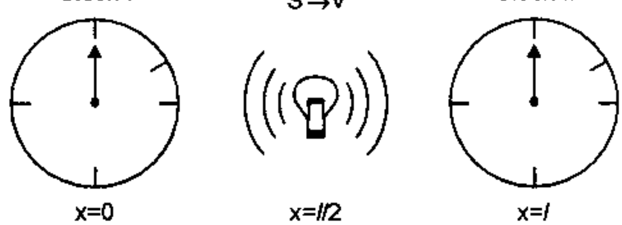

Fig. 4

If stationary frame of reference, the observer records two events, one the receipt of a light signal by clock–I at $x = 0$, $t_1 = \frac{l}{2c}$ and second the receipt of a light signal by clock–II at $x_2 = l$, $t_2 = \frac{l}{2c}$. But an observer in moving frame of reference S', *i.e.,* for a moving observer clock–I receives its signals at

$$t_1' = \frac{t_1 - \left(\frac{v}{c}\right)^2 x_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{l}{2c}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \left[\text{as } x_1 = 0,\ t_1 = \frac{l}{2c}\right]$$

with clock–II receives its signal at

$$t_2' = \frac{t_2 - \left(\frac{v}{c}\right)^2 x_2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{l}{2c} - \left(\frac{v}{c}\right)^2 l}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \left[\text{as } x_2 = l,\ t_2 = \frac{l}{2c}\right]$$

Thus $t_2' < t_1'$ and clock–II appears to receive its signal earlier than clock–I so that clocks do not start at the same time whereas these were so for stationary observer in frame S. This means that the two events that are simultaneous in one reference frame are not simultaneous in another frame moving with respect to first, unless the two events occur at the same point in space. ($x_2 = x$, so that $t_2 = t_1'$)

4.9. EQUIVALENCE OF MASS AND ENERGY

The famous relation between mass and energy was given by Einstein. It is expressed as

$$E = mc^2$$

where E is total energy of a moving body, m is its effective mass and c is the velocity of light.

Now let us prove it.

Let a force F be applied on a particle of mass m then the force F is given by

$$F = \text{Rate of change of momentum}$$

$$F = \frac{d}{dt}(mv)$$

$$F = m \cdot \frac{dv}{dt} + v\frac{dm}{dt}$$

$$\text{Work done} = \text{Force} \times \text{displacement}$$

$$= F \cdot dx$$

$$\text{Work done} = \text{Change in kinetic energy}$$

$$\text{Change in kinetic energy} = \left(m \cdot \frac{dv}{dt} + v\frac{dm}{dt}\right)dx$$

Since, $\quad v = \dfrac{dx}{dt}$

$\therefore$ Change in kinetic energy

$$dK = mvdv + v^2\, dm \quad \dots (1)$$

We have, the realistic mass is given by

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$\therefore \quad m^{-2} = \frac{m_0^2\, c^2}{c^2 - v^2}$$

$$\therefore \quad m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

On differentiating we get

$$c^2 \cdot 2m\, dm - v^2\, 2m\, dm - m^2\, 2v\, dv = 0$$

$$\therefore \quad c^2\, dm - v^2\, dm - mv\, dv = 0$$

$$\therefore \quad mv\, dv + v^2\, dm = c^2\, dm$$

Putting this value in equation (1) we get

$$dK = c^2\, dm \qquad \dots (2)$$

On integrating

$$\int dK = \int_{m_0}^{m} c^2\, dm = c^2 (m - m_0)$$

$$K = c^2 (m - m_0) \qquad \dots (3)$$

Since the total energy E possessed by a moving body is made up of the kinetic energy and energy stored up as internal energy, *i.e.*,

$$E = K + m_0 c^2 = (m - m_0) c^2 + m_0 c^2$$

$m_0 c^2$ is known as **rest mass energy.**

$$E = mc^2 - m_0 c^2 + m_0 c^2$$

$$\boxed{E = mc^2}$$

This is Einstein's mass energy relation. This shows clearly that a system with total energy E has associated with it a mass E/c^2, or any mass m has energy mc^2.

Examples : There are large number of examples which verify this relation ($E = mc^2$).

(1) Compton Effect : Compton found that when a light ray is incident on an electron with elastic collision between a photon and electron, then the scattered photon moves in the new direction and the electron recoils with a velocity comparable to the velocity of light. Compton determined the value for the wavelength shift by using conservation of energy and momentum. This value agreed with the experimental result.

(2) Fine Structure of Spectral Lines : Fine structure of spectral lines is explained by Sommerfeld, on the basis of relativistic variation of mass. The good agreement of his theory with experiment provides another verification of mass–energy relation.

• 4.10. RELATIVISTIC RELATION BETWEEN ENERGY AND MOMENTUM

In classical mechanics a particle must have rest mass in order to have energy and momentum but in relativistic mechanics this requirement does not hold. In this position the total energy is given by

$$E = mc^2$$

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or} \quad E^2 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} \qquad \dots (1)$$

where m_0 is rest mass.

and momentum is

$$P = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \qquad Pc = \frac{m_0 v c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \qquad P^2 c^2 = \frac{m_0^2 v^2 c^2}{1 - \frac{v^2}{c^2}} \qquad \dots (2)$$

Substituting eqn. (2) from eqn. (1) we get

$$E^2 - P^2 c^2 = \frac{m_0^2 c^4 - m_0^2 v^2 c^2}{\left(1 - \frac{v^2}{c^2}\right)} = \frac{m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}}$$

$$= m_0^2 c^4$$

$$\boxed{E^2 = m_0^2 c^4 + P^2 c^2}$$

• 4.11. RELATIVISTIC LAW OF ADDITION OF VELOCITIES

According to special relativity postulates the speed of light c in free space has the same value for all observers due to their relative motion. This law tells us that velocities can not be added directly as in classical mechanics because velocities must be added in a manner consistent with Lorentz transformation.

Suppose there are two frames of reference S and S' (fig. 5), frame S' is moving with velocity v relative to S along x–axis. An observer in S measures the velocity of the body as

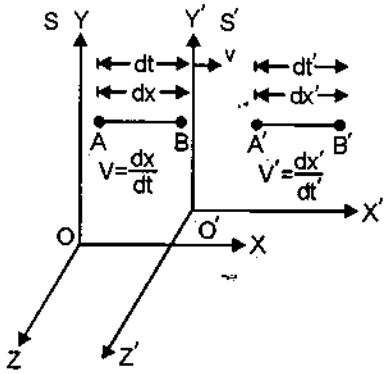

Fig. 5

$$V = \frac{dx}{dt} \qquad \ldots (1)$$

where dx is the distance moved by the body in time dt.
while to an observer in S' the distance is dx' and time interval is dt' then

$$V' = \frac{dx'}{dt'} \qquad \ldots (2)$$

We have Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

On differentiating we get

$$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$\therefore$ by (2)

$$V' = \frac{dx'}{dt'} = \frac{\frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}}}{\frac{dt - \frac{vdx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}} = \frac{\frac{dx - vdt}{dt - \frac{vdx}{c^2}}}{} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

From (1) $V = \frac{dx}{dt}$ so $V' = \frac{V - v}{1 - \frac{Vv}{c^2}}$

This is the relativistic addition of velocities V and v.

If $V' = c$, that is, if light is emitted in the moving frame S' in its direction of motion relative to S, an observer in frame S will measure the speed

$$V = \frac{V' + v}{1 + \frac{V' v}{c^2}}$$

$$= \frac{c + v}{1 + \frac{vc}{c^2}} = \frac{c\,(c + v)}{c + v} = c$$

i.e., $$\boxed{V = c}$$

Thus both observers measure the same value for the velocity of light is vacuum is constant and independent of the frame of reference.

From above expression it is clear that **if any velocity v is added relativistically to the velocity of light c then again velocity of light is obtained.**

• 4.12. RELATIVITY OF MASS

If a body of rest mass, m_0, moves with very high speed v relative to an observer then its mass m appears to be increased by factor $\frac{1}{\sqrt{-1 - \frac{v^2}{c^2}}}$.

That is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

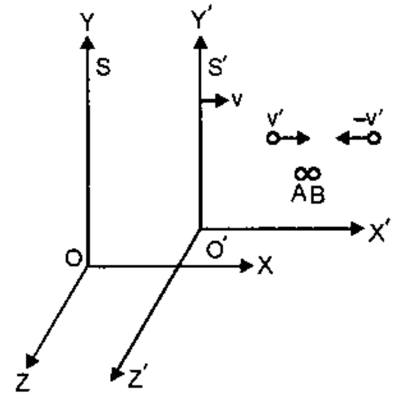

Fig. 6

where m_0 is the rest mass of the body.

This is the relativistic mass of the body.

Proof : Let us consider two frames of reference S and S' (fig. 6), S' is moving with velocity v relative to S. Let two particles A and B of mass m_1 and m_2 be moving with the velocity v' and $-v'$ in the frame S' approach each other.

The velocities of the particles seen by an observer from the frame S will be

$$u_1 = \frac{v' + v}{1 + \frac{v' v}{c^2}} \quad \ldots (1)$$

and $$u_2 = \frac{-v' + v}{1 - \frac{v' v}{c^2}} \quad \ldots (2)$$

when particles collide then they are momentarily at rest with respect to the frame S' but as seen from S they are still moving with velocity v. Since the total momentum of the particles is conserved so

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\, v$$

$$\therefore \quad m_1\left(\frac{v'+v}{1+\frac{v'v}{c^2}}\right)+m_2\left(\frac{-v'+v}{1-\frac{v'v}{c^2}}\right)=(m_1+m_2)\,v$$

$$\therefore \quad m_1\left(\frac{v'+v}{1+\frac{v'v}{c^2}}-v\right)=m_2\left(v-\frac{v'+v}{1-\frac{v'v}{c^2}}\right)$$

$$\frac{m_1}{m_2}=\frac{v'-\frac{v'v^2}{c^2} \quad 1+\frac{v'v}{c^2}}{v'-\frac{v'v^2}{c^2} \quad 1+\frac{v'v}{c^2}}$$

$$\frac{m_1}{m_2}=\frac{1+\frac{v'v}{c^2}}{1-\frac{v'v}{c^2}} \Rightarrow \left(\frac{m_1}{m_2}\right)^2=\frac{\left(1+\frac{v'v}{c^2}\right)^2}{\left(1-\frac{v'v}{c^2}\right)^2} \qquad \ldots (3)$$

From eqn. (1) we have

$$u_1^2=\left(\frac{v'+v}{1+\frac{v'v}{c^2}}\right)^2$$

$$\therefore \quad 1-\frac{u_1^2}{c^2}=1-\frac{1}{c^2}\left(\frac{v'+v}{1+\frac{v'v}{c^2}}\right)^2$$

$$=\frac{\left(1+\frac{v'v}{c^2}\right)^2-\left(\frac{v'+v}{c}\right)^2}{\left(1+\frac{v'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2}=\frac{\left(1-\frac{v'^2}{c^2}\right)-\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{v'v}{c^2}\right)^2}$$

$$\left(1+\frac{v'v}{c^2}\right)^2=\frac{\left(1-\frac{v'^2}{c^2}\right)-\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_1^2}{c^2}\right)}$$

Similarly, $\left(1+\frac{v'v}{c^2}\right)^2=\dfrac{\left(1-\frac{v'^2}{c^2}\right)-\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}$

Putting these values in eqn. (3) we get

$$\left(\frac{m_1}{m_2}\right)^2 = \sqrt{\frac{\frac{1-u_2^2}{c^2}}{\frac{1-u_1^2}{c^2}}} \qquad \ldots (4)$$

If the particle B is moving with zero velocity in frame S before collision then $v_2 = 0$ and $m = m_0$ (rest mass) so

$$m_1 = \frac{m_0}{\sqrt{1 - \frac{u_1^2}{c^2}}}$$

Since both particles are exactly identical so replacing m_1 by m and u_1 by v we get

$$\boxed{m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

This is the desired result.

