मङ्गलायतन विश्वविद्यालय

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CONTENTS

UNIT

1

RIEMANN INTEGRATION

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STRUCTURE

1.1. INTRODUCTION

In elementary calculus, the process of integration is treated as the inverse operation of differentiation and the integral of a function is called an anti-derivative. Historically, however, the subject of integration was developed is connection with areas of plane regions. It was based on the concept of the limit of a sum when the number of terms in the sum tends to infinity and each term tends to zero. This notion of integral as summation is associated with intuitive dependence on geometrical concepts. The first satisfactory rigorous arithmetic treatment of definite integral was given by a German mathematician George Friedrich Bernhard Riemann (1826–1866). Many refinements and generalisations of the subject have appeared since then. However, we shall confine ourselves to the discussion of Riemann integration only.

We shall be dealing with closed finite intervals $[a, b]$ so that $(b - a) \in R$ and $x \in [a, b]$ implies $a \le x \le b$. Moreover, all functions f will be assumed to be real valued function defined and bounded on $[a, b]$.

Thus $f : [a, b] \to R$ and $| f(x) | \le k$, where k is a positive real number.

1.2. PARTITION OF A CLOSED INTERVAL

Let $I = [a, b]$ be a finite closed interval. If $a = x_0 < x_1 < x_2 < \ldots\ldots < x_n = b$, then the finite ordered set $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ is called a partition of I.

The $n + 1$ points $x_0, x_1, x_2, \ldots\ldots, x_n$ are called partition points of P.

The n closed sub-intervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$, ..., $I_r = [x_{r-1}, x_r]$, ..., $I_n = [x_{n-1}, x_n]$ determined by P are called the segments of the partition P.

Clearly, $$\bigcup_{r=1}^{n} I_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a, b] = I$$

The length of the rth sub-interval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r. Thus $\delta_r = x_r - x_{r-1}$.

Note 1. Partition is also known as dissection or net.

Note 2. By changing the partition points, the partition can be changed and hence, there can be an infinite number of partitions of the interval I.

We shall denote by $P[a, b]$ the set (or family) of all partitions of $[a, b]$.

1.3. NORM OF A PARTITION

The maximum of the lengths of the sub-intervals of a partition P is called the **norm or mesh** of the partition P and is denoted by $\| P \|$ or μ (P).

Thus $$\| P \| = \max. \{\delta_r : r = 1, 2, \ldots\ldots, n\}$$
$$= \max. (x_r - x_{r-1} : r = 1, 2, \ldots\ldots, n\}.$$

1.4. REFINEMENT OF A PARTITION

If P, P′ be two partitions of $[a, b]$ and $P \subset P'$, then the partition P′ is called a refinement of partition P on $[a, b]$. We also say P′ is finer than P.

Thus, if P′ is finer than P, then every point of P is used in the construction of P′ and P′ has at least one additional point.

If P_1, P_2 are two partitions of $[a, b]$, then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. Therefore, $P_1 \cup P_2$ is called a **common refinement** of P_1 and P_2.

Note 1. If $P_1, P_2 \in P[a, b]$ and $P_1 \subset P_2$, then $\| P_2 \| \le \| P_1 \|$.

Note 2. If $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ is a partition of $[a, b]$,

then $$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \ldots\ldots + \delta_n = (x_1 - x_0) + (x_2 - x_1) + \ldots\ldots + (x_n - x_{n-1}) = x_n - x_0 = b - a.$$

1.5. UPPER AND LOWER DARBOUX SUMS

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Let $f : [a, b] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a, b]$.

Since f is bounded on $[a, b]$, f is also bounded on each of the sub-intervals. Let M, m be the supermum and infimum of f in $[a, b]$ and M_r, m_r be the supermum and infimum of f in the rth sub-interval $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots\ldots, n$.

The sum $M_1\delta_1 + M_2\delta_2 + \ldots\ldots + M_r\delta_r + \ldots\ldots + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$ is called the **upper Darboux sum of** f corresponding to the partition P and is denoted by $U(P, f)$ or $U(f, P)$.

The sum $m_1\delta_1 + m_2\delta_2 + \ldots\ldots + m_r\delta_r + \ldots\ldots + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is called the **lower Darboux sum of f** corresponding to the partition P and is denoted by $L(P, f)$ or $L(f, P)$.

Thus $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r ; \quad L(P, f) = \sum_{r=1}^{n} m_r\delta_r$$

Clearly, these sums depend upon the function f and the partition P, and do exist for every bounded function.

1.6. OSCILLATORY SUM

Let $f : [a, b] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a, b]$.

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots\ldots, n$. Then

$$U(P, f) - L(P, f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r = \sum_{r=1}^{n} O_r\delta_r$$

where $O_r = M_r - m_r$ denotes the oscillation of f in I_r.

$U(P, f) - L(P, f) = \sum_{r=1}^{n} O_r\delta_r$ is called the oscillatory sum of f corresponding to the partition P and is denoted by $\omega(P, f)$.

Since $O_r = M_r - m_r \geq 0$, $r = 1, 2, \ldots\ldots, n$, each oscillatory sum consists of a finite number of non-negative terms.

$\therefore$ $$\omega(P, f) \geq 0.$$

Theorem 1. *If $f : [a, b] \to R$ is a bounded function and $P \in P[a, b]$, then*

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$$

where m, M are the infimum and supermum of f on $[a, b]$.

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Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$.

f is bounded on $[a, b]$ $\Rightarrow$ f is bounded on each sub-interval $[x_{r-1}, x_r]$, $r = 1, 2, ..., n$.

Let m_r and M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$.

Clearly $\quad m \le m_r \le M_r \le M \;\Rightarrow\; m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$

$$\Rightarrow \quad \sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$

$$\Rightarrow \quad m \sum_{r=1}^{n} \delta_r \le L(P, f) \le U(P, f) \le M \sum_{r=1}^{n} \delta_r$$

$$\Rightarrow \quad m(b-a) \le L(P, f) \le U(P, f) \le M(b-a) \qquad \left[\because \sum_{r=1}^{n} \delta_r = b - a\right]$$

Note. The above theorem implies that $L(P, f)$ and $U(P, f)$ are bounded if f is bounded.

Theorem 2. *If $f : [a, b] \to R$ is a bounded function and $P \in P(a, b)$, then*

(i) $L(P, f) \le U(P, f)$ *(ii)* $L(P, -f) = -U(P, f)$ *and* $U(P, -f) = -L(P, f)$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Let m, M be the infimum and supermum of f on $[a, b]$ and m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$.

(*i*) We have $\quad m_r \le M_r, \quad r = 1, 2, ..., n$

$$\Rightarrow \quad m_r\delta_r \le M_r\delta_r \;\Rightarrow\; \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \;\Rightarrow\; L(P, f) \le U(P, f).$$

(*ii*) f is bounded on $[a, b]$ $\Rightarrow$ $-f$ is bounded on $[a, b]$. m_r, M_r are the infimum and supermum of f on I_r.

$\Rightarrow$ $-M_r$, $-m_r$ are the infimum and supermum of $-f$ on I_r. **(Note this step)**

$$\therefore \quad \text{By definition, } L(P, -f) = \sum_{r=1}^{n} (-M_r)\delta_r = -\sum_{r=1}^{n} M_r\delta_r = -U(P, f)$$

$$U(P, -f) = \sum_{r=1}^{n} (-m_r)\,\delta_r = -\sum_{r=1}^{n} m_r\delta_r = -L(P, f).$$

Theorem 3. *If $f : [a, b] \to R$, $g : [a, b] \to R$ are bounded functions and $P \in P[a, b]$, then*

(i) $U(P, f + g) \le U(P, f) + U(P, g)$ *(ii)* $L(P, f + g) \ge L(P, f) + L(P, g)$

(iii) $\omega(P, f + g) \le \omega(P, f) + \omega(P, g)$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

f, g are bounded on $[a, b]$ $\Rightarrow$ $f + g$ is bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supermum of f on I_r,

m_r'', M_r'' be the infimum and supermum of g on I_r and

m_r, M_r be the infimum and supermum of $f + g$ on I_r.

(*i*) M_r', M_r'' are supermum of f, g on I_r

$$\Rightarrow \quad f(x) \le M_r',\; g(x) \le M_r'' \quad \forall\, x \in I_r$$

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$\Rightarrow \quad f(x) + g(x) \le M_r' + M_r'' \quad \forall\, x \in I_r$

$\Rightarrow \quad (f+g)(x) \le M_r' + M_r'' \quad \forall\, x \in I_r$

$\Rightarrow$ $M_r' + M_r''$ is an upper bound of $f + g$ on I_r

But M_r is the least upper bound of $f + g$ on I_r

$\therefore \quad M_r \le M_r' + M_r''$ on I_r, $r = 1, 2, \ldots\ldots, n \Rightarrow \quad M_r\delta_r \le M_r'\delta_r + M_r''\delta_r$

$$\Rightarrow \quad \sum_{i=1}^{\infty} M_r\delta_r \le \sum_{r=1}^{n} M_r'\delta_r + \sum_{r=1}^{n} M_r''\delta_r \quad \Rightarrow \quad U(P, f+g) \le U(P, f) + U(P, g).$$

(ii) m_r', m_r'' are infimum of f, g on I_r.

$\Rightarrow \quad f(x) \ge m_r',\ g(x) \ge m_r'' \quad \forall\, x \in I_r$

$\Rightarrow \quad f(x) + g(x) \ge m_r' + m_r'' \quad \forall\, x \in I_r$

$\Rightarrow \quad (f+g)(x) \ge m_r' + m_r'' \quad \forall\, x \in I_r$

$\Rightarrow$ $m_r' + m_r''$ is a lower bound of $f + g$ on I_r.

But m_r is the greatest lower bound of $f + g$ on I_r.