Now the following results are obtained.

(i) mass of the particle m depends upon the velocity v.

(ii) when $v \to c$, $m \to \infty$ this means that no particle can have the velocity equal to or greater than the velocity of light.

(iii) when $v << c$ then $m \approx m_0$. This means that at ordinary velocities the difference between m and m_0 is very small.

• SUMMARY

- A co-ordinate system relative to which the position and motion of an object are specified is called a frame of reference
- The non-accelerated frames are known as inertial frames or we can say that the frames in which Newton's laws hold good are called inertial frames.
- The accelerated frames are called non-inertial frames.
- The negative result of the Michelson Morley experiment suggested that the velocity of light is constant in all directions.
- Basic Postulates of special theory of relativity :

  (1) The laws of physics are the same in all inertial frames of reference. It is also known as **principle of relativity.**

  (2) The speed of light in free space (c) has the same value in all inertial frames of reference, i.e., the speed of light is constant.
- Lorentz transformation equations of relativity are :

  (i) $x'd = \dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $x = \dfrac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$ (ii) $t' = \dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $t = \dfrac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

  (iii) $y' = y$ (iv) $z' = z$
- The length contraction or the Lorentz-Fitzgerald contraction is given by the factor $\sqrt{1 - \frac{v^2}{c^2}}$.
- The relation $E = m_0c^2$ is known as rest mass energy.

- The law of relativistic addition of velocities states that if any velocity v is added relativistically to the velocity of light c then again velocity of light is obtained.

• TEST YOURSELF

1. Describe Michelson-Morley experiment. How were its negative results interpreted ?
2. What are the basic postulates of special theory of relativity ? Deduce Lorentz transformation equations from them.
3. Obtain mass energy relation $E = mc^2$ and explain it.
4. Derive the relativistic relation $E = m_0^2c^4 + P^2c^2$ between energy and momentum.
5. Obtain an expression for the relativistic law of addition of velocities.
6. A body moving with velocity v has mass m. Show that

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the rest mass of the body and c is the speed of light.

7. Write short notes on :
   (a) Length contraction (b) Time dilation
   (c) Simultaneity (d) Frame of reference
8. Calculate the velocity of a particle when its rest mass energy is double of its K.E. **[Ans.** 0.866 c]
9. Calculate the speed of a particle of rest mass 3.33×10^{-27} gram whose energy is 2 MeV. **[Ans.** 1.047×10^8 ms^{-1}]
10. The energy of massless particles ($m_0 = 0$) is given by :
    (a) $E = Pc$ (b) $E = \frac{h\nu}{c}$ (c) $E = h\,\lambda$ (d) none of these
11. Mass of moving photon is equal to :
    (a) $\frac{E}{c}$ (b) $\frac{E}{P}$ (c) $\frac{E}{c^2}$ (d) none of these
12. A star is receding from the earth. This will give rise to :
    (a) red shift (b) blue shift (c) both shifts (d) no shift
13. The speed of light in vacuum in two inertial system :
    (a) is same
    (b) is different
    (c) depends upon the direction of propagation of source and observer
    (d) depends upon the relative velocity between the source and the observer
14. A particle of zero rest travels with speed :.
    (a) greater than light (b) less than light
    (c) equal to light (d) none of these
15. One atomic mass unit is equivalent to :
    (a) 931 joule (b) 931 MeV (c) 231 eV (d) none of these
16. One joule of energy is equivalent to :
    (a) 0.1×10^{-17} kg (b) 0.1×10^{17} kg
    (c) 0.1×10^{17} kg (d) none of these

ANSWERS

10. (a) **11.** (c) **12.** (a) **13.** (a) **14.** (c) **15.** (b) **16.** (a).

UNIT 5

OSCILLATIONS

STRUCTURE

LEARNING OBJECTIVES

After going through this unit you will learn :

- Oscillations and Harmonic motions of a body.
- Motion of different types of pendulum *i.e.*, simple, compound, torsion.
- Energy stored in body while in Harmonic motion.
- The curve traced by the particle while in Harmonic motion or the Lissajous figures.

• 5.1. SIMPLE HARMONIC OSCILLATIONS

(a) Periodic Motion : The motion of a body is said to be periodic motion if its motion is repeated identically after a fixed interval of time and this fixed interval of time is known as **period of motion**.

Examples : (i) The revolution of earth around the sun is an example of periodic motion. Its period of revolution is one year.

(ii) The rotation of earth about its polar axis is a periodic motion whose period of rotation is **one day**.

(iii) The revolution of moon around the earth is also an example of periodic motion whose period of motion is 27.3 days.

(b) Oscillatory Motion : When a body moves to and fro repeatedly about its mean position in a definite interval of time then this motion is known as oscillatory or vibratory motion.

Thus we can say that a periodic and bounded motion of a body about a fixed point is called an oscillatory motion. The oscillatory motion can be expressed in terms of sine and cosine functions or their combinations. Due to this, the **oscillatory motion is also called harmonic motion.**

Examples : (i) The motion of the pendulum of a wall clock is an example of oscillatory motion.

(ii) When the bob of simple pendulum is displaced from its mean position and left to itself then the motion of bob is known as oscillatory motion.

(c) Time Period : In periodic motion the time taken by the body in one period is known as time period. It is denoted by T and S.I. unit of T is second.

(d) Frequency : The number of periodic motions made by the body in one second is known as frequency. Its S.I. unit is hertz. Thus the frequency is the reciprocal of the periodic time.

$$\boxed{\text{Frequency} = \frac{1}{\text{Time period}}}$$

Phase : Phase of an oscillatory particle at any instant is a physical quantity which completely expresses the position and direction of motion of the particle at that instant with respect to its mean position.

• 5.2. SIMPLE HARMONIC MOTION

When a particle moves to and fro repeatedly about its mean position under a restoring force, which is always directed towards the mean position and whose magnitude at any instant is directly proportional to the displacement of the particle from the mean position at that instant, this motion of the particle is known as **simple harmonic motion.** *i.e.*

$$\text{restoring force } (F) \propto - \text{ (displacement)}$$

$$F = -kx$$

where k is constant and this constant is known as force constant.

Here negative sign shows that the restoring force is always directed towards the mean position.

Characteristics of Simple Harmonic Motion

(i) Displacement : The distance of the particle from the mean position at any instant, is known as displacement of the particle at that instant.

(ii) Amplitude : The maximum displacement of the particle from mean position is known as amplitude of motion.

(iii) Velocity : In simple harmonic motion the velocity of the particle at any instant is equal to the rate of change of displacement at that instant.

The displacement of the particle at time t is given by

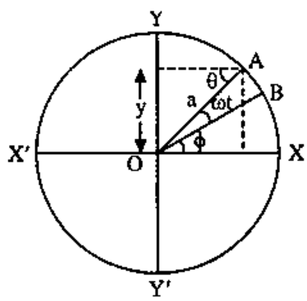

Fig. 1.

$$y = a \sin \omega t$$

$$\therefore \quad \text{velocity} \quad v = \frac{dy}{dt}$$

$$= \frac{d}{dt}(a \sin \omega t)$$

$$v = a\omega \cos \omega t$$

$$= a\omega \sqrt{1 - \sin^2 \omega t}$$

$$= a\omega \sqrt{1 - \frac{y^2}{a^2}}$$

$$\boxed{v = \omega \sqrt{a^2 - y^2}}$$

This is the expression for the velocity of the particle at any instant.

(iv) Acceleration : The acceleration of a particle in simple harmonic motion at any instant is equal to the rate of change of velocity at that instant *i.e.*

$$\text{acceleration } (\alpha) = \frac{dv}{dt}$$

$$\alpha = \frac{d}{dt}(a\omega \cos \omega t)$$

$$= -\omega^2 a \sin \omega t$$

$$\boxed{\alpha = -\omega^2 y}$$

This is the expression for acceleration. This is the condition of S.H.M.

At mean position $y = 0$ $[\because \theta = 0]$

$\therefore$ $\boxed{\alpha = 0}$ *i.e.* $\alpha = \text{min}$

At extreme position $y = a$ $[\because \theta = 90°]$

$\therefore$ $\boxed{\alpha = -\omega^2 a}$ *i.e.* max.

Thus, from above it is clear that acceleration and velocity are not uniform in the whole motion. The maximum value of velocity is known as **velocity amplitude** in S.H.M and the maximum value of acceleration is called **acceleration amplitude.**

(v) Time Period : The time taken by a particle in simple harmonic motion to complete one period is known as time period.

We know

acceleration $\alpha = \omega^2 y$ (neglecting negative sign)

$$\therefore \quad \omega = \sqrt{\frac{\alpha}{y}}$$

$$\therefore \text{ time period, } \quad T = \frac{2\pi}{\omega}$$

or
$$T = 2\pi\sqrt{\frac{y}{\alpha}}$$

or
$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

and frequency is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

• 5.3. EQUATION OF MOTION OF A SIMPLE HARMONIC OSCILLATOR

A particle executing simple harmonic motion is known as harmonic oscillator. Consider a particle of mass m executing simple harmonic motion along a straight line (fig. 2).

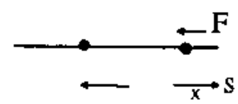

Fig. 2

Let x be the displacement of the particle from mean position O at any time t, then from the basic condition of simple harmonic motion, the restoring force F is proportional to the displacement x with negative sign *i.e.*,

$$F \propto -x$$

$$F = -kx \qquad \ldots (1)$$

where k is constant and this constant is known as force constant. The equation of motion can be obtained by Newton's second law. *i.e.*

$$F = m\frac{d^2x}{dt^2}$$

From eq. (1), $F = -kx$

Therefore

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Putting $\frac{k}{m} = \omega^2$, then we get

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \dots (2)$$

This is the differential equation of simple harmonic oscillator.

Let the solution of equation (2) be

$$x = Ce^{\alpha t} \quad \dots (3)$$

where C and α are constants

From equation (3),

$$\frac{dx}{dt} = C\alpha e^{\alpha t}$$

and

$$\frac{d^2x}{dt^2} = C\alpha^2 e^{\alpha t}$$

Putting these values in equation (2), we get

$$C\alpha^2 e^{\alpha t} + \omega^2 Ce^{\alpha t} = 0$$

$$Ce^{\alpha t}(\alpha^2 + \omega^2) = 0$$

as

$$Ce^{\alpha t} \neq 0$$

$\therefore$

$$\alpha^2 + \omega^2 = 0$$

$\therefore$

$$\alpha = \pm\sqrt{(-\omega^2)} = \pm i\omega$$

where i is imaginary number, $i = \sqrt{(-1)}$. Thus two solutions of equation (2) are possible. Therefore

$$x = Ce^{i\omega t} \text{ and } x = Ce^{-i\omega t}$$

The general solution will be

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \quad \dots (4)$$

where C_1 and C_2 are constants.

From equation (4),

$$x = C_1[\cos \omega t + i \sin \omega t] + C_2[\cos \omega t - i \sin \omega t]$$

$$= (C_1 + C_2)\cos \omega t + (iC_1 - iC_2)\sin \omega t$$

Let $$C_1 + C_2 = a \sin \phi$$

and $$i(C_1 - C_2) = a \cos \phi$$

where a and ϕ are constants.

Therefore

$$x = a \sin \phi \cos \omega t + \cos \phi \sin \omega t$$

$$\boxed{x = a \sin (\omega t + \phi)}$$

This is the required solution of equation (2). This equation gives displacement of particle executing simple harmonic motion at any time t.

(a) Velocity : We have

$$x = a \sin (\omega t + \phi)$$

Differentiating with respect to time t, we get

$$v = \frac{dx}{dt} = a \cos (\omega t + \phi) \,.\, \omega$$

$$= a\omega \sqrt{1 - \sin^2 (\omega t + \phi)}$$

$$= \omega \sqrt{(a^2 - a^2 \sin^2 (\omega t + \phi))}$$

$$\boxed{v = \omega \sqrt{a^2 - x^2}}$$

This is the expression for the velocity at any displacement x.

(b) Period : We have,

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\therefore \qquad \boxed{T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}}$$

This is the expression for the time period.

(c) Frequency :

$$\because \qquad f = \frac{1}{T}$$

$$f = \frac{\omega}{2\pi}$$

$$\therefore \qquad \boxed{f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}}$$

This is the expression for frequency.

(d) Importance of S.H.M : The importance of S.H.M. in physics is due to the following reasons :

(i) The physical problems in mechanics, optics, electricity and in atomic and molecular physics in which the force is directly proportional to its displacement from some equilibrium position the resulting motion is represented by the simple harmonic model.

(ii) The complicated periodic motions occuring in physical problems can also be represented by the combination of a number of simple harmonic motions having frequencies which are multiples of the complicated motion. The vibrations of atoms in solids, the electrical and acoustical oscillations in a cavity can be analysed in this manner.

• 5.4. ENERGY OF A PARTICLE EXECUTING SIMPLE HARMONIC MOTION

When a particle oscillates about its mean position then it has potential energy as well as kinetic energy. The potential energy is due to displacement from the mean position and kinetic energy is due to its velocity. These energies vary during oscillation while the total energy of the particle remains conserved.

(a) Potential Energy : Let us consider a particle of mass m executing S.H.M. Let x be its displacement from the mean position at any time 't'. In this position force F acting on the particle is given by

$$F = -kx \qquad \ldots (1)$$

where k is the force constant. In terms of potential energy, the force is given by

$$F = -\frac{dU}{dx}$$

so from eq. (1)

$$\frac{dU}{dx} = kx \qquad \ldots (2)$$

On integrating eq. (2), we get

$$U = \frac{1}{2}kx^2 + C \qquad \ldots (3)$$

where C is constant of integration.

Now at $x = 0$, $U = 0$, then $C = 0$

so by eq. (3)

$$U = \frac{1}{2}kx^2 \qquad \ldots (4)$$

But for simple harmonic motion, we have

$$x = a \sin(\omega t + \phi) \qquad \ldots (5)$$

where $\omega^2 = \frac{k}{m}$ From eqs. (4) and (5), we get

$$\boxed{U = \frac{1}{2}ka^2 \sin^2(\omega t + \phi)}$$

This is the expression for the potential energy of the particle at any time 't'.

From this expression it is clear that when $\sin^2(\omega t + \phi) = 1$, then V is maximum.

(b) Kinetic Energy : Kinetic energy of the particle at any time 't' is

$$K.E. = \frac{1}{2}mv^2 \qquad \ldots (6)$$

We have

$$x = a \sin(\omega t + \phi)$$

differentiating with respect to 't',
we get

$$v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi)$$

so by eq. (6)

$$K.E. = \frac{1}{2}mv^2$$

$$= \frac{1}{2} m [\omega a \cos(\omega t + \phi)]^2$$

$$= \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi)$$

$$\boxed{K.E. = \frac{1}{2} k a^2 \cos^2(\omega t + \phi)} \quad \ldots (7)$$

This is the expression for kinetic energy. From this expression it is clear that when $\cos^2$ then kinetic energy will be maximum which is $\frac{1}{2} kA^2$.

Total energy of the particle is given by

$$E = P.E. + K.E.$$

$$E = \frac{1}{2} ka^2 \sin^2(\omega t + \phi) + \frac{1}{2} ka^2 \cos^2(\omega t + \phi)$$

$$E = \frac{1}{2} ka^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

$$E = \frac{1}{2} ka^2$$

But $k = \omega^2 m$ and $\omega = 2\pi x$

where n is the frequency.

$$\therefore \quad E = \frac{1}{2} ka^2$$

$$E = \frac{1}{2} \omega^2 ma^2$$

$$E = \frac{1}{2} (2\pi n)^2 ma^2$$

$$\boxed{E = 2\pi^2 n^2 ma^2}$$

This is the expression for the total energy of the oscillating particle.

From this expression we see that total energy is proportional to the square of the amplitude (a^2) and also inversely proportional to the square of the time period (T^2) $\left(\because \quad n \propto \frac{1}{T}\right)$

(c) Since $x = a \sin(\omega t + \phi)$

$$\therefore \quad v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi)$$

$$= \pm \sqrt{\left(\frac{k}{m}\right)} \sqrt{[a^2 - a^2 \sin^2(\omega t + \phi)]}$$

$$= \pm \sqrt{\left(\frac{k}{m}\right)} \sqrt{(a^2 - x^2)}$$

Total energy of the particle is

$$E = K.E. + P.E.$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} kx^2$$

$$= \frac{1}{2} k(a^2 - x^2) + \frac{1}{2} kx^2$$

$$= \frac{1}{2} ka^2$$

When the displacement is one half of the amplitude *i.e.* $x = \frac{a}{2}$, then we get

$$(U)_{a/2} = \frac{1}{2}kx^2 + \frac{1}{2}k\left(\frac{1}{2}a\right)^2$$
$$= \frac{1}{8}ka^2$$
$$= \frac{1}{4}E$$

and
$$(K)_{a/2} = \frac{1}{2}k(a^2 - x^2)$$
$$= \frac{1}{2}k\left(a^2 - \frac{a^2}{4}\right)$$
$$= \frac{3}{8}ka^2$$
$$= \frac{3}{4}E.$$

Thus, the potential energy is one fourth and the kinetic energy is three fourths of the total energy.

Now let x be the displacement at which the energy is half potential and half kinetic energy *i.e.*

$$U = K = \frac{1}{2}E$$
$$U = \frac{1}{2}E = \frac{1}{2}kx^2$$
$$\frac{1}{2}\left(\frac{1}{2}ka^2\right) = \frac{1}{2}kx^2$$
$$\therefore \quad \boxed{x = \frac{a}{\sqrt{2}}}$$

• 5.5. TIME-AVERAGE OF KINETIC ENERGY AND POTENTIAL ENERGY

The kinetic energy of a particle of mass m executing S.H.M. under a force constant k is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$
$$= \frac{1}{2}m\omega^2a^2\cos^2(\omega t + \phi), \text{ where } \omega^2 = \frac{k}{m}$$

The time-average of the kinetic energy over a period T of the motion is

$$\overline{K} = \frac{\int_0^T K\,dt}{T} = \frac{1}{2}m\omega^2a^2\frac{\int_0^{2\pi/\omega}\cos^2(\omega t + \phi)\,dt}{2\pi/\omega}$$
$$= \frac{m\omega^3a^2}{4\pi}\int_0^{2\pi/\omega}\frac{1 - \sin^2(\omega t + \phi)}{2}\,dt$$
$$= \frac{m\omega^3a^2}{8\pi}\left[\int_0^{2\pi/\omega}dt + \int_0^{2\pi/\omega}\cos 2(\omega t + \phi)\,dt\right]$$
$$= \frac{m\omega^3a^2}{8\pi}\left[\frac{2\pi}{\omega} - 0\right]$$
$$= \frac{1}{4}m\omega^2a^2 = \frac{1}{4}ka^2$$

The potential energy of the particle is

$$U = \frac{1}{2} kx^2 = \frac{1}{2} ka^2 \sin^2 (\omega t + \phi)$$

The time-average of the potential energy over a period T of the motion is

$$\overline{U} = \frac{\int_0^T Udt}{T} = \frac{\frac{1}{2} ka^2 \int_0^{2\pi/\omega} \sin^2 (\omega t + \phi)}{2\pi/\omega}$$

$$= \frac{k\omega a^2}{4\pi} \cdot \int_0^{2\pi/\omega} \frac{1 - \cos 2(\omega t + \phi)}{2} dt$$

$$= \frac{k\omega a^2}{8\pi} \left[\int_0^{2\pi/\omega} dt - \int_0^{2\pi/\omega} \cos 2(\omega t + \phi)\, dt \right]$$

$$= \frac{k\omega a^2}{8\pi} \left[\frac{2\pi}{\omega} + 0 \right]$$

$$= \frac{1}{4} ka^2$$

Hence, the time-average kinetic energy $\overline{K}$ is equal to the time average potential energy $\overline{U}$ and each is equal to $\frac{1}{4} ka^2$.