$\therefore \quad m_r \ge m_r' + m_r'' \quad \Rightarrow \quad m_r\delta_r \ge m_r'\delta_r + m_r''\delta_r$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r\delta_r \ge \sum_{r=1}^{n} m_r'\delta_r + \sum_{r=1}^{n} m_r''\delta_r \quad \Rightarrow \quad L(P, f+g) \ge L(P, f) + L(P, g).$$

(iii) $\omega(P, f+g) = U(P, f+g) - L(P, f+g) \le [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)]$

$\qquad = [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)] = \omega(P, f) + \omega(P, g)$

$\therefore \quad \omega(P, f+g) \le \omega(P, f) + \omega(P, g).$

Theorem 4. *If P' is a refinement of P containing p points more than P and $|f(x)| \le k \quad \forall\ x \in [a, b]$, then*

(i) $L(P, f) \le L(P', f) \le L(P, f) + 2pk\delta$ \qquad (ii) $U(P, f) \ge U(P', f) \ge U(P, f) - 2pk\delta$

(iii) $\omega(P, f) - \omega(P', f) \le 4pk\delta$

where $\|P\| = \delta$.

Proof. Let P' contain just one point ξ(say) more than

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_r = b\}$$

and $\quad x_{r-1} < \xi < x_r$, then $\quad P' = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, \xi, x_r, \ldots\ldots, x_n = b\}$

Let m_r', m_r'' and m_r be the infimum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Let M_r', M_r'' and M_r be the supermum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Since $\quad |f(x)| \le k \quad \forall \quad x \in [a, b]$ *i.e.*, $\ -k \le f(x) \le k \quad \forall\ x \in [a, b]$

$\therefore \quad -k \le m_r \le m_r' \le k,\ -k \le m_r \le m_r'' \le k$

$\qquad -k \le M_r' \le M_r \le k,\ -k \le M_r'' \le M_r \le k$

$\Rightarrow \quad 0 \le m_r' - m_r \le 2k,\ 0 \le m_r'' - m_r \le 2k$

$\qquad 0 \le M_r - M_r' \le 2k,\ 0 \le M_r - M_r'' \le 2k.$

(i) $\quad L(P', f) - L(P, f) = [m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi)] - m_r(x_r - x_{r-1})$

$\qquad = m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi) - m_r[(x_r - \xi) + (\xi - x_{r-1})]$

$\qquad = (m_r' - m_r)(\xi - x_{r-1}) + (m_r'' - m_r)(x_r - \xi)$

$\qquad \le 2k(\xi - x_{r-1}) + 2k(x_r - \xi) = 2k(x_r - x_{r-1}) = 2k\delta_r$

$\qquad \le 2k\delta \qquad [\because \quad \delta_r \le \|P\| = \delta]$

$\Rightarrow$ $L(P', f) \le L(P, f) + 2k\delta$

If P′ contains p points more than P, then introducing the additional points one by one and proceeding as above p times, we have $L(P', f) \le L(P, f) + 2pk\delta$

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Also $L(P, f) \le L(P', f)$

$\therefore$ $L(P, f) \le L(P', f) \le L(P, f) + 2pk\delta.$

(*ii*)
$$\begin{aligned} U(P, f) - U(P', f) &= M_r(x_r - x_{r-1}) - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)] \\ &= M_r[(x_r - \xi) + (\xi - x_{r-1})] - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)] \\ &= (M_r - M_r'')(x_r - \xi) + (M_r - M_r')(\xi - x_{r-1}) \\ &\le 2k(x_r - \xi) + 2k(\xi - x_{r-1}) = 2k(x_r - x_{r-1}) = 2k\delta_r \\ &\le 2k\delta \qquad [\because \ \delta_r \le \|P\| = \delta] \end{aligned}$$

$\Rightarrow$ $U(P', f) \ge U(P, f) - 2k\delta.$

If P′ contains p points more than P, then introducing the additional points one by one and proceeding as above p times, we have

$$U(P', f) \ge U(P, f) - 2pk\delta$$

Also $U(P, f) \ge U(P', f)$

$\therefore$ $U(P, f) \ge U(P', f) \ge U(P, f) - 2pk\delta.$

(*iii*) Now $U(P, f) - U(P', f) \le 2pk\delta$ and $L(P', f) - L(P, f) \le 2pk\delta$

Adding, $[U(P, f) - L(P, f)] - [U(P', f) - L(P', f)] \le 4pk\delta$

$\Rightarrow$ $\omega(P, f) - \omega(P', f) \le 4pk\delta.$

Theorem 5. *If $P_1, P_2 \in P[a, b]$, then*

(*i*) $L(P_1, f) \le U(P_2, f)$ (*ii*) $L(P_2, f) \le U(P_1, f)$

i.e., no lower sum can exceed any upper sum.

Proof. Let $P = P_1 \cup P_2$ be a common refinement of P_1 and P_2.

(*i*) Since any refinement does not lower the lower sum and does not raise the upper sum.

$\therefore$ $L(P_1, f) \le L(P, f)$ and $U(P, f) \le U(P_2, f)$

Also $L(P, f) \le U(P, f)$

Combining, we have $L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f)$

$\Rightarrow$ $L(P_1, f) \le U(P_2, f).$

(*ii*) *Please try yourself.*

1.7. UPPER AND LOWER RIEMANN INTEGRALS

Let $f : [a, b] \to R$ be a bounded function. Then for every $P \in P[a, b]$, we have

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$$

where m and M are the infimum and supremum of f on $[a, b]$.

Thus for every $P \in P[a, b]$, we have $L(P, f) \le M(b - a)$ and $U(P, f) \ge m(b - a)$

$\Rightarrow$ The set $\{L(P, f)\}_{P \in P[a, b]}$ of lower sums is bounded above by $M(b - a)$ and, therefore, has the least upper bound.

The set $\{U(P, f)\}_{P \in P[a, b]}$ of upper sums is bounded below by $m(b - a)$ and, therefore, has the greatest lower bound.

Lower Riemann Integral of f on $[a, b]$ is defined as $\sup \{L(P, f)\}_{P \in P[a, b]}$ and is denoted by $\underline{\int_a^b} f(x)\,dx$.

Upper Riemann Integral of f on $[a, b]$ is defined as $\inf \{U(P, f)\}_{P \in P[a, b]}$ is denoted by $\overline{\int_a^b} f(x)\,dx$.

1.8. DEFINITIONS AND EXISTENCE OF THE RIEMANN-STEIELTJE'S INTEGRAL

A bounded function f is said to be Riemann integrable (or simply R-integrable) on $[a, b]$ if its lower and upper Riemann integrals are equal *i.e.*, if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

The common value of these integrals is called the Riemann integral of f on $[a, b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

The interval $[a, b]$ is called the range of integration. The numbers a and b are called the lower and upper limits of integration respectively.

Note 1. Riemann integral is based on the notion of bounds and is subject to two conditions :

(*i*) f is bounded on the interval, and (*ii*) the interval is closed.

2. The family of all bounded functions which are R-integrable on the closed intervals $[a, b]$ is denoted by $R[a, b]$. If f is R-integrable on $[a, b]$, then we write $f \in R[a, b]$.

3. f is R-integrable on $[a, b]$ $\Rightarrow$ (*i*) f is bounded on $[a, b]$

(*ii*) $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Theorem 6. *If $f : [a, b] \to R$ is a bounded function, then* $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

Proof. Let $P_1, P_2 \in P[a, b]$, then

$$L(P_1, f) \le U(P_2, f) \quad (\because \text{ no lower sum can exceed any upper sum})$$

This is true for each $P_1 \in P[a, b]$. Keeping P_2 fixed, the set $\{L(P_1, f)\}_{P_1 \in P[a, b]}$ has an upper bound $U(P_2, f)$.

Also $$\sup \left\{L(P_1, f)\right\}_{P_1 \in P[a,b]} = \underline{\int_a^b} f(x)\,dx$$

Since supermum $\le$ any upper bound

$$\therefore \quad \underline{\int_a^b} f(x)\,dx \le U(P_2, f)$$

This is true for each $P_2 \in P[a, b]$. Thus the set $\left\{U(P_2, f)\right\}_{P_2 \in P[a,b]}$ has a lower bound $\int_{\underline{a}}^{b} f(x)\,dx$.

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But. $$\inf\left\{U(P_2, f)\right\}_{P_2 \in P[a,b]} = \int_{a}^{\bar{b}} f(x)\,dx.$$

Since any lower bound ≤ infimum.

$$\therefore \qquad \int_{\underline{a}}^{b} f(x)\,dx \le \int_{a}^{\bar{b}} f(x)\,dx.$$

Theorem 7. *If $f : [a, b] \to R$ is bounded function, then*

$$m(b-a) \le \int_{\underline{a}}^{b} f(x)\,dx \le \int_{a}^{\bar{b}} f(x)\,dx \le M(b-a)$$

where m and M are the infimum and supermum of f on [a, b].

Proof. For every $P \in P[a, b]$, we have

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a) \qquad \ldots(1)$$

Now. $$\sup\left\{L(P, f)\right\}_{P \in P[a,b]} = \int_{\underline{a}}^{b} f(x)\,dx$$

$$\Rightarrow \qquad L(P, f) \le \int_{\underline{a}}^{b} f(x)\,dx \qquad \ldots(2)$$

$$\inf\left\{U(P, f)\right\}_{P \in P[a,b]} = \int_{a}^{\bar{b}} f(x)\,dx \quad \Rightarrow \quad \int_{a}^{\bar{b}} f(x)\,dx \le U(P, f) \qquad \ldots(3)$$

Also. $$\int_{\underline{a}}^{b} f(x)\,dx \le \int_{a}^{\bar{b}} f(x)\,dx \qquad \ldots(4)$$

From (1), (2), (3) and (4), we have

$$m(b-a) \le L(P, f) \le \int_{\underline{a}}^{b} f(x)\,dx \le \int_{a}^{\bar{b}} f(x)\,dx \le U(P, f) \le M(b-a)$$

$$\Rightarrow \qquad m(b-a) \le \int_{\underline{a}}^{b} f(x)\,dx \le \int_{a}^{\bar{b}} f(x)\,dx \le M(b-a).$$

Theorem 8. *If $f \in R[a, b]$, then*

(*i*) $m(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a)$ *if* $b \ge a$

(*ii*) $m(b-a) \ge \int_{a}^{b} f(x)\,dx \ge M(b-a)$ *if* $b \le a$ *where m and M are the infimum and supermum of f on [a, b].*

Proof. For $a = b$, the result is trivial.

If $b > a$, then for every $P \in P[a, b]$, we have

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a) \qquad \ldots(1)$$

Now $$\sup\left\{L(P, f)\right\}_{P \in P[a,b]} = \int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{b} f(x)\,dx \qquad | \because \ f \in R\,[a, b]$$

$$\Rightarrow \qquad L(P, f) \le \int_{a}^{b} f(x)\,dx \qquad \ldots(2)$$

Also $\quad \inf\{U(P,f)\}_{P \in P[a,b]} = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad | \because \quad f \in R[a,b]$

$$\Rightarrow \qquad \int_a^b f(x)\,dx \le U(P,f) \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$m(b-a) \le L(P,f) \le \int_a^b f(x)\,dx \le U(P,f) \le M(b-a)$$

$$\therefore \qquad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

If $b < a$, then $a > b$.

$\therefore$ Interchanging a and b in the above result, we have

$$m(a-b) \le \int_b^a f(x)\,dx \le M(a-b)$$

$$\Rightarrow \qquad -m(a-b) \ge -\int_a^b f(x)\,dx \ge -M(a-b)$$

$$\Rightarrow \qquad m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a).$$

ILLUSTRATIVE EXAMPLES

Example 1. *Let $f(x) = x$ for $x \in [0, 1]$ and let $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0, 1]$. Compute $U(P, f)$ and $L(P, f)$.*

Sol. Partition P divides the interval [0, 1] into sub-intervals

$$I_1 = \left[0, \frac{1}{3}\right], \ I_2 = \left[\frac{1}{3}, \frac{2}{3}\right], \ I_3 = \left[\frac{2}{3}, 1\right]$$

$$\therefore \qquad \delta_1 = \frac{1}{3} - 0 = \frac{1}{3};\ \delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3};\ \delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$$

Since $f(x) = x$ is increasing on [0, 1].

$$\therefore \qquad M_1 = \frac{1}{3},\ m_1 = 0\ ;\ M_2 = \frac{2}{3},\ m_2 = \frac{1}{3},\ M_3 = 1,\ m_3 = \frac{2}{3}$$

$$\therefore \qquad U(P,f) = \sum_{r=1}^{3} M_r\,\delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$

$$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = \frac{2}{3}$$

$$L(P,f) = \sum_{r=1}^{3} m_r\delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{3}.$$

Example 2. *Compute $L(P, f)$ and $U(P, f)$ for the function f defined by $f(x) = x^2$ on $[0, 1]$ and $P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$.*

NOTES

Sol. Partition P divides the interval [0, 1] into sub-intervals

$$I_1 = \left[0, \frac{1}{4}\right], I_2 = \left[\frac{1}{4}, \frac{2}{4}\right], I_3 = \left[\frac{2}{4}, \frac{3}{4}\right], I_4 = \left[\frac{3}{4}, 1\right], \delta_1 = \delta_2 = \delta_3 = \delta_4 = \frac{1}{4}.$$

Since $f(x) = x^2$ is increasing on [0, 1]

$$\therefore \quad m_1 = 0, M_1 = \frac{1}{16}; m_2 = \frac{1}{16}, M_2 = \frac{4}{16}, m_3 = \frac{4}{16}, M_3 = \frac{9}{16}, m_4 = \frac{9}{16}, M_4 = 1.$$

$$\therefore \quad L(P, f) = \sum_{r=1}^{4} m_r\delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$$

$$= \left(0 + \frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right) \times \frac{1}{4} = \frac{7}{32}$$

$$U(P, f) = \sum_{r=1}^{4} M_r\delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$

$$= \left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + 1\right) \times \frac{1}{4} = \frac{15}{32}.$$

Example 3. *If f is defined on $[a, b]$ by $f(x) = k \quad \forall \quad x \in [a, b]$ where k is constant, then $f \in$ **R[a, b]** and $\int_a^b f(x)\,dx = k(b-a)$.*

Or

A constant function is R-integrable.

Sol. Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any sub-interval $[x_{r-1}, x_r]$; $r = 1, 2, \ldots, n$, we have

$$M_r = m_r = k \qquad [\because \quad f(x) = k = \text{constant}]$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1})$$

$$= k\sum_{r=1}^{n} (x_r - x_{r-1}) = k(b-a) = \text{constant}$$

and

$$L(P, f) = \sum_{r=1}^{n} m_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1})$$

$$= k\sum_{r=1}^{n} (x_r - x_{r-1}) = k(b-a) = \text{constant}$$

$$\therefore \quad \underline{\int_a^b} f(x)\,dx = \sup \{L(P, f)\}_{P\in P[a,b]} = k(b-a)$$

$$\overline{\int_a^b} f(x)\,dx = \inf \{U(P, f)\}_{P\in P[a,b]} = k(b-a)$$

Since $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = k(b-a)$

$\therefore$ $f \in \mathbf{R[a, b]}$ and $\int_a^b f(x)\, dx = k(b-a)$.

Example 4. *If f is defined on [0, 1] by f(x) = x ∀ x ∈ [0, 1] then f ∈* **R[0, 1]** *and* $\int_0^1 f(x)\, dx = \frac{1}{2}$.

Sol. Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{r-1}{n}, \frac{r}{n}, ..., \frac{n}{n} = 1\right\}$ be *any partition* of [0, 1]. Then for any sub-interval

$$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right], r = 1, 2, ..., n.$$

We have $M_r = \frac{r}{n}, m_r = \frac{r-1}{n}$ and $\delta_r = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r}{n}.\frac{1}{n}$$

$$= \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{1}{n^2}(1 + 2 + 3 + ... + n) = \frac{1}{n^2}.\frac{n(n+1)}{2} = \frac{n+1}{2n}$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{r-1}{n}.\frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n}(r-1)$$

$$= \frac{1}{n^2}[0 + 1 + 2 + ... + (n+1)] = \frac{1}{n^2}.\frac{(n-1).n}{2} = \frac{n-1}{2n}$$

$$\therefore \quad \underline{\int_0^1} f(x)\, dx = \sup\{L(P, f)\}_{P \in P[0,1]} = \lim_{n\to\infty}\left[\frac{n-1}{2n}\right] = \lim_{n\to\infty}\frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$$

and
$$\overline{\int_0^1} f(x)\, dx = \inf\{U(P, f)\}_{P \in P[0,1]} = \lim_{n\to\infty}\left[\frac{n+1}{2n}\right] = \lim_{n\to\infty}\frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}$$

Since, $\underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x)\, dx = \frac{1}{2}$

$\therefore$ $f \in \mathbf{R[0, 1]}$ and $\int_0^1 f(x)\, dx = \frac{1}{2}$.

Example 5. *If f is defined on [0, a], a > 0 by f(x) = x^3 ∀ x ∈ [0, a], then*

$$f \in \mathbf{R[0, a]} \text{ and } \int_0^a f(x)\, dx = \frac{a^4}{4}.$$

Sol. Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, ..., \frac{(r-1)a}{n}, \frac{ra}{n}, ..., \frac{na}{n} = a\right\}$ be any partition of [0, a]. Then for any sub-interval $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$, $r = 1, 2, ..., n$, we have

$$M_r = \frac{r^3a^3}{n^3}, m_r = \frac{(r-1)^3a^3}{n^3} \quad [\because \; f(x) = x^3 \text{ is increasing on } [0, a]]$$

NOTES

Also, $\delta_r = \dfrac{a}{n}$

$$\therefore \quad U(P; f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r^3a^3}{n^3}.\frac{a}{n}$$

$$= \frac{a^4}{n^4}\sum_{r=1}^{n} r^3 = \frac{a^4}{n^4}.\frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{n^2}.\frac{a^4}{4}$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{(r-1)^3\, a^3}{n^3}.\frac{a}{n}$$

$$= \frac{a^2}{n^4}\sum_{r=1}^{n} (r-1)^3 = \frac{a^4}{n^4}.\frac{(n-1)^2\,n^2}{4} = \frac{(n-1)^2}{n^2}.\frac{a^4}{4}$$

$$\therefore \quad \underline{\int_0^a} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P\in P[0,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n-1)^2}{n^2}.\frac{a^4}{4}\right] = \lim_{n\to\infty}\frac{a^4}{4}\left(1-\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

and

$$\overline{\int_0^a} f(x)\,dx = \inf\left\{U(P, f)\right\}_{P\in P[0,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{n^2}.\frac{a^4}{4}\right] = \lim_{n\to\infty}\frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

Since, $\underline{\int_0^a} f(x)\,dx = \overline{\int_0^a} f(x)\,dx = \dfrac{a^4}{4}$

$$\therefore \quad f \in \mathbf{R[0, a]} \quad \text{and} \quad \int_0^a f(x)\,dx = \frac{a^4}{4}.$$

Example 6. *Let $f(x) = \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$ and let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots\ldots, \frac{n\pi}{2n}\right\}$ be the partition of $\left[0, \frac{\pi}{2}\right]$.*

Compute U(P, f) and L(P, f). Hence prove that $f \in R\left[0, \frac{\pi}{2}\right]$.