Now the position-average of the kinetic and potential energy is obtained as follows. The kinetic energy is

$$K = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 = \frac{1}{2} k (a^2 - x^2)$$

The position-average over the displacement from $x = 0$ to $x = a$ is

$$\overline{K} = \frac{\int_0^a Kdx}{a} = \frac{\frac{1}{2} k \int_0^a (a^2 - x^2)\, dx}{a}$$

$$= \frac{1}{2} \frac{k}{a} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= \frac{1}{3} ka^2$$

the potential energy is

$$U = \frac{1}{2} kx^2$$

The position-average over the displacement from $x = 0$ to $x = a$ is

$$\overline{U} = \frac{\int_0^a Udx}{a}$$

$$= \frac{\frac{1}{2} k \int_0^a x^2 dx}{a}$$

$$= \frac{1}{2}\frac{k}{a}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{6}ka^2$$

Hence, the position-average kinetic energy $\left(\frac{1}{2}ka^2\right)$ is not equal to the position-average potential energy $\left(\frac{1}{6}ka^2\right)$.

• 5.6. POTENTIAL ENERGY CURVE AND SMALL OSCILLATIONS IN ONE DISMENSIONAL POTENTIAL WELL

The curve which gives the variation of the potential energy of a particle with its position is known as **potential energy curve.** The potential energy curve of a particle is moving in one dimension against its distance from the origin along the line of its motion.

In this position the force on the particle is given by

$$F = -\frac{dV(x)}{dx}$$

From this expression it is clear that it $V(x)$ decreases with increasing x, then the slope $\frac{dV(x)}{dx}$ is negative and hence F is positive. On the other hand, if $V(x)$ increases with increase of x, then the slope $\frac{dV(x)}{dx}$ is positive and hence the force F is towards the negative x-axis. **In this position the particle at any point experiences a force which tends to bring it in the region of lower potential energy.**

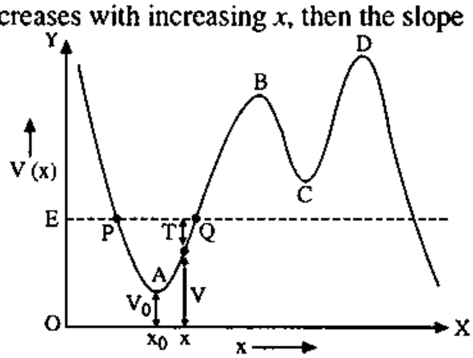

Fig. 3.

Positions of Equilibrium : From fig. 3 we see that the slope $\frac{dV(x)}{dx}$ is zero at points A, B, C and D and hence the force acting on the particle at these points will be zero. In this position the particle remains in a state of equilibrium.

Positions of Unstable Equilibrium : In the fig. at points B and D the potential energy is maximum and slope is zero. Due to this the force acting on the particle is zero. If we displace the particle on either side, then the force $F = \frac{dV}{dx}$ acts on the particle which tends to displace the particle further away from equitibrium position. **Hence the positions at which the potential energy is maximum are the positions of unstable equilibrium.**

Positions of Stable Equilibrium : The points at which the potential energy of the particle is minimum are the positions of stable equilibrium. From fig. 3 the points A and C are the states of stable equibibrium.

Bounded Region : Potential Well : The particle can be displaced from x_0 only if its total energy E is more than the minimum potential energy at x_0. As the particle moves, its potential energy goes on changing but it can never be more than E, because the kinetic energy K can not be negative. Thus, if the total energy E corresponds to the horizontal line drawn on the potential energy curve then particle must remain confined between the points x_1 and x_2. **Thus, the particle oscillates between points P and Q with a certain period. In this position the points P and Q are known as turning points for the given particle. The motion of the particle is confined to region between points P and Q. This region is called bounded region or potential well.**

Nature of Small Oscillations in one Dimensional Potential Well : At point x_0, the potential energy function $U(x)$ can be expanded into Taylor's series

$$U(x) = U_0 + \left(\frac{dU}{dx}\right)_{x_0}(x - x_0) + \left(\frac{d^2U}{dx^2}\right)_{x_0}\frac{(x-x_0)^2}{2!} + \ldots\ldots\ldots$$

where U_0, $\left(\frac{dU}{dx}\right)_{x_0}$ are the values at the points $x = x_0$. If x_0 is a point of stable equilibrium, then $\left(\frac{dU}{dx}\right)_{x_0}$ is zero and $\left(\frac{d^2U}{dx^2}\right)_{x_0}$ is a positive quantity. Therefore, putting C_2, C_3 .... for $\left(\frac{d^2U}{dx^2}\right)_{x_0}$, $\left(\frac{d^3U}{dx^3}\right)_{x_0}$ .... we have

$$U(x) = U_0 + \frac{C_2}{2!}(x - x_0)^2 + \frac{C_3}{3!}(x - x_0)^3 \ldots\ldots\ldots$$

If we shift the origin to x_0, then we can put x in place of $x - x_0$, then we get

$$U(x) = U_0 + \frac{C_2}{2!}x^2 + \frac{C_3}{3!}x^3 + \ldots\ldots\ldots\ldots$$

For small oscillations, the term containing x^3 and higher can be ignored. Then

$$U(x) = U_0 + \frac{1}{2}C_2 x^2 \qquad \ldots (1)$$

From this equation it is clear that the particle oscillates in parabolic region. **Thus, for small-amplitude oscillations the potential well is parabolic.**

$$\because \qquad F = -\frac{dU(x)}{dx}$$

$\therefore$ From equation (1), we get

$$F = -\frac{d}{dx}\left[U_0 + \frac{1}{2}C_2 x^2\right]$$

$$\boxed{F = -C_2 x}$$

This is the characteristic of S.H.M. **Thus, for small oscillations, the motion in a potential well is simple harmonic.**

• 5.7. PERIOD OF OSCILLATION OF A MASS SUSPENDED BY A SPRING

Let us consider a weightless spring of length l, hanging vertically as shown in fig. 4 when a weight of mass m is attached to its lower end then its length increases by x'. In this position the spring exerts a restoring vertical force F on the mass m.

Now from Hooke's law

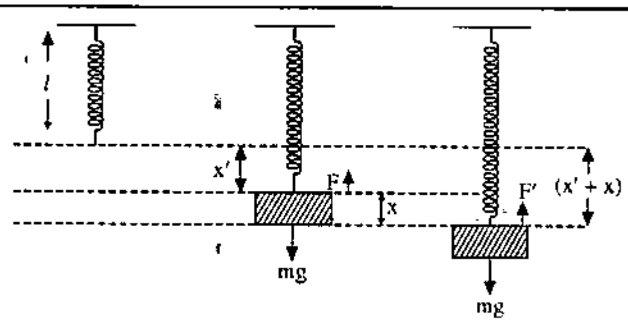

Fig. 4.

$$F = -kx' \qquad \ldots (1)$$

where k is constant and this constant is known **as force constant of the spring.** In equation (1) the negative sign shows that the direction is opposite to the expansion in length of the spring. The force due to mass m is mg acting downward.

Since there is no acceleration in the body so net force should be zero *i.e.*

$$F + mg = 0$$

$$-kx' + mg = 0 \qquad \text{[from eq. (1)]}$$

$$x' = \frac{mg}{k} \qquad \dots (2)$$

when the body is displaced by small distance and left, then the body starts to vibrate about its mean position. Let x be the displacement from its equilibrium position. In this position the tolat increment in the length of the spring is $(x' + x)$.

Again, from Hooke's law

$$F' = -k\,(x' + x)$$

$$F' = -k\left(\frac{mg}{k} + x\right) \qquad \text{[from eq. (2)]}$$

$$F' = -mg - kx \qquad \dots (3)$$

Therefore, total force is given by

$$F'' = F' + mg$$

$$F'' = (-mg - kx) + mg \qquad \text{[From eq. (3)]}$$

$$F'' = -kx \qquad \dots (4)$$

From Newton's second law force F'' will be equal to the product of mass m and acceleration $\frac{d^2x}{dt^2}$ *i.e.*

$$F'' = m\frac{x^2x}{dt^2} = -kx \qquad \text{[From eq. (4)]}$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $\omega^2 = \frac{k}{m}$

This equation shows that the motion of spring is simple harmonic.

In this position the time period is given by

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

5.8. FREQUENCY OF MASS CONNECTED WITH TWO SPRINGS IN HORIZONTAL POSITION

When the mass m oscillates, then at any instant one spring is stretched and the other is compressed and vice-versa.

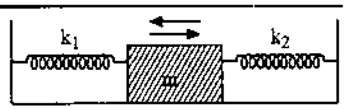

Fig.5.

Let x be the displacement of mass m from its mean position then

$$m\frac{d^2x}{dt^2} = -k_1x - k_2x$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{k_1 + k_2}{m}x$$

or
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $\omega^2 = \dfrac{k_1 + k_2}{m}$

This shows that, the motion is simple harmonic.

The time period is given by

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}}$$

and frequency is given by

$$\boxed{n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}}$$

• 5.9. SIMPLE PENDULUM

When a heavy particle is suspended by an inextensible, weightless and flexible string from a rigid support, then this system is known as simple pendulum.

Let us consider a particle of mass m which is suspended by an inextensible, weightless and flexible string of length l.

Let a pendulum be displaced from its mean position and allowed to oscillate as shown in fig. 6.

Let at time 't' the particle be at point P. In this position the force acting on the particle vertically downward is mg.

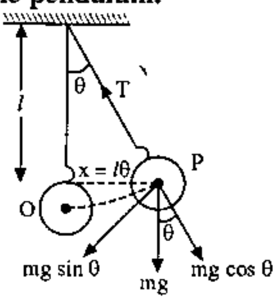

Fig. 6.

Now resolving mg into two components :

(1) Force along the string = $mg \cos\theta$

(2) Force perpendicular to the string = $mg \sin\theta$

Let the tension in the string be T which is balanced by the component $mg \cos\theta$.

i.e.
$$T = mg\cos\theta$$

Hence $-mg\sin\theta$ is the only force which acts on the oscillating particle

$$\therefore \qquad F = -mg\sin\theta \qquad \ldots (1)$$

Here the negative sign shows that the acceleration is directed towards the mean position.

$$\because \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\ldots\ldots\ldots\ldots$$

If θ is very small then $\sin\theta = \theta$.
so by eq. (1).

$$F = -mg\theta \qquad \ldots (2)$$

The displacement is

$$x = l\theta$$

acceleration $\dfrac{d^2x}{dt^2} = l\dfrac{d^2\theta}{dt^2}$

$$\therefore \qquad \text{Force} = ml\frac{d^2\theta}{dt^2} \qquad [\because \; F = ma] \quad \ldots (3)$$

From eqns. (2) and (3), we get

$$ml\frac{d^2\theta}{dt^2} = -mg\,\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0} \quad \ldots (4)$$

This is the expression for the equation of motion of simple pendulum which is similar to the eq. of simple harmonic motion *i.e.*,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \ldots (5)$$

From eqns. (4) and (5)

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Time period $\qquad T = \dfrac{2\pi}{\omega}$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

This is the expression for the time period of simple pendulum.

• 5.10. COMPOUND PENDULUM

When a rigid body which is capable of oscillating freely in a vertical plane about a fixed axis, passing through the body but not through its centre of gravity then the system is called compound pendulum and this fixed point is known as point of suspension

Let us consider a rigid body of mass m. Let G be the centre of gravity of the body and S be the point of suspension. When this rigid body is displaced from its mean position, then SG makes an angle θ with the vertical.

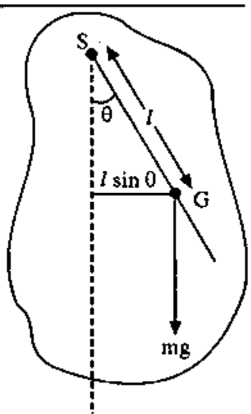

Fig. 7.

The force acting vertically downwards

$= mg$

and restoring moment of this force

$= -mg\,l\sin\theta$

This restoring moment produces angular acceleration in the pendulum.

Let I be the moment of inertia of the pendulum about an axis passing through S and perpendicular to its length. In this position the angular acceleration is $\dfrac{d^2\theta}{dt^2}$.

so $$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

where $I\dfrac{d^2\theta}{dt^2}$ is the torque.

Here negative sign shows that the force is directed towards the mean position.

If θ is very small than sin θ = θ

$$\therefore \qquad I\frac{d^2\theta}{dt^2} = -mgl\,\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

This is the equation of motion of compound pendulum. Thus, motion of compound pendulum is simple harmonic.

Here $\qquad \omega^2 = \frac{mgl}{I} \Rightarrow \omega = \sqrt{\frac{mgl}{I}}$

∴ Time period

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgl}}}$$

From theorem of parallel axes the moment of inertia of the pendulum about an axis passing through S and perpendicular to its plane

$$= mk^2 + ml^2$$

$$= m\,(k^2 + l^2)$$

where k is the radius of gyration about an axis passing through the centre of gravity G of pendulum.

so $\qquad \frac{d^2\theta}{dt^2} + \left(\frac{lg}{k^2 + l^2}\right)\theta = 0$

$\therefore \qquad \omega^2 = \frac{lg}{k^2 + l^2}$

$$\omega = \sqrt{lg/k^2 + l^2}$$

∴ Time period $T = \frac{2\pi}{\omega}$

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{lg}}$$

$$\boxed{T = 2\pi\sqrt{\frac{\frac{k^2}{l} + l}{g}}}$$

Here $\left(\frac{k^2}{l} + l\right)$ is called the equivalent length of simple pendulum.

• 5.11. TORSION PENDULUM

When one end of a very thin and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy disc or sphere, then this arrangement is known as torsion pendulum.

If the disc is turned through an angle θ then the wire is also twisted through the same angle θ. In this position the restoring torsional couple $(-c\theta)$ begins to act which tends to bring the pendulum to its initial position, where c is the torsional constant or **couple per unit twist.**

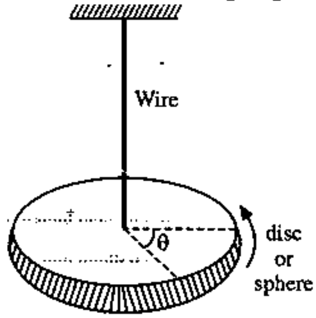

Fig. 8.

Thus torsional couple $\tau = -c\theta$ ... (1)

Let I be the moment of inertia of the disc about the wire as the axis. Here $\frac{d^2\theta}{dt^2}$ is the angular acceleration and the couple due to the acceleration is given by $I\frac{d^2\theta}{dt^2}$. This couple is balanced by the restoring torsional couple. Thus,

$$I\frac{d^2\theta}{dt^2} = -c\theta$$

$$I\frac{d^2\theta}{dt^2} + c\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$$

or
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

where $\omega^2 = \frac{c}{I}$.

This equation represents the simple harmonic motion where period is given by

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{c}}}$$

From mechanics of materials, 'c' in case of wire is given by

$$c = \frac{\pi\eta r^4}{2l}$$

where r = radius of the wire

l = length of the wire

η = modulus of rigidity

• STUDENT ACTIVITY

1. What is the importance of S.H.M. ?

2. What do you mean by restoring force ?

5.12. COMPOSITION OF TWO S.H.M'S. OF EQUAL PERIODS

Let the equations of two simple harmonic motions of equal frequencies are given by.

$$x_1 = a_1 \sin(\omega t + \phi_1) \quad \ldots (1)$$

and

$$x_2 = a_2 \sin(\omega t + \phi_2) \quad \ldots (2)$$

The resultant displacement is given by

$$x = x_1 + x_2$$

$$x = a_1 \sin(\omega t + \phi_1) + a_2 \sin(\omega t + \phi_2)$$

or

$$x = a_1 (\sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1) + a_2 (\sin \omega t \cos \phi_2 + \cos \omega t \sin \phi_2)$$

$$= \sin \omega t (a_1 \cos \phi_1 + a_2 \cos \phi_2) + \cos \omega t (a_1 \sin \phi_1 + a_2 \sin \phi_2)$$

Let

$$a_1 \cos \phi_1 + a_2 \cos \phi_2 = R \cos \theta \quad \ldots (3)$$

and

$$a_1 \sin \phi_1 + a_2 \sin \phi_2 = R \sin \theta \quad \ldots (4)$$

Therefore

$$x = \sin \omega t (R \cos \theta) \cos \omega t (R \sin \theta)$$

$$x = R \sin(\omega t + \theta)$$

This equation represents simple harmonic motion. Hence the resultant motion of the particle is simple harmonic.

Now squaring and adding equations (3) and (4), we get

$$a_1^2 + a_2^2 + 2a_1 a_2 (\cos \phi_1 . \cos \phi_2 + \sin \phi_1 . \sin \phi_2) = R^2$$

or

$$a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2) = R^2$$

so the resultant amplitude

$$\boxed{R = \sqrt{[a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)]}} \quad \ldots (5)$$

Dividing equation (4) by (3), we get

$$\boxed{\tan \theta = \frac{a_1 \sin \phi_1 + a_2 \sin \phi_2}{a_1 \cos \phi_1 + a_2 \cos \phi_2}}$$

This is the expression for phase θ.

Here there may be two cases.

(i) Maximum Amplitude : When $\phi_1 - \phi_2 = 2n\pi$ where $n = 0, 1, 2, \ldots\ldots$, then from equation (5), we get

$$R = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(2n\pi)}$$

$$= \sqrt{a_1^2 + a_2^2 + 2a_1 a_2} = \sqrt{(a_1 + a_2)^2}$$

$$\boxed{R = a_1 + a_2}$$

(ii) Minimum Amplitude : When $\phi_1 - \phi_2 = (2n + 1)\pi$ where $x = 0, 1, 2$ then from equation (5), we get

$$R = \sqrt{[a_1^2 + a_2^2 + 2a_1 a_2 \cos(2n + 1)\pi]}$$

$$= \sqrt{(a_1^2 + a_2^2 - 2a_1 a_2)} \qquad [\because \cos(2n+1)\pi = -1]$$

$$= \sqrt{(a_1 - a_2)^2}$$

$$\boxed{R = (a_1 - a_2)}$$

If $a_1 = a_2$, then we get

$$R = a_1 - a_2 = 0$$

$$\boxed{R = 0}$$

This means that when two simple harmonic motions of same amplitude, same time period but of opposite phase act at a point then particles at rest.

• 5.13. LISSAJOU'S FIGURES

When a particle is acted upon by two mutually perpendicular simple harmonic motions simultaneously then due to the effect of these motions the particle traces a curve. These curves are known as Lissajou's Figures.

These curves may be circle, ellipse, parabola, straight line etc. It depends upon the amplitudes, frequencies and phase difference between them.

When Frequencies (Time Periods) are equal and Amplitudes are different : Let us consider two simple harmonic motions in X and Y directions acting on a particle simultaneously. Then equations are

$$x = a \sin (\omega t + \phi) \qquad \ldots (1)$$

and

$$y = b \sin \omega t \qquad \ldots (2)$$

where a and b are the amplitudes and ϕ is the phase difference.

From eq. (1)

$$\frac{x}{a} = \sin (\omega t + \phi)$$

$$\frac{x}{a} = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

From eq. (2), $\sin \omega t = \frac{y}{a}$

and

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t} = \sqrt{1 - \frac{y^2}{b^2}}$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b} \cos \phi + \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \sin \phi$$

or

$$\frac{x}{a} - \frac{y}{b} \cos \phi = \sqrt{\left(1 - \frac{y^2}{b^2}\right)} \sin \phi$$

Squaring on both sides, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi - \frac{2xy}{ab} \cos \phi = \left(1 - \frac{y^2}{b^2}\right) \sin^2 \phi$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi = \sin^2 \phi \qquad \ldots (3)$$

This equation represents the general equation of an ellipse, which is situated between a rectangle of arms $2a$ and $2b$.

Now the following cases may asise

Case I : When $\phi = 0°$; then $\cos \phi = 1$, and $\sin \phi = 0$, From equation (3), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos 0^\circ = 0 \qquad \left[\because \begin{array}{l}\cos 0^\circ = 1\\ \sin 0^\circ = 0\end{array}\right]$$

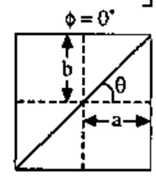

Fig. 9.(a)

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 = 0$$

or
$$y = \frac{b}{a}x$$

This is the equation of straight line, fig. 9 (a).

Case II : When $\phi = \pi/4$, then $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ from eq. (3), we get

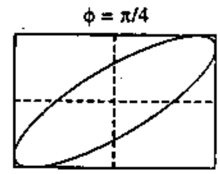

Fig. 9.(b)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\frac{\pi}{4} = \sin^2\frac{\pi}{4}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{\sqrt{2}\,xy}{ab} = \frac{1}{2}$$

This is the equation of oblique ellipse as shown in the fig. 9 (b).

Case III : When $\phi = \frac{\pi}{2}$, then $\cos\phi = 0$ and $\sin\phi = 1$ then from eq. (3), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

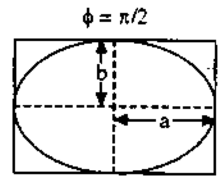

Fig. 9.(c)

This equation represents the ellipse whose axes are along the coordinate axis as shown in the fig. 9 (c).

Case IV : When $\phi = \frac{3\pi}{4}$, then $\cos\phi = -\frac{1}{\sqrt{2}}$ so by eq. (3), we get

$\sin\phi = \frac{1}{\sqrt{2}}$

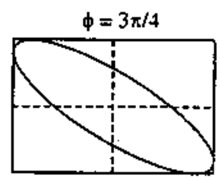

Fig. 9.(d)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{\sqrt{2}\,xy}{ab} = \frac{1}{2}$$

This is the equation of oblique ellipse as shown in fig. 9 (d).