Sol. Here $$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots\ldots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \ldots\ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

For any sub-interval $I_r = \left[\dfrac{(r-1)\pi}{2\pi}, \dfrac{r\pi}{2n}\right]$, $r = 1, 2, \ldots, n$

$$M_r = \sin\frac{r\pi}{2n},\ m_r = \sin\frac{(r-1)\pi}{2n}$$

$\left|\because\ f(x) = \sin x \text{ is increasing on } \left[0, \frac{\pi}{2}\right]\right.$

Also $$\delta_r = \frac{r\pi}{2n} - \frac{(r-1)\pi}{2n} = \frac{\pi}{2n}$$

NOTES

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \sin\frac{r\pi}{2n}\cdot\frac{\pi}{2n} = \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin\frac{n\pi}{2n}\right]$$

$$= \frac{\pi}{2n}\cdot\frac{\sin\left[\frac{\pi}{2n} + \frac{n-1}{2}\cdot\frac{\pi}{2n}\right]\sin\left(\frac{n}{2}\cdot\frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)}$$

$$\left[\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \frac{\pi}{2n}\cdot\frac{\sin\frac{(n+1)\pi}{4\pi}\sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n}\cdot\sin\frac{\frac{(n+1)\pi}{4\pi}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n}\cdot\frac{\sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n}\cdot\frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4n} + \cos\frac{\pi}{4}\sin\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n}\cdot\frac{1}{\sqrt{2}}\cdot\left[\cot\frac{\pi}{4n} + 1\right] = \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right]$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{2n}\cdot\frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[0 + \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots\ldots + \sin\frac{(n-1)\pi}{2n}\right]$$

$$= \frac{\pi}{2n}\cdot\frac{\sin\left[\frac{\pi}{2n} + \frac{n-2}{2}\cdot\frac{\pi}{2n}\right]\sin\left(\frac{n-1}{2}\cdot\frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)}$$

$$= \frac{\pi}{2n}\cdot\frac{\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n}\cdot\frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4n} - \cos\frac{\pi}{4}\sin\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n}\cdot\frac{1}{\sqrt{2}}\cdot\left[\cot\frac{\pi}{4n} - 1\right] = \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right]$$

$$\therefore \quad \int_0^{\pi/2} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P\in P[0,\pi/2]}$$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right] = \lim_{n\to\infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} - \frac{\pi}{4n}\right] = 1$$

NOTES

and $$\overline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = \inf\left\{U(P, f)\right\}_{P\in P[0,\pi/2]}$$

$$= \lim_{n\to\infty} \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right] = \lim_{n\to\infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} + \frac{\pi}{4n}\right] = 1$$

Since $\underline{\int_0^{\pi/2}} f(x)\,dx = \overline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = 1$

$\therefore$ $$f \in \mathbf{R}\left[\mathbf{0}, \frac{\pi}{\mathbf{2}}\right] \text{ and } \int_0^{\pi/2} f(x)\,dx = 1.$$

Example 7. *Show by an example that every bounded function need not be R-integrable.*

Sol. Consider a function f defined on $[0, 1]$ by $f(x) = \begin{cases} 0, \text{ when } x \text{ is rational} \\ 1, \text{ when } x \text{ is irrational} \end{cases}$

Clearly, $f(x)$ is bounded in $[0, 1]$ because $0 \le f(x) \le 1 \quad \forall\, x \in [0, 1]$

If $P = \{0, x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ is any partition of $[0, 1]$, then for any sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, we have $M_r = 1$, $m_r = 0$

$\therefore$ $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} 1.(x_r - x_{r-1}) = x_n - x_0 = 1$$

and $$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} 0\,(x_r - x_{r-1}) = 0$$

$\therefore$ $$\underline{\int_0^1} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P\in P[0,1]} = 0$$

and $$\overline{\int_0^1} f(x)\,dx = \inf\left\{U(P, f)_{P\in P[0,1]}\right\} = 1$$

Since, $$\underline{\int_0^1} f(x)\,dx \ne \overline{\int_0^1} f(x)\,dx, f \notin \mathbf{R[0, 1]}.$$

Example 8. *If f be a function defined on* $\left[0, \frac{\pi}{4}\right]$ *by* $f(x) = \begin{cases} \cos x, \text{ if } x \text{ is rational} \\ \sin x, \text{ if } x \text{ is irrational} \end{cases}$, *then* $f \notin \mathbf{R}\left[\mathbf{0}, \frac{\pi}{\mathbf{4}}\right]$.

Sol. Let $P = \left\{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots\ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots\ldots, \frac{n\pi}{4\pi} = \frac{\pi}{4}\right\}$ be any partition of $\left[\mathbf{0}, \frac{\pi}{\mathbf{4}}\right]$.

Then for any sub-interval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$, $r = 1, 2, \ldots\ldots, n$, we have

$$M_r = \cos\frac{(r-1)\pi}{4n} \text{ and } m_r = \sin\frac{(r-1)\pi}{4n} \quad \left\{\because \cos x \ge \sin x \text{ on } \left[0, \frac{\pi}{4}\right]\right\}$$

Also, $\delta_r = \frac{\pi}{4\pi}$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{\pi}{4n}\cos\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos\frac{\pi}{4n} + \ldots\ldots + \cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\cdot\frac{\cos\left[0+\frac{n-1}{2}\cdot\frac{\pi}{4n}\right]\sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{4n}\right)} = \frac{\pi}{4n}\cdot\frac{\cos\frac{(n-1)\pi}{8n}\cdot\sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}$$

and

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{\pi}{4n}\sin\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\sin 0 + \sin\frac{\pi}{4n} + \ldots\ldots + \sin\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\cdot\frac{\sin\left[0+\frac{n-1}{2}\cdot\frac{\pi}{4n}\right]\sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{4n}\right)} = \frac{\pi}{4n}\cdot\frac{\sin\frac{(n-1)\pi}{8n}\sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}$$

$$\therefore \quad \int_0^{\pi/4} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P\in P[0,\pi/4]}$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\sin\frac{(n-1)\pi}{8n}\sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\sin\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1\times 2\sin^2\frac{\pi}{8} = 1-\cos\frac{\pi}{4} = 1-\frac{1}{\sqrt{2}}$$

and

$$\overline{\int_0^{\frac{\pi}{4}}} f(x)\,dx = \inf\left\{U(P, f)\right\}_{P\in P[0,\pi/4]}$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\cos\frac{(n-1)\pi}{8n}\sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\cos\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1 \times 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

NOTES

Since, $\underline{\int_0^{\frac{\pi}{4}}} f(x)\,dx \neq \overline{\int_0^{\frac{\pi}{4}}} f(x)\,dx$. $\therefore$ $f \notin R\left[0, \frac{\pi}{4}\right]$.

TEST YOUR KNOWLEDGE 1.1

1. If f is defined on $[0, a]$: $a > 0$ by $f(x) = x^2\ \forall\ x \in [0, a]$, then $f \in R[0, a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

2. If $f(x)$ be defined $[0, 1]$, as follows: $f(x) = \begin{cases} 1, \text{ when } x \text{ is rational} \\ -1, \text{ when } x \text{ is irrational} \end{cases}$. Show that f is not R-integrable over $[0, 1]$.

   [**Hint.** For any sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots\ldots, n$, we have $M_r = 1$, $m_r = -1$].

3. Prove that $\int_1^2 f(x)\,dx = 7$, where $f(x) = 2x + 4$.

4. Prove that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$.

5. Show that $f(x) = 2 - 3x$ is integrable on $[1, 3]$ and $\int_1^3 (2-3x)\,dx = -8$.

6. Show that $f(x) = x^2$ is integrable on $[1, 4]$ and $\int_1^4 x^2\,dx = 21$.

7. Let f be defined on $[0, 1]$ by $f(x) = \begin{cases} \frac{1}{2}, \text{ when } x \in Q \\ \frac{1}{3}, \text{ when } x \in R - Q. \end{cases}$

   Then show that f is bounded but not R-integrable on $[0, 1]$.

   $\left[\textbf{Hint. } m_r = \frac{1}{3}, M_r = \frac{1}{2}\right]$

1.9. NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY

Theorem 9. *A bounded function f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.*

Proof. (The condition is necessary)

Let f be integrable on $[a, b]$ so that $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

Let $\varepsilon > 0$ be given

Since $\overline{\int_a^b} f(x)\,dx = \inf \left\{U(P, f)\right\}_{P \in P[a,b]}$ and $\underline{\int_a^b} f(x)\,dx = \sup \left\{U(P, f)\right\}_{P \in P[a,b]}$

therefore, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad ...(1)$$

and
$$L(P_2, f) > \int_{\underline{a}}^b f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2} \quad ...(2)$$

NOTES

Let $P = P_1 \cup P_2$,

then $U(P, f) \le U(P_1, f) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2} < L(P_2, f) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le L(P, f) + \varepsilon$

[Using (1) and (2)]

$\Rightarrow \quad U(P, f) - L(P, f) < \varepsilon$

Conversely. (The condition is sufficient)

Let $\varepsilon > 0$ be given. Let P be a partition of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$ ...(3)

Since
$$L(P, f) \le \int_{\underline{a}}^b f(x)\,dx \le \int_a^{\bar{b}} f(x)\,dx \le U(P, f)$$

$\therefore \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx \le U(P, f) - L(P, f) < \varepsilon$ [by (3)]

But $\varepsilon > 0$ is arbitrary

$\therefore \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^b f(x)\,dx = 0 \quad \Rightarrow \quad \int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^b f(x)\,dx$

$\Rightarrow$ f is integrable.

Theorem 10. *A bounded function f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for every partition P of $[a, b]$ with $\| P \| < \delta$, $U(P, f) - L(P, f) < \varepsilon$.*

Proof. (The condition is necessary)

Let f be integrable on $[a, b]$ so that $\int_{\underline{a}}^b f(x)\,dx = \int_a^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx$

Let $\varepsilon > 0$ be given. By Darboux's theorem, there exists $\delta > 0$, such that for each partition P of $[a, b]$, with $\| P \| < \delta$,

$$U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad ...(1)$$

and
$$L(P, f) > \int_{\underline{a}}^b f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2}$$

or
$$-L(P, f) < -\int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad ...(2)$$

Adding (1) and (2), we have $U(P, f) - L(P, f) < \varepsilon$ for each partition P with $\| P \| < \delta$.

Conversely. (The condition is sufficient)

Let $\varepsilon > 0$ be given. Then for each partition P with $\| P \| < \delta$ (where δ is a positive number depending on ε).

$$U(P, f) - L(P, f) < \varepsilon \quad ...(3)$$

Also, for any partition P, $L(P, f) \le \int_{\underline{a}}^b f(x)\,dx \le \int_a^{\bar{b}} f(x)\,dx \le U(P, f)$

$$\Rightarrow \qquad \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx \le U(P, f) - L(P, f) < \varepsilon \qquad \text{[by (3)]}$$

But $\varepsilon > 0$ is arbitrary

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$$\therefore \qquad \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx = 0 \quad \Rightarrow \quad \int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx$$

$\Rightarrow$ f is integrable.

1.10. SOME CLASSES OF BOUNDED INTEGRABLE FUNCTIONS

A bounded function $f : [a, b] \to R$ is integrable on $[a, b]$ if

(*i*) f is continuous on $[a, b]$ (*ii*) f is monotonic on $[a, b]$

(*iii*) f has a finite number of points of discontinuity on $[a, b]$

(*iv*) the set of points of discontinuity of f on $[a, b]$ has a finite number of limit points.

Now we prove these assertions in the following theorems.

Theorem 11. *If $f : [a, b] \to R$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. f is continuous on **closed interval** $[a, b]$

$\Rightarrow$ f is uniformly continuous on $[a, b]$

$\Rightarrow$ For each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $| f(x') - f(x'') | < \dfrac{\varepsilon}{b-a}$ ...(1)

for all $x', x'' \in [a, b]$ and $| x' - x'' | < \delta$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a, b]$ such that $\| P \| < \delta$.

Since f is continuous on $[a, b]$, therefore, f is bounded on $[a, b]$.

$\Rightarrow$ f is continuous on $I_r = [x_{r-1}, x_r]$ and attains its infimum m_r and supermum M_r at some points c_r and d_r of $[x_{r-1}, x_r]$ so that $m_r = f(c_r)$ and $M_r = f(d_r)$

Since $| c_r - d_r | \le | x_r - x_{r-1} | = \delta_r < \delta$ and $c_r, d_r \in [x_{r-1}, x_r] \subset [a, b]$

$\Rightarrow$ c_r, d_r satisfy the conditions imposed on x', x'' in (1).

$\therefore$ From (1), $| f(c_r) - f(d_r) | < \dfrac{\varepsilon}{b-a}$

But $| f(c_r) - f(d_r) | = | m_r - M_r | = M_r - m_r$

$$\therefore \qquad M_r - m_r < \frac{\varepsilon}{b-a}$$

$$\text{Now,} \quad U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$< \sum_{r=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\delta_r = \frac{\varepsilon}{b-a} \sum_{r=1}^{n} \delta_r = \frac{\varepsilon}{b-a}\,(b-a) = \varepsilon$$

$\Rightarrow$ $U(P, f) - L(P, f) < \varepsilon$ with $\| P \| < \delta$

$\Rightarrow$ f is integrable on $[a, b]$.

Note. There exist functions which are integrable but not continuous. So continuity is a sufficient but not necessary condition.

Theorem 12. *If $f : [a, b] \to R$ is monotonic on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. Let f be monotonically increasing on $[a, b]$, then

$$f(a) \le f(x) \le f(b) \quad \forall\ x \in [a, b]$$

$\Rightarrow$ f is bounded on $[a, b]$ and $\inf f = f(a)$ and $\sup f = f(b)$

Let $\varepsilon > 0$ be given and $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition $[a, b]$ such that

$$\delta_r < \frac{\varepsilon}{f(b) - f(a) + 1} \quad \text{for } r = 1, 2, \ldots\ldots, n.$$

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$.

Since f is monotonically increasing, $m_r = f(x_{r-1})$ and $M_r = f(x_r)$

$$\text{Now} \quad U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\ \delta_r$$

$$< \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\varepsilon}{f(b) - f(a) + 1}$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(x_n) - f(x_0)]$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(b) - f(a)] = \frac{f(b) - f(a)}{f(b) - f(a) + 1}\,\varepsilon < \varepsilon$$

$\therefore$ for each $\varepsilon > 0$, $\exists$ a partition P such that $U(P, f) - L(P, f) < \varepsilon$

$\Rightarrow$ f is integrable on $[a, b]$.

Similarly, when f is monotonically decreasing on $[a, b]$, we can prove that f is integrable on $[a, b]$.

Hence, f is monotonic on $[a, b]$ $\Rightarrow$ f is integrable on $[a, b]$.

Theorem 13. *If the set of points of discontinuity of a bounded function $f : [a, b] \to R$ is finite, then f is integrable on $[a, b]$.*

Proof. Let $c_1, c_2, \ldots\ldots, c_p$ be the finite number of points of discontinuity of f on $[a, b]$ such that

$$c_1 < c_2 < \ldots\ldots < c_p.$$

Let $\varepsilon > 0$ be given.

Enclose the points $c_1, c_2, \ldots\ldots, c_p$ in p non-overlapping sub-intervals

$$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2],\ \ldots\ldots,\ I_p = [a_p, b_p]$$

such that the sum of their lengths $= \sum_{i=1}^{p} (b_i - a_i)$ is $< \frac{\varepsilon}{2(M - m)}$, where m and M are the infimum and supermum of f on $[a, b]$.

Since the oscillation of f in each of these sub-intervals is $\leq M - m$, therefore, the total contribution of these p sub-intervals to the oscillatory sum is

$$\sum_{i=1}^{p} (M_i - m_i)(b_i - a_i) \leq (M - m) \cdot \frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}$$

The $(p + 1)$ sub-intervals in $[a, b]$ that are formed by deleting the above p sub-intervals are

$$I_1' = [a, a_1], I_2' = [b_1, a_2], I_3' = [b_2, a_3], \ldots\ldots, I_p' = [b_{p-1}, a_p], I_{p+1}' = [b_p, b].$$

f is continuous on each of these sub-intervals. Therefore, there exists a partition P_r of I_r', $r = 1, 2, \ldots\ldots, p + 1$ such that the part of the oscillatory sum arising from each of these $(p + 1)$ sub-intervals is

$$< \frac{\varepsilon}{2(p+1)}$$

$\therefore$ The total contribution of these $(p + 1)$ sub-intervals to the oscillatory sum is

$$< \frac{\varepsilon}{2(p+1)} \cdot (p+1) = \frac{\varepsilon}{2}$$

Thus, for the partition $P = \{a, \ldots\ldots, a_1, b_1, \ldots\ldots, a_2, b_2, \ldots\ldots, a_p, b_p, \ldots\ldots b\}$ of $[a, b]$, we have

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Hence f is integrable on $[a, b]$.

Note. There are integrable functions having an infinite number of points of discontinuity in $[a, b]$.

Theorem 14. *If the set of points of discontinuity of a bounded function $f : [a, b] \to R$ has a finite number of limit points, then f is integrable on $[a, b]$.*

Proof. Let $c_1, c_2, \ldots\ldots, c_p$ be the finite number of limit points of the set of points of discontinuity of f on $[a, b]$ such that $c_1 < c_2 < \ldots\ldots < c_p$.

Let $\varepsilon > 0$ be given.

Enclose the points $c_1, c_2, \ldots\ldots, c_p$ in p non-overlapping sub-intervals

$I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ..., $I_p = [a_p, b_p]$ such that the sum of their lengths $= \sum_{i=1}^{p} (b_i - a_i)$ is $< \frac{\varepsilon}{2(M-m)}$, where m and M are the infimum and supermum of f on $[a, b]$.

Since the oscillation of f in each of these sub-intervals is $\leq M - m$, therefore, the total contribution of these p sub-intervals to the oscillatory sum is $\sum_{i=1}^{p} (M_i - m_i)(b_i - a_i)$

$$\leq (M - m) \cdot \frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}$$

The $(p + 1)$ sub-intervals in $[a, b]$ that are formed by deleting the above p sub-intervals are

$$I_1' = [a, a_1], I_2' = [b_1, b_2], I_3' = [b_2, a_3], \ldots\ldots, I_p' = [b_{p-1}, a_p],$$
$$I_{p+1}' = [b_p, b].$$

In each of these sub-intervals, f has only a finite number of points of discontinuity [for otherwise the interval which contains an infinite number of points of discontinuity of f will have a limit point (by Bolzano Weierestrass Theorem). Therefore, each of these sub-intervals can be further sub-divided such that the part of the oscillatory sum arising from each of these $(p + 1)$ sub-intervals is

$$< \frac{\varepsilon}{2(p+1)}$$

∴ The total contribution of these $(p + 1)$ sub-intervals to the oscillatory sum is

$$< \frac{\varepsilon}{2(p+1)} \cdot (p+1) = \frac{\varepsilon}{2}.$$

Thus, for the partition $P = \{a, \ldots\ldots, a_1, b_1, \ldots\ldots, a_2, b_2, \ldots\ldots, a_p, b_p, \ldots\ldots, b\}$ of $[a, b]$, we have

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Hence f is integrable on $[a, b]$.

1.11. RIEMANN SUM

Let f be a real valued function defined on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of $[a, b]$.