Case V : When $\phi = \pi$, then $\cos\phi = -1$ and $\sin\phi = 0$; from eq. (3)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab} = 0$$

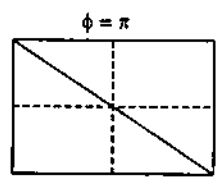

Fig. 9.(e)

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2 = 0$$

$$y = -\frac{b}{a}x$$

This is the equation of straight line as shown in fig. 9 (e).

Frequencies 'Nearly' Equal : If the frequencies of the two simple harmonic motions are exactly equal, the Lissajou's figures remain perfectly steady. If the frequencies are nearly equal (not exactly) the phase difference ϕ between the two motions will not remain steady but it will change slowly *i.e.* starting from $\phi = 0$ to $\pi/2$ and again $\pi/2$. to π, again from π to 2π. We know at $\phi = 0$, straight line is obtained, at $\phi = \pi/4$, oblique ellipse is obtained, at $\phi = \pi/2$ ellipse is obtained and so on. This means that in one motion (0 to 2π) all the possible forms of Lissajou's figure are obtained.

The frequency of this complete motion (cycle) is equal to the difference in the frequencies of the component motions.

Thus, **"if the frequencies are nearly equal (not exactly) then all the possible forms of Lissajous' figures will be obtained."**

• 5.14. COMPOSITION OF TWO RECTANGULAR S.H.M.'s IN FREQUENCY RATIO 2 : 1

Let the simple harmonic motions of frequencies in the ratio 2 : 1 be given by

$$x = a \sin(2\omega t + \phi) \qquad \ldots (1)$$

and

$$y = b \sin \omega t \qquad \ldots (2)$$

From equation (1), we get

$$\frac{x}{a} = \sin 2\omega t \cos \phi + \cos 2\omega t \,.\, \sin \phi$$

$$\frac{x}{a} = 2 \sin \omega t \cos \omega t \cos \phi + (1 - 2\sin^2 \omega t) \sin \phi$$

From equation (2) $\sin \omega t = \frac{y}{b}$

$\therefore$ and $\cos \omega t = \sqrt{1 - \sin^2 \omega t} = \sqrt{1 - \frac{y^2}{b^2}}$

$$\therefore \quad \frac{x}{a} = \frac{2y}{b}\sqrt{\left(1 - \frac{y^2}{b^2}\right)} \cos \phi + \left(1 - \frac{2y^2}{b^2}\right) \sin \phi$$

or

$$\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \phi = \frac{2y}{b}\sqrt{\left(1 - \frac{y^2}{b^2}\right)} \cos \phi$$

Squaring on both sides, we get

$$\frac{x^2}{a^2} + \left(1 - \frac{2y^2}{b^2}\right)^2 \sin^2 \phi - 2\frac{x}{a}\left(1 - \frac{2y^2}{b^2}\right) \sin \phi = \frac{4y^2}{b^2}\left(1 - \frac{y^2}{b^2}\right) \cos^2 \phi$$

or

$$\frac{x^2}{a^2} + \sin^2 \phi + \frac{4y^4}{b^4} \sin^2 \phi - \frac{4y^2}{b^2} \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4xy^2}{ab^2} \sin \phi = \frac{4y^2}{b^2} \cos^2 \phi - \frac{4y^4}{b^4} \cos^2 \phi$$

or

$$\frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} (\sin^2 \phi + \cos^2 \phi) - \frac{4y^2}{b^2} (\sin^2 \phi + \cos^2 \phi) + \frac{4xy^2}{ab^2} \sin \phi = 0$$

or

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

or

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{a} \sin \phi\right) = 0 \qquad \ldots (3)$$

This is the general equation of the path of the particle. This is the equation of a curve having two loops as shown in the fig.

Now following cases may arises.

Case I : When $\phi = 0$, then $\sin\phi = 0$, from eq. (iii), we get

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0$$

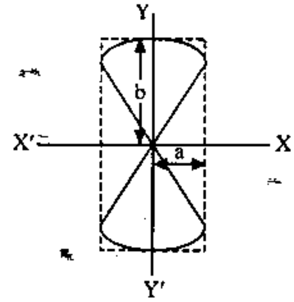

Fig. 10.(a)

This equation represents the English digit 8 type figure which is symmetrical about both axes, as shown in the fig. 10(a).

Case II : When $\phi = \pi/4$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ then from equation (3), we get

$$\left(\frac{x}{a} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{a^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{\sqrt{2}a}\right) = 0$$

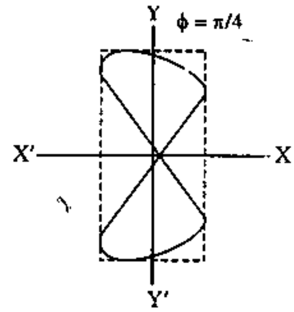

Fig. 10.(b)

This is the equation of curve having two loops as shown in the fig. 10 (b).

Case III : When $\phi = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$, from equation (3), we get

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{a}\right) = 0$$

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2} + \frac{4y^4}{b^4}\left(\frac{x}{a} - 1\right) = 0$$

$$\left[\left(\frac{x}{a} - 1\right)^2 + \frac{2y^2}{b^2}\right]^2 = 0$$

or $$y^2 = -\frac{b^2}{2a}(x - a)$$

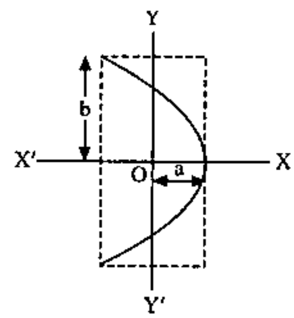

Fig. 10.(c)

This is the equation of parabola as shown in the fig. 10 (c).

Case IV : When $\phi = \frac{3\pi}{4}$ and $\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, from equation (3)

$$\left(\frac{x}{a} + \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{a^2}\left(\frac{y^2}{b^2} - 1 - \frac{x}{a\sqrt{2}}\right) = 0$$

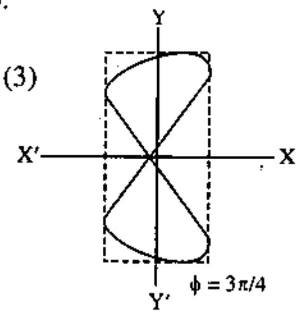

Fig. 10.(d)

This is the equation of a curve having two loops as shown in fig. 10(d).

Case V : When $\phi = \pi$. In this position the path of the particle is similar to that in the case I.

Case VI : When $\phi = \frac{3\pi}{2}$, then from equation (3), we get

$$\left(\frac{x}{a} + 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 - \frac{x}{a}\right) = 0$$

or $$\left(\frac{x}{a} + 1\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2}\left(\frac{x}{a} + 1\right) = 0$$

or $$\left[\left(\frac{x}{a} + 1\right) - \frac{2y^2}{b^2}\right]^2 = 0$$

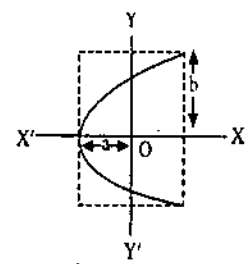

Fig. 10.(e)

$$y^2 = \frac{b^2}{2a}(x+a)$$

This is the equation of parabola as shown in the fig. 10 (e).

If there is a small change in the frequency ratio 2 : 1, then Lissajou's figure changes slowly and all the forms of figures will be obtained slowly.

• 5.15. LISSAJOU'S FIGURES PRODUCED BY TUNING FORKS OF FREQUENCIES 100 AND 200 cps

Let the equation of motion produced by two tuning forks be

$$x = a \sin \omega t \quad \ldots (1)$$

and

$$y = a \sin (2\omega t + \phi) \quad \ldots (2)$$

where a is the amplitude of each tuning forks, ω is the frequency and ϕ is the initial phase difference.

From equation (2)

$$\frac{y}{a} = \sin 2\omega t \, . \cos \phi + \cos 2\omega t \sin \phi$$

$$= 2 \sin \omega t \, . \cos \omega t \, . \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi$$

From eqn. (1), $\sin \omega t = \frac{x}{a}$ and $\cos \omega t = \sqrt{1 - \frac{x^2}{a^2}}$

so

$$\frac{y}{a} = \frac{2x}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cos \phi + \left[1 - \frac{2x^2}{a^2}\right] \sin \phi$$

or

$$\frac{y}{a} - \left(1 - \frac{2x^2}{a^2}\right) \sin \phi = \frac{2x}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cos \phi$$

Taking square on both sides, we get

$$\frac{y^2}{a^2} + \left(1 - \frac{2x^2}{a^2}\right)^2 \sin^2 \phi - \frac{2y}{a}\left(1 - \frac{2x^2}{a^2}\right) \sin \phi = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right) \cos^2 \phi$$

or

$$\frac{y^2}{a^2} + \sin^2 \phi + \frac{4x^4}{a^4} - \frac{4x^2}{a^2} - \frac{2y}{a} \sin \phi + \frac{4yx^2}{a^3} \sin \phi = 0$$

or

$$\left(\frac{y}{a} - \sin \phi\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 + \frac{y}{a} \sin \phi\right) = 0 \quad \ldots (3)$$

This is the general equation of a curve having two loops.

Now following cases may arise :

Case I : When $\phi = 0$ then $\sin \phi = 0$ and equation (3) reduces to

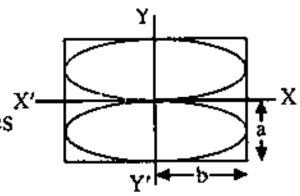

Fig. 11.(a)

$$\frac{y^2}{a^2} + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1\right) = 0$$

This equation represents the curve similar to English figure 8 as shown in the fig. 11(a).

Case II : When $\phi = \pi/4$, then $\sin \phi = \frac{1}{\sqrt{2}}$ and from equation (3), we get

$$\left(\frac{y}{a}-\frac{1}{\sqrt{2}}\right)^2+\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}-1+\frac{y}{a\sqrt{2}}\right)=0$$

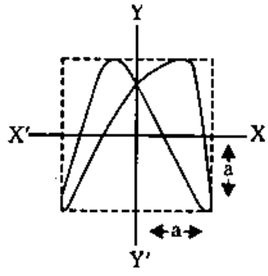

Fig. 11.(b)

This is the equation of the curve having two loops as shown in the fig 11. (b).

Case III : When $\phi=\pi/2$, then $\sin\phi=1$ so by, (3), we get

$$\left(\frac{y}{a}-1\right)^2+\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}-1+\frac{y}{a}\right)=0$$

or
$$\left(\frac{y}{a}-1\right)^2+\frac{4x^4}{a^4}+\frac{4x^2}{a^2}\left(\frac{y}{a}-1\right)=0$$

or
$$\left[\left(\frac{y}{a}-1\right)+\frac{2x^2}{a^2}\right]^2=0$$

or
$$x^2=-\frac{a}{2}(y-a)$$

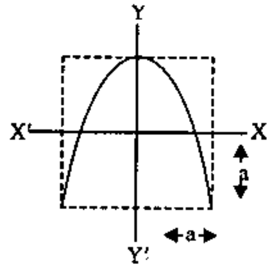

Fig. 11.(c)

This is the equation of parabola as shown in the fig. 11 (c).

Frequencies 100 and 201 (Nearly 1 : 2)

We know that, if the frequencies of the two component wibrations are exactly in the ratio 1 : 2 (as 100 and 200), the path of resultant vibration remains steady. In this position, although the phase difference between the two components changes, but after each $\frac{1}{100}$ second it attains its initial value. But if the ratio is nearly 1 : 2 (as 100 and 201). then it causes its own variation in the phase difference ϕ. During each second, one component completes one full vibration more than double of the other and assumes all values between 0 and 2π. Due to this, the resultant vibration describes a continuously changing path passing through all the terms.

• 5.16. RESULTANTANT OF TWO RECTANGULAR SIMPLE HARMONIC MOTIONS WHOSE AMPLITUDES AS WELL AS PERIODS ARE IN THE RATIO 1 : 2 AND PHASE DIFFERENCE IS 90°

Let us suppose that a particle be subjected to two simple harmonic motions of periods as well as amplitudes in the ratio 1 : 2, and acting along the axes of x and y respectively. Let the x-motion lead over the y-motion by 90° in phase. The equations of these motions would be

$$x=a\sin\left(2\omega t+\frac{\pi}{2}\right)$$

and
$$y=2a\sin\omega t, \qquad \ldots (2)$$

where a and $2a$ are the amplitudes, and 2ω and ω the frequencies respectively.

The equation of the resultant path of the particle is obtained by eliminating t between eqs. (1) and (2).

Eq. (1) gives

$$\frac{x}{a}=\sin\left(2\omega t+\frac{\pi}{2}\right)$$

$$=\cos 2\omega t=1-2\sin^2\omega t.$$

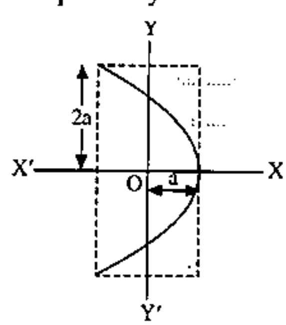

Fig. 12.

From eq. (2),

$$\sin\omega t=y/2a.$$

$$\therefore \quad \frac{x}{a} = 1 - \frac{y^2}{2a^2}$$

or $$\frac{y^2}{2a^2} = 1 - \frac{x}{a}$$

$$y^2 = 2a(a - x)$$

This is the equation of a parabola as shown in fig. 12.

Now, consider the case when the y-motion leads over the x-motion by $\pi/2$. Now the equations of motions will be

$$x = a \sin 2\omega t \quad \ldots (3)$$

and $$y = 2a \sin\left(\omega t + \frac{\pi}{2}\right) \quad \ldots (4)$$

From eq. (3), we have

$$\frac{x}{a} = 2\sin \omega t \cos \omega t$$

$$= 2\sqrt{(1 - \cos^2 \omega t)} \cos \omega t.$$

Putting for $\cos \omega t$ in the last eq., we get

$$\frac{x}{a} = 2\sqrt{\left(1 - \frac{y^2}{4a^2}\right)} \frac{y}{2a}$$

or $$\frac{x^2}{a^2} = \frac{y^2}{a^2}\left(1 - \frac{y^2}{4a^2}\right)$$

or $$x^2 + y^2\left(\frac{y^2}{4a^2} - 1\right) = 0$$

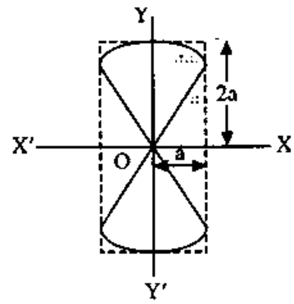

Fig. 13.

This equation represents the curve of '8' as shown in fig. 13.

• 5.17. DEMONSTRATION OF LISSAJOU'S FIGURES

(i) Optical Method : Two electrically maintained tuning forks F_1 and F_2 are placed such that the vibrations of one take place in a vertical plane and those of the other in a horizontal plane. A thin strip of mica carrying mirror is fastened to the side of one prong of each of the forks as at M_1 and M_2. Light from a source (convergent by a lens) falls on the mirror M_1. This light reflected from M_1 strikes the mirror M_2 from where it is reflected on to the screen O. The position of the lens is adjusted so that image is produced on the screen.

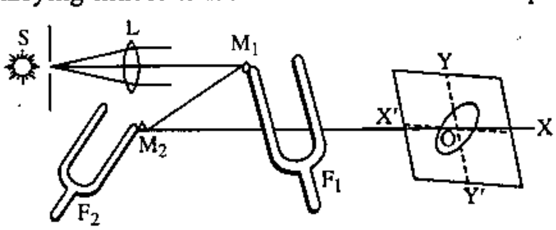

Fig. 14.

Now, if the vertical fork F_1 alone is in vibration, the spot of light on the screen executes a vertical S.H.M. so rapidly that only a bright vertical line is observed. If both the forks vibrate *simultaneously*, the two rectangular vibrations of the spot of light are compounded and the spot describes the path of the resultant motion.

(ii) Blackburn's Pendulum : It is a mechanical device to demonstrate the Lissajou's figures. It consists of a string with its two ends attached to a rigid beam at two points M and N, fig. 15. The string is cut at its centre and the ends so formed are attached to a heavy funnel C. The exit tube of the funnel is very thin so that when it is filled with sand, a fine stream falls on the ground. A clip B brings the two strings together, which can be slipped on the string.

It is equivalent to two pendula. If the funnel is vibrated in the plane, the effective length of the pendulum will be BC and AC when it is vibrated perpendicular to the plane of paper. If the funnel is pulled out slantwise and released, it moves under the action of two S.H.M.'s at right angles. A stream of sand escapes from the funnel which traces a Lissajou's figure on the ground.

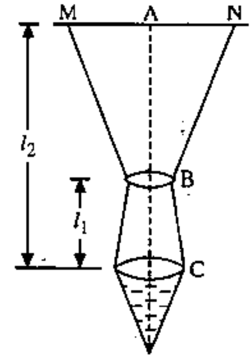

Fig. 15.

Let $BC = l_1$ and $AC = l_2$. The periods of the two S.H.M.'s are given by

$$T_1 = 2\pi\sqrt{\frac{l_1}{g}}$$

and

$$T_2 = 2\pi\sqrt{\frac{l_2}{g}}$$

$$\therefore \qquad \frac{T_1^2}{T_2^2} = \frac{l_1}{l_2}.$$

Changing the position of the clip B, the effective lengtlis of the pendula, l_1 and l_2 change and we get different ratios of the periods.

If $\qquad \frac{l_1}{l_2} = \frac{1}{4}$, we get $\frac{T_1}{T_2} = \frac{1}{2}$.

The sand traces a curve having two loops. Similarly different types may be obtained.

(iii) Cathode Ray Oscilloscope (C.R.O.) It is an instrument for plotting varying physical quantity potential difference, sound pressure, heart beat-against another-current, displacement, time. It is so called because it traces the desired wave-form with a beam of electrons, and beams of electrons were originally called cathode rays.

Principle : It makes use of two properties of electrons namely

(a) An electron beam is deflected in magnetic and electric field.

(b) When fast moving electrons strike the glass screen coated with zinc sulphide, they produce fluorescence.

Construction : C.R.O. is essentially an electrostatic instrument. It consists of a highly evacuated glass tube, one end of which opens out to form a screen S which is coated internally with zinc sulphide. A hot cathode filament C at the other end of the tube, emits electrons. These are then attracted by the cylinders A_1, A_2 and A_3 which have increasing positive potentials with respect to the filament. Many of the electrons shoot through the cylinders and strike the screen S to produce fluorescence in a green spot. On their way to screen, the electrons pass through two pairs of metal plates, X_1X_2 and $Y_1 Y_2$, called the deflecting plates.

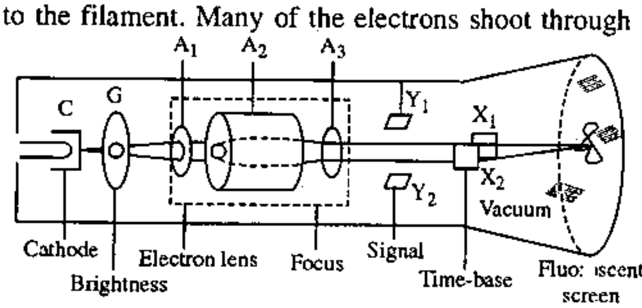

Fig. 16.

Working : If an A.C. voltage is applied only to $X_1 X_2$ plates, the spot would be simply drawn out into a horizontal line. If the voltage is applied only to $Y_1 Y_2$ plates, the spot would be drawn out into a vertical line. If the voltage is applied between $X_1 X_2$ and $Y_1 Y_2$ both, the spot traces out the Lissajous' figures. The wave form of the A.C. voltage can also be traced.

Uses : It is used in television, Radar and the only instrument used in wireless and electronics to study the A.C. voltage, comparison of frequency and to measure the phase.

• 5.18. DETERMINATION OF UNKNOWN FREQUENCY BY LISSAJOU'S FIGURES

The Lissajous' figures provide an important method of comparing the forks frequencies of two tuning forks. These are arranged to produce Lissajou's figures which are carefully studied.

(1) When the frequencies of two forks are in a whole number ratio. ($n_1/n_2 = 1, 2, 3, ...$). Let the frequenciess of the two forks be n_1 and n_2. In this case a *steady* figure is obtained, (say fig. 17).

The fork of frequency n_1 is vibrating along the X-axis (say) and the fork of frequency n_2 along the Y-axis. If the complete figure is traced in t second; then in t second the number of vibrations along the X-axis will be $n_1 t$, while the number along the Y-axis will be $n_2 t$.

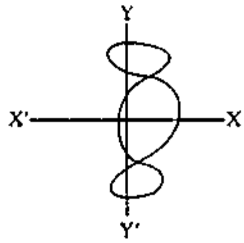

Fig. 17.