Let $\xi_r \in [x_{r-1}, x_r]$, $r = 1, 2, \ldots\ldots, n$. Then the sum $\sum_{r=1}^{n} f(\xi_r)\delta_r$ is called a Riemann sum of f on $[a, b]$ relative to P.

Since ξ_r is any arbitrary point of $[x_{r-1}, x_r]$, therefore, corresponding to each partition P of $[a, b]$, there exist infinitely many Riemann sums.

1.12. INTEGRAL AS THE LIMIT OF A SUM (SECOND DEFINITION OF INTEGRABILITY)

A function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ of $[a, b]$ with $\| P \| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$ arbitrarily, $\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < \varepsilon$.

The number I is the Riemann integral of f on $[a, b]$ *i.e.*, $I = \int_a^b f(x)\, dx$

Thus a function f is integrable on $[a, b]$ if $\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ exists and is independent of the choice of sub-interval $[x_{r-1}, x_r]$ and of the point $\xi_r \in [x_{r-1}, x_r]$.

This limit, if it exists, is $I = \int_a^b f(x)\,dx$.

1.13. EQUIVALENCE OF THE TWO DEFINITIONS OF RIEMANN INTEGRAL

Definition 1. A bounded function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if its lower and upper integrals are equal and the common value of these integrals is called the Riemann integral of f on $[a, b]$.

Thus $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Definition 2. A function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ of $[a, b]$ with $\| P \| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$ arbitrarily,

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\delta_r - I \right| < \varepsilon \text{ where } I = \int_a^b f(x)\,dx.$$

Definition 1 $\Rightarrow$ Definition 2

Let a bounded function f be integrable on $[a, b]$ according to definition 1, so that

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

Let $\varepsilon > 0$ be given.

Then, by Darbourx's theorem, there exists a $\delta > 0$ such that for every partition P with $\| P \| < \delta$.

$$U(P, f) < \overline{\int_a^b} f(x)\,dx + \varepsilon = \int_a^b f(x)\,dx + \varepsilon \quad \ldots(1)$$

and $$L(P, f) > \underline{\int_a^b} f(x)\,dx - \varepsilon = \int_a^b f(x)\,dx - \varepsilon \quad \ldots(2)$$

If m_r, M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$, then for $\xi_r \in [x_{r-1}, x_r]$, we have

$$m_r \le f(\xi_r) \le M_r \quad \Rightarrow \quad m_r\delta_r \le f(\xi_r)\,\delta_r \le M_r\delta_r$$

$$\Rightarrow \quad \sum_{r=1}^{m} m_r\delta_r \le \sum_{r=1}^{n} f(\xi_r)\,\delta_r \le \sum_{r=1}^{n} M_r\delta_r \quad \Rightarrow \quad L(P, f) \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le U(P, f) \quad \ldots(3)$$

From (1), (2) and (3), we have

$$\int_a^b f(x)\,dx - \varepsilon < L(P, f) \le \sum_{r=1}^{n} f(\xi_r)\,\delta_r \le U(P, f) < \int_a^b f(x)\,dx + \varepsilon$$

$$\Rightarrow \quad I - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\,\delta_r < I + \varepsilon \quad \text{where} \quad I = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad \left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < \varepsilon.$$

$\Rightarrow$ f is integrable according to definition 2.

Conversely. Definition 2 $\Rightarrow$ Definition 1

Let f be integrable on $[a, b]$ according to definition 2.

We shall show that f is bounded on $[a,b]$ and its lower and upper integrals on $[a, b]$ are equal.

If possible, let f be not bounded on $[a, b]$.

By definition 2, for $\varepsilon = 1$, there exists $\delta > 0$ and a number I such that for each partition P of $[a, b]$ with $\| P \| < \delta$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < 1, \ \forall\ \xi_r \in [x_{r-1}, x_r]$$

$$\therefore \quad \left| \left| \sum_{r=1}^{n} f(\xi_r)\delta_r \right| - |I| \right| \le \left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < 1$$

$$\Rightarrow \quad |I| - 1 < \left| \sum_{r=1}^{n} f(\xi_r)\delta_r \right| < |I| + 1$$

$$\Rightarrow \quad \left| \sum_{r=1}^{n} f(\xi_r)\delta_r \right| < |I| + 1 \ \ \forall\ \xi_r \in [x_{r-1}, x_r] \qquad \ldots(4)$$

Since f is not bounded on $[a, b]$, it is not bounded on at least one sub-interval of P, say $[x_{m-1}, x_m]$.

Taking $\xi_r = x_r$ for $r \ne m$, each term of $\sum_{r=1}^{n} f(\xi_r)\,\delta_r$ except $f(\xi_m)\,\delta_m$ is fixed,

$$i.e., \quad \sum_{r=1}^{m-1} f(\xi_r)\delta_r + \sum_{r=m+1}^{n} f(\xi_r)\delta_r \text{ is fixed.}$$

Since f is not bounded on $[x_{m-1}, x_m]$, we can choose $\xi_m \in [x_{m-1}, x_m]$ such that

$$\left| \sum_{r=1}^{m-1} f(\xi_r)\,\delta_r + \sum_{r=m+1}^{n} f(\xi_r)\,\delta_r + f(\xi_m)\,\delta_m \right| < |I| + 1$$

$$i.e., \quad \left| \sum_{r=1}^{n} f(\xi_r)\delta_r \right| > |I| + 1 \quad \text{which contradicts (4).}$$

$\therefore$ f cannot be unbounded on any sub-interval of $[a, b]$ and hence f is bounded on $[a, b]$.

Now, let $\varepsilon > 0$ be given. Then, by definition 2, there exists $\delta > 0$ and a number I such that for every partition P with $\| P \| < \delta$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < \frac{\varepsilon}{2} \ \ \forall\ \xi_r \in [x_{r-1}, x_r]$$

$$\Rightarrow \quad I - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\xi_r)\delta_r < I + \frac{\varepsilon}{2} \ \ \forall\ \xi_r \in [x_{r-1}, x_r] \qquad \ldots(5)$$

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Let m_r, M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$, then there exist points $\alpha_r, \beta_r \in [x_{r-1}, x_r]$ such that

$$f(\alpha_r) < m_r + \frac{\varepsilon}{2(b-a)} \quad \text{and} \quad f(\beta_r) > M_r - \frac{\varepsilon}{2(b-a)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(\alpha_r)\delta_r < \sum_{r=1}^{n} m_r\delta_r + \frac{\varepsilon}{2(b-a)} \sum_{r=1}^{n} \delta_r$$

and
$$\sum_{r=1}^{n} f(\beta_r)\delta_r > \sum_{r=1}^{n} M_r\delta_r - \frac{\varepsilon}{2(b-a)} \sum_{r=1}^{n} \delta_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(\alpha_r)\delta_r < L(P, f) + \frac{\varepsilon}{2(b-a)} . (b-a)$$

and
$$\sum_{r=1}^{n} f(\beta_r)\delta_r < U(P, f) - \frac{\varepsilon}{2(b-a)} . (b-a)$$

$$\Rightarrow \quad \left.\begin{aligned} &\sum_{r=1}^{n} f(\alpha_r)\delta_r < L(P, f) + \frac{\varepsilon}{2} \\ \text{and} \quad &\sum_{r=1}^{n} f(\beta_r)\delta_r > U(P, f) - \frac{\varepsilon}{2} \end{aligned}\right] \quad \ldots(6)$$

From (5) and (6), taking $\xi_r = \alpha_r$ and β_r, we have

$$I - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\xi_r)\delta_r < L(P, f) + \frac{\varepsilon}{2}$$

and
$$I + \frac{\varepsilon}{2} > \sum_{r=1}^{n} f(\xi_r)\delta_r > U(P, f) - \frac{\varepsilon}{2}$$

$\Rightarrow$ $I - \varepsilon < L(P, f)$ and $I + \varepsilon > U(P, f)$ for every partition P with $\| P \| < \delta$.

But
$$L(P, f) \leq \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx \leq U(P, f)$$

$$\therefore \quad I - \varepsilon < \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx < I + \varepsilon$$

$$\Rightarrow \quad \left| \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \right| < (I + \varepsilon) - (I - \varepsilon) = 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary.

$$\therefore \quad \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx = 0 \quad \Rightarrow \quad \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$$

$\Rightarrow$ f is integrable according to definition I.

Note. From the above theorem, we conclude that f is integrable on $[a, b]$ if $\lim_{\| P \| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ exists, where $\xi_r \in [x_{r-1}, x_r]$ and $\int_a^b f(x)\, dx = \lim_{\| P \| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$.

ILLUSTRATIVE EXAMPLES

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Example 1. *From definition, prove that*

(*i*) $\int_1^2 f(x)\,dx = 6$ *where* $f(x) = 2x + 3$ (*ii*) $\int_0^1 (2x^2 - 3x + 5)\,dx = \frac{25}{6}$

Sol. (*i*) Since $f(x) = 2x + 3$ is bounded and continuous on [1, 2].

$\therefore$ f is integrable on [1, 2].

Consider a partition $P = \{1 = x_0, x_1, x_2, \ldots\ldots, x_n = 2\}$ of [1, 2] dividing it into n equal sub-intervals, each of length

$$\frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \quad \text{so that} \quad \|P\| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Also $\qquad x_r = 1 + \frac{r}{n}$ and $\delta_r = \frac{1}{n}$, $r = 1, 2, \ldots\ldots, n.$

$$\therefore \quad \int_1^2 f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(1 + \frac{r}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty} \sum_{r=1}^{n} \left[2\left(1+\frac{r}{n}\right)+3\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{5}{n} + \frac{2r}{n^2}\right) = \lim_{n\to\infty}\left[\frac{5}{n}\cdot n + \frac{2}{n^2}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty}\left[5 + \frac{2}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[5 + \frac{n+1}{n}\right] = \lim_{n\to\infty}\left(6 + \frac{1}{n}\right) = 6.$$

(*ii*) Since $f(x) = 2x^2 - 3x + 5$ is bounded and continuous on [0, 1], therefore, f is integrable on [0, 1].