A vibrating point passes through any point in its path twice in one complete vibration, therefore during $n_1 t$ vibrations along the X-axis, the figure will cut the Y-axis $2n_1 t$ times. Similarly, vibrations will cut the X-axis $2n_2 t$ times. Thus

$$\frac{\text{number of times } P_y \text{, the figure cuts the } Y\text{-axis}}{\text{number of times } P_x \text{, the figure cuts the } X\text{-axis}} = \frac{2n_1 t}{2n_2 t} = \frac{n_1}{n_2}.$$

$$\therefore \quad \frac{p_y}{p_x} = \frac{n_1}{n_2} = \frac{\text{frequency of } X\text{-vibrations}}{\text{frequency of } Y\text{-vibrations}}$$

If the number p_x and p_y are counted, the ratio of the frequencies of the two tuning forks is obtained.

From fig. 17, $\quad p_y = 6, p_x = 2.$

$$\therefore \quad \frac{n_1}{n_2} = \frac{6}{2} = \frac{3}{1}.$$

Therefore, the ratio of the frequencies is 3 : 1.

(2) When the Frequencies are Nearly Equal : Suppose the frequency of one fork is n_1 and that of the other is n_2, such that n_1 is only slightly different from n_2. In this case the motion of the higher frequency fork continuously goes ahead of the other. The Lissajou's figure will gradually change, assuming in turn the form of a straight line, ellipse or circle. Suppose the complete cycle is traced in t second. During t second the higher fork has made one vibration more than that of the lower. But the number of vibrations made by the forks in t second are $n_1 t$ and $n_2 t$.

$$\therefore \quad n_1 t \sim n_2 t = 1$$

or $$n_1 \sim n_2 = 1/t$$

or $$n_1 = n_2 \pm 1/t.$$

• 5.19. COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS HAVING SLIGHTLY DIFFERENT FREQUENICES

Given that

$$x_1 = a \sin \omega_1 t$$
$$x_2 = a \sin \omega_2 t$$

After superposition

$$x = x_1 + x_2$$

$$x = a(\sin \omega_1 t + \sin \omega_2 t)$$

$$= 2a \sin\left(\frac{\omega_1 + \omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_1 - \omega_2}{2}\right)t$$

$$= \left[2a \cos\frac{(\omega_1 - \omega_2)\,t}{2}\right] \sin\left(\frac{\omega_1 + \omega_2}{2}\right)t$$

$$= A \sin\frac{(\omega_1 + \omega_2)\,t}{2}, \text{ where } A = 2a \cos\frac{(\omega_1 - \omega_2)}{2}t$$

This equation represents a periodic motion not the simple harmonic. Its amplitude is given by

$$A = 2a \cos\frac{(\omega_1 - \omega_2)t}{2}$$

or $$A = 2a \cos 2\pi \frac{(n_1 - n_2)t}{2} \qquad [\because \quad \omega = 2\pi n]$$

Thus, the resultant amplitude of motion varies periodically between 0 and $\pm 2a$ as a cosine function.

Amplitude A will be maximum if

$$\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t = \pm 1$$

or $$2\pi\left(\frac{n_1 - n_2}{2}\right) = m\pi, \qquad \text{where } m = 0, 1, 2, 3.........$$

$\therefore$ $$t = \frac{m}{n_1 - n_2}$$

$$= 0, \frac{1}{n_1 - n_2}, \frac{2}{n_1 - n_2}, \frac{3}{n_1 - n_2} ....$$

Hence, the time interval between two consecutive maxima is, $\frac{1}{n_1 - n_2}$.

Amplitude A will be minimum when

$$\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t = 0$$

or $$2\pi \frac{(n_1 - n_2)\,t}{2} = \left(m + \frac{1}{2}\right)\pi$$

where $m = 0, 1, 2, 3....$

or $$t = \frac{m + \frac{1}{2}}{n_1 - n_2}$$

$$t = \frac{1}{2(n_1 - n_2)}, \frac{3}{2(n_1 - n_2)}, \frac{5}{2(n_1 - n_2)}, ...$$

Hence, the time interval between two consecutive minima is also, $\frac{1}{n_1 - n_2}$.

• SUMMARY

- The motion of a body is said to be periodic motion if its motion is repeated identically after a fixed interval of time and this fixed interval of time is known as **period of motion**.
- When a body moves to and fro repeatedly about its mean position in a definite interval of time then this motion is known as oscillatory or vibratory motion.
- When a particle moves to and fro repeatedly about its mean position under a restoring force, which is always directed towards the mean position and whose magnitude at any instant is directly proportional to the displacement of the particle from the mean position at that instant, this motion of the particle is known as **simple harmonic motion.**
- Equations of motion for a simple harmonic oscillator are :
  (i) Displacement $(x) = a \sin(\omega t + \phi)$
  (ii) Velocity $(v) = \omega\sqrt{a^2 - x^2}$
  (iii) Time period $(T) = 2\pi\sqrt{\frac{M}{K}}$
  (iv) Frequency $(f) = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$
- Energy of a particle executing simple harmonic motion.

$$U = \frac{1}{2} Ka^2 \sin^2(\omega t + \phi)$$

$$\text{K.E.} = \frac{1}{2} Ka^2 \cos^2(\omega t + \phi)$$

Total energy $(E) = 2\pi^2 n^2 ma^2$

- When a heavy particle is suspended by an inextensible, weightless and flexible string from a rigid support, then this system is known as simple pendulum.
- When a rigid body which is capable of oscillating freely in a vertical plane about a fixed axis, passing through the body but not through its centre of gravity then the system is called compound pendulum and this fixed point is known as point of suspension
- The time period T of a compound pendulum is given by $T = 2\pi\sqrt{\dfrac{\frac{k^2}{l} + l}{g}}$
- When one end of a very thin and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy disc or sphere, then this arrangement is known as torsion pendulum.
- The time period T of a Torsion Pendulum is given by $T = 2\pi\sqrt{\frac{I}{c}}$.
- When a particle is acted upon by two mutually perpendicular simple harmonic motions simultaneously then due to the effect of these motions the particle traces a curve. These curves are known as Lissajou's Figures.

• STUDENT ACTIVITY

1. What are the factors on which the shape of Lissajou's figures depends ?

2. What are the uses of Lissajous' figures ?

• TEST YOURSELF

1. Explain the terms :
   (a) Periodic motion (b) Oscillatory motion
   (c) Time period (d) Frequency (e) Phase
2. State and explain simple harmonic motion. Give its characteristics and explain the terms related to it.
3. Obtain the equation of motion of a simple harmonic oscillator and solve it. Obtain the expressions for its velocity, period and frequncy.
4. Derive an expression for the energy of a body executing S.H.M. and show that it is proportional to the square of the amplitude of motion.
5. What do you understand by a potential well ?
6. Find an expression for period of oscillation of a mass m connected with a massless vertical spring.

7. Explain simple pendulum and obtain an expression for its time period.
8. What is a compound pendulum ? Show that its motion is simple harmonic and obtain its period.
9. Explain torsional pendulum and obtain the expression for its time period.
10. Two S.H.M.'s are imposed on a particle in same direction with same time period. Obtain the expressions for the resultant amplitude and phase.
11. What are Lissajou's figures ? Calculate the resultant of two S.H.M.'s of equal time period when they act at right angles to each other.
12. Calculate the resultant of two S.H.M.'s at right angles to each other having periods in the ratio 1 : 2.
13. The necessary and sufficient condition for simple harmonic motion is :
    (a) constant acceleration
    (b) constant time period
    (c) restoring force directly proportional to displacement
    (c) restoring acceleration directly proportional to displacement.
14. The work done by a simple pendulum in one complete oscillation is :
    (a) zero (b) equal to E_t
    (c) equal to E_k (d) equal to v.
15. A particle is moving such that its acceleration is represented by the equation $a = -bx$, where x is its displacement from mean position and b is a constant. Its time period will be :
    (a) $2\pi\sqrt{b}$ (b) $\frac{2\pi}{\sqrt{b}}$ (c) $\frac{2\pi}{b}$ (d) $2\sqrt{\frac{\pi}{b}}$.
16. The total energy of a particle executing simple harmonic motion is directly proprotional to :
    (a) the square of amplitude of motion
    (b) the amplitude of motion
    (c) the frequency of oscillator
    (d) the displacement from mean position.
17. A particle is executing simple harmonic motion with frequency f. Its energy will oscillate with a frequency :
    (a) $2f$ (b) $4f$ (c) f (d) $f/2$.
18. The time-period and amplitude of a particle executing simple harmonic motion are 6 second and 3 cm respectively. Its maximum velocity in cm/sec will be :
    (a) $\frac{\pi}{2}$ (b) π (c) 2π (d) 3π.
19. The potential energy of an oscillating simple pendulum is maximum at :
    (a) mean position (b) extreme position
    (c) at the mid point on the right of mean position
    (d) at the mid point on the left of mean position.
20. The time period of a simple pendulum in a stationary train is T. This train is accelerated with an acceleration 'a', then its time period will :
    (a) decrease (b) increase
    (c) become infinite (d) remain unchanged.
21. The physical quantity conserved in simple harmonic motion is :
    (a) potential energy (b) kinetic energy
    (c) time period (d) total energy.
22. The phase difference between two waves of frequencies n and $2n$ is 4π. The Lissajou's, figure resulting from the superposition of these waves will be:
    (a) circle (b) ellipse
    (c) figure of 8 (d) straight line.

23. The Lissajou's figure obtained by two forks changes from a parabola to the figure of 8. The ratio of frequencies of forks will be :
(a) 2 : 1 (b) 1 : 1
(c) approximately 2 : 1 (d) approximatley 1 : 1.

24. The amplitudes, frequencies and phase of two mutually perpendicular simple harmonic motions are same. The resultant vibration obtained from the superposition of these will be :
(a) parabola (b) straight line (c) ellipse (d) circle.

25. Two waves are given as $x = 4 \sin 50\, \omega t$ and $y = \cos 25\, \omega t$. The Lissajou's, figure obtained from the superposition of these waves will be :
(a) figure of 8 (b) parabola
(c) circle (d) ellipse.

26. Two mutually perpendicular simple harmonic motions having same amplitudes and frequencies superimpose to produce Lissajou's, figures is the form of a circle. The phase difference between the two is :
(a) 0 (b) 180 (c) 45 (d) 90.

27. The ratio of frequencies of two forks is 1 : 2. The phase difference between two perpendicular sound waves emitted by them is zero. The shape of Lissajou's obtained will be a :
(a) circle (b) ellipse (c) straight line (d) figure of 8.

28. Which Lissajou's figure is represented by the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 :$$

(a) oblique ellipse (b) parabola
(c) straight ellipse (d) circle.

ANSWERS

13. (c) **14.** (a) **15.** (b) **16.** (a) **17.** (a) **18.** (b) **19.** (b) **20.** (a) **21.** (d) **22.** (c)
23. (c) **24.** (b) **25.** (b) **26.** (d) **27.** (d) **28.** (c)

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