Consider a partition $P = \{0 = x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ of [0, 1] dividing it into n equal sub-intervals, each of length

$$\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \quad \text{so that } \|P\| \to 0 \text{ as } n \to \infty.$$

Also, $\qquad x_r = 0 + \frac{r}{n} = \frac{r}{n}$ and $\delta_r = \frac{1}{n}$, $r = 1, 2, \ldots\ldots, n$

$$\therefore \quad \int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n\to\infty}\sum_{r=1}^{n} f(x_r)\,.\,\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left[2\left(\frac{r}{n}\right)^2 - 3\left(\frac{r}{n}\right) + 5\right]\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{2r^2}{n^3} - \frac{3r}{n^2} + \frac{5}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{3}{n^2}\cdot\frac{n(n+1)}{2} + \frac{5}{n}\cdot n\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{3}{2}\left(\frac{n+1}{n}\right) + 5\right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{3}{2}\left(1 + \frac{1}{n}\right) + 5 \right]$$

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$$= \frac{1}{3}(1)(2) - \frac{3}{2}(1) + 5 = \frac{25}{6}.$$

Example 2. *Evaluate* $\int_{-1}^{1} f(x)\, dx$, *where* $f(x) = |\, x \,|$.

Sol. Since $f(x) = |\, x \,| = \begin{cases} -x, \text{ when } x \le 0 \\ x, \text{ when } x > 0 \end{cases}$

$\therefore$ f is bounded and continuous on $[-1, 1]$

$\Rightarrow$ f is integrable on $[-1, 1]$.

Consider a partition $P = \{-1 = x_0, x_1, \ldots\ldots, x_n = 0, x_{n+1}, x_{n+2}, \ldots\ldots, x_{2n} = 1\}$ of $[-1, 1]$ dividing it into $2n$ equal sub-intervals, each of length $\frac{b-a}{2n} = \frac{1-(-1)}{2n} = \frac{1}{n}$ so that $\| P \| \to 0$ as $n \to \infty$.

Also, $x_r = -1 + \frac{r}{n}$ and $\delta_r = \frac{1}{n}$, $r = 1, 2, \ldots\ldots, 2n$

$$\therefore \quad \int_{-1}^{1} f(x)\, dx = \lim_{\|P\| \to 0} \sum_{r=1}^{2n} f(\xi_r)\delta_r = \lim_{n \to \infty} \sum_{r=1}^{2n} f(x_r).\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f(x_r)\delta_r + \sum_{r=n+1}^{2n} f(x_r)\delta_r \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f\left(-1 + \frac{r}{n}\right).\frac{1}{n} + \sum_{r=n+1}^{2n} f\left(-1 + \frac{r}{n}\right).\frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} -\left(-1 + \frac{r}{n}\right).\frac{1}{n} + \sum_{r=n+1}^{2n} \left(-1 + \frac{r}{n}\right).\frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} \left(\frac{1}{n} - \frac{r}{n^2}\right) + \sum_{r=n+1}^{2n} \left(-\frac{1}{n} + \frac{r}{n^2}\right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{n}.n - \frac{1}{n^2} \sum_{r=1}^{n} r + \left(-\frac{1}{n}\right).n + \frac{1}{n^2} \sum_{r=n+1}^{2n} r \right]$$

$$= \lim_{n \to \infty} \left[-\frac{1}{n^2} \cdot \frac{n(n+1)}{2} + \frac{1}{n^2} \left\{ (n+1) + (n+2) + \ldots + 2n \right\} \right]$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2}\left(\frac{n+1}{n}\right) + \frac{1}{n^2} \cdot \frac{n}{2}(n + 1 + 2n) \right]$$

$$\left[\because \text{ in an A.P. } S_n = \frac{n}{2}(a + l) \right]$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2n}(3n+1) \right]$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(3+\frac{1}{n}\right) \right] = -\frac{1}{2} + \frac{3}{2} = 1.$$

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Example 3. *Evaluate* $\int_{-1}^{2} f(x)\,dx$, *where* $f(x) = |\ x\ |$.

Sol. Since $f(x) = |\ x\ | = \begin{cases} -x, \text{ when } x \le 0 \\ x, \text{ when } x > 0 \end{cases}$

$\therefore$ f is bounded and continuous on $[-1, 2]$

$\Rightarrow$ f is integrable on $[-1, 2]$

Consider a partition $P = \{-1 = x_0, x_1, x_2, \ldots\ldots, x_n = 0, x_{n+1}, x_{n+2}, \ldots, x_{3n} = 2\}$ of $[-1, 2]$ dividing it into $3n$ equal sub-intervals, each of length $\frac{b-a}{3n} = \frac{2-(-1)}{3n} = \frac{1}{n}$ so that $\| P \| \to 0$ as $n \to \infty$.

Also. $x_r = -1 + \frac{r}{n}$ and $\delta_r = \frac{1}{n}$, $r = 1, 2, \ldots\ldots, 3n$.

$$\therefore \quad \int_{-1}^{2} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{3n} f(\xi_r)\delta_r = \lim_{n \to \infty} \sum_{r=1}^{3n} f(x_r)\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f(x_r)\delta_r + \sum_{r=n+1}^{3n} f(x_r)\delta_r \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f\left(-1+\frac{r}{n}\right).\frac{1}{n} + \sum_{r=n+1}^{3n} f\left(-1+\frac{r}{n}\right).\frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} -\left(-1+\frac{r}{n}\right).\frac{1}{n} + \sum_{r=n+1}^{3n} \left(-1+\frac{r}{n}\right).\frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} \left(\frac{1}{n}-\frac{r}{n^2}\right) + \sum_{r=n+1}^{3n} \left(-\frac{1}{n}+\frac{r}{n^2}\right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{n}.n - \frac{1}{n^2}\sum_{r=1}^{n} r + \left(-\frac{1}{n}\right).2n + \frac{1}{n^2}\sum_{r=n+1}^{3n} r \right]$$

$$= \lim_{n \to \infty} \left[1 - \frac{1}{n^2}.\frac{n(n+1)}{2} - 2 + \frac{1}{n^2}\left\{(n+1)+(n+2)+\ldots+3n\right\} \right]$$

$$= \lim_{n \to \infty} \left[-1 - \frac{1}{2}\left(\frac{n+1}{n}\right) + \frac{1}{n^2}.\frac{2n}{2}(n+1+3n) \right]$$

$$= \lim_{n \to \infty} \left[-1 - \frac{1}{2}\left(1+\frac{1}{n}\right) + \left(4+\frac{1}{n}\right) \right] = -1 - \frac{1}{2} + 4 = \frac{5}{2}.$$

Example 4. *Show that* $\int_0^a \sin x\,dx = 1 - \cos a$, *where a is a fixed real number.*

Sol. Since $f(x) = \sin x$ is bounded and continuous on $[0, a]$, therefore, f is integrable on $[0, a]$.

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Consider a partition $P = \{0 = x_0, x_1, x_2, \ldots\ldots, x_n = a\}$ of $[0, a]$ dividing it into n equal sub-intervals, each of length $\frac{a-0}{n} = \frac{a}{n}$ so that $\|P\| \to 0$ as $n \to \infty$.

Also, $x_r = 0 + \frac{ra}{n} = \frac{ra}{n}$ and $\delta_r = \frac{a}{n}$, $r = 1, 2, \ldots\ldots, n$.

$$\therefore \quad \int_0^a f(x)\,dx = \lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n\to\infty}\sum_{r=1}^{n} f(x_r)\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} f\left(\frac{ra}{n}\right)\cdot\frac{a}{n} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{a}{n}\sin\frac{ra}{n}$$

$$= \lim_{n\to\infty}\frac{a}{n}\left[\sin\frac{a}{n} + \sin\frac{2a}{n} + \ldots\ldots + \sin\frac{na}{n}\right]$$

$$= \lim_{n\to\infty}\frac{a}{n}\cdot\frac{\sin\left(\frac{a}{n} + \frac{n-1}{2}\cdot\frac{a}{n}\right)\sin\left(\frac{n}{2}\cdot\frac{a}{n}\right)}{\sin\frac{a}{2n}}$$

$$\left[\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \lim_{n\to\infty} 2\cdot\frac{\frac{a}{2n}}{\sin\frac{a}{2n}}\cdot\sin\frac{a}{2}\left(\frac{2}{n} + \frac{n-1}{n}\right)\cdot\sin\frac{a}{2}$$

$$= \lim_{n\to\infty} 2\cdot\frac{\frac{a}{2n}}{\sin\frac{a}{2n}}\cdot\sin\frac{a}{2}\left(1 + \frac{1}{n}\right)\cdot\sin\frac{a}{2}$$

$$= 2\times 1\times\sin\frac{a}{2}\times\sin\frac{a}{2} \qquad \left[\begin{array}{l}\because \text{ as } n\to\infty,\ \theta = \frac{a}{2n}\to 0 \\ \text{and } \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1\end{array}\right.$$

$$= 2\sin^2\frac{a}{2} = 1 - \cos a.$$

Example 5. *Prove that* $\int_0^{\pi/2}\cos x\,dx = 1.$

Sol. Since $f(x) = \cos x$ is bounded and continuous on $\left[0, \frac{\pi}{2}\right]$, therefore, f is integrable on $\left[0, \frac{\pi}{2}\right]$.

Consider a partition $P = \left\{0 = x_0, x_1, x_2, ..., x_n = \frac{\pi}{2}\right\}$ of $\left[0, \frac{\pi}{2}\right]$ dividing it into n equal sub-intervals, each of length $\frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$ so that $\| P \| \to 0$ as $n \to \infty$.

Also, $$x_r = 0 + \frac{r\pi}{2n} = \frac{r\pi}{2n} \quad \text{and} \quad \delta_r = \frac{\pi}{2n}, \quad r = 1, 2, ..., n.$$

$$\therefore \quad \int_0^{\pi/2} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} f\left(\frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\pi}{2n} \cos \frac{r\pi}{2n}$$

$$= \lim_{n \to \infty} \frac{\pi}{2n} \left[\cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + ... + \cos \frac{n\pi}{2n}\right]$$

$$= \lim_{n \to \infty} \frac{\pi}{2n} \cdot \frac{\cos\left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right) \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\left[\because \quad \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... \text{ to } n \text{ terms} = \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}\right]$$

$$= \lim_{n \to \infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(\frac{2}{n} + \frac{n-1}{n}\right) \sin \frac{\pi}{4}$$

$$= \lim_{n \to \infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(1 + \frac{1}{n}\right) \sin \frac{\pi}{4}$$

$$= 2 \times 1 \times \cos \frac{\pi}{4} \times \sin \frac{\pi}{4} = \sin \frac{\pi}{2} = 1.$$

Example 6. *Show that the greatest integer function $f(x) = [x]$ is integrable on $[0, 4]$ and $\int_0^4 [x]dx = 6$.*

Sol. $$f(x) = [x] \text{ on } [0, 4] \quad \Rightarrow \quad f(x) = \begin{cases} 0 \text{ when } 0 \le x < 1 \\ 1 \text{ when } 1 \le x < 2 \\ 2 \text{ when } 2 \le x < 3 \\ 3 \text{ when } 3 \le x < 4 \end{cases}$$

$\Rightarrow$ f is bounded and has *only four* points of finite discontinuity at 1, 2, 3, 4.

Since the points of discontinuity of f on [0, 4] are finite in number, therefore, f is integrable on [0, 4] and

$$\int_0^4 [x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \int_3^4 [x]dx$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$$

$$= 0 + (2-1) + 2(3-2) + 3(4-3) = 6.$$

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Example 7. *Show that the function f defined by* $f(x) = \begin{cases} 0, \text{if } x \text{ is an integer} \\ 1, \text{otherwise} \end{cases}$ *is integrable on [0, m], m being a positive integer.*

Sol. $$f(x) = \begin{cases} 0, \text{if } x = 0, 1, 2, \ldots\ldots, m \\ 1, \text{if } r-1 < x < r,\ r = 1, 2, \ldots\ldots, m \end{cases}$$

$\Rightarrow$ f is bounded and has only $m + 1$ points of finite discontinuity at $0, 1, 2, \ldots\ldots, m$.

Since the points of discontinuity of f on $[0, m]$ are finite in number, therefore, f is integrable on $[0, m]$.

Note. $$\int_0^m f(x)\,dx = \int_0^1 f(x)\,dx + \int_0^2 f(x)\,dx + \ldots\ldots + \int_{m-1}^m f(x)\,dx$$

$$= \int_0^1 1\,dx + \int_1^2 1\,dx + \ldots\ldots + \int_{m-1}^m 1\,dx$$

$$= (1-0) + (2-1) + \ldots\ldots + (m-(m-1)) = 1 + 1 + \ldots\ldots + 1 = m.$$

Example 8. *Show that the function f defined by*

$$f(x) = \frac{1}{2^n}, \quad \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \ (n = 0, 1, 2, \ldots\ldots)$$

$$f(a) = 0$$

is integrable on [0, 1], although it has an infinite number of points of discontinuity.

Also evaluate $\int_0^1 f(x)\,dx$.

Sol.
$$\begin{aligned} f(x) &= 1, && \text{when } \tfrac{1}{2} < x \le 1 \\ &= \frac{1}{2}, && \text{when } \frac{1}{2^2} < x \le \frac{1}{2} \\ &= \frac{1}{2^2}, && \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ &\ \vdots \\ &= \frac{1}{2^{n-1}}, && \text{when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ &\ \vdots \\ &= 0, && \text{when } x = 0 \end{aligned}$$

Thus we notice that f is bounded and continuous on [0, 1] except at the points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

The set of points of discontinuity of f on [0, 1] is $\left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots\right\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, therefore, f is integrable on [0, 1].

Now $\int_{1/2^n}^{1} f(x)\,dx$

$$= \int_{1/2}^{1} f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + \ldots\ldots + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \int_{1/2}^{1} 1\,dx + \int_{1/2^2}^{1/2} \frac{1}{2}\,dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}\,dx + \ldots\ldots + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}\,dx$$

$$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2}-\frac{1}{2^3}\right) + \ldots\ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + \ldots\ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \ldots\ldots + \left(\frac{1}{2^2}\right)^{n-1}\right]$$

$$= \frac{1}{2}\cdot\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}} = \frac{2}{3}\left(1-\frac{1}{4^n}\right)$$

Proceeding to the limit when $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{2}{3}$.

Example 9. *Show that a function f defined on [0, 1] by* $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}, \\ 0, & x = 0 \end{cases}$

(n = 1, 2, ...) is integrable on [0, 1]. Also show that $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1$.

Sol.

$$\begin{aligned} f(x) &= 1, && \text{when } \tfrac{1}{2} < x \le 1 \\ &= \tfrac{1}{2}, && \text{when } \tfrac{1}{3} < x \le \tfrac{1}{2} \\ &= \tfrac{1}{3}, && \text{when } \tfrac{1}{4} < x \le \tfrac{1}{3} \\ &\ \vdots \\ &= \tfrac{1}{n}, && \text{when } \tfrac{1}{n+1} < x \le \tfrac{1}{n} \\ &\ \vdots \\ &= 0, && \text{when } x = 0 \end{aligned}$$

Thus we notice that f is bounded and continuous on [0, 1] except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$

The set of points of discontinuity of f on $[0, 1]$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$ which has only one limit point 0.

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Since the set of points of discontinuity of f on $[0, 1]$ has a finite number of limit points, therefore, f is integrable on $[0, 1]$.

Now $\int_{1/(n+1)}^{1} f(x)\,dx$

$$= \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + ... + \int_{1/(n+1)}^{1/n} f(x)\,dx$$

$$= \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} \frac{1}{2}\,dx + \int_{1/4}^{1/3} \frac{1}{3}\,dx + ... + \int_{1/n+1}^{1/n} \frac{1}{n}\,dx$$

$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{4}\right) + ... + \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)}\right)$$

$$= \frac{\pi^2}{6} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= \frac{\pi^2}{6} - \left(1 - \frac{1}{n+1}\right)$$

Proceeding to the limit as $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1$.

Example 10. *Show that the function f defined on $[0, 1]$ as $f(x) = 2rx$ if $\frac{1}{r+1} < x < \frac{1}{r}$, $r \in N$ is integrable over $[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.*

Sol.
$$\begin{aligned} f(x) &= 2x, \text{ when } \frac{1}{2} < x < 1 \\ &= 4x, \text{ when } \frac{1}{3} < x < \frac{1}{2} \\ &= 6x, \text{ when } \frac{1}{4} < x < \frac{1}{3} \\ &\vdots \\ &= 2(n-1)\,x, \text{ when } \frac{1}{n} < x < \frac{1}{n-1} \\ &\vdots \end{aligned}$$

Thus we notice that f is bounded and continuous on $[0, 1]$ except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, ...$

The set of points of discontinuity of f on $[0, 1]$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on $[0, 1]$ has a finite number of limit points, therefore, f is integrable on $[0, 1]$.

Now $\int_{1/n}^{1} f(x)\,dx$

$$= \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + ... + \int_{1/n}^{1/n-1} f(x)\,dx$$

$$= \sum_{r=1}^{n-1} \int_{1/(r+1)}^{1/r} 2rx\,dx = \sum_{r=1}^{n-1} \left[rx^2\right]_{1/(r+1)}^{1/r} = \sum_{r=1}^{n-1} r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n} \frac{2r+1}{r(r+1)^2} = \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right] \qquad \text{(Partial Fractions)}$$

$$= \sum_{r=1}^{n-1} \left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1} \frac{1}{(r+1)^2}$$

$$= \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] + \left[\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}\right]$$

$$= \left(1 - \frac{1}{n}\right) + \left(\frac{\pi^2}{6} - 1\right) \qquad \left[\because \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} = \frac{\pi^2}{6}\right]$$

$$= \frac{\pi^2}{6} - \frac{1}{n}$$

Proceeding to the limit as $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

Example 11. *Show that* $\lim_{n \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{1}{8n}\right] = \frac{3}{8}$.

Sol. $\lim_{n \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{1}{8n}\right]$

$$= \lim_{n \to \infty} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{n^2}{(n+n)^3}\right]$$

$$= \lim_{n \to \infty} \sum_{r=0}^{n} \frac{n^2}{(n+r)^3} = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{\frac{1}{n}}{\left(1 + \frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$= \left[\frac{-1}{2(1+x)^2}\right]_0^1 = -\frac{1}{2}\left(\frac{1}{4} - 1\right) = \frac{3}{8}.$$

NOTES

Example 12. *Show that:* $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+......+\frac{1}{2n}\right)=\frac{\pi}{4}$.

Sol. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+......+\frac{1}{2n}\right)$

$$= \lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+......+\frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{1}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x^2} \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

Example 13. *Show that* $\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+......+\sin\frac{n\pi}{n}\right]=\frac{2}{\pi}$.

Sol. $\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+......+\sin\frac{n\pi}{n}\right] = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\sin\frac{r\pi}{n}$

$$= \int_0^1 \sin \pi x \, dx \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$= -\left.\frac{\cos \pi x}{\pi}\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}.$$

Example 14. *Prove that* $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{4n}{n}\right)\right]^{1/n}=5\left[\frac{5}{e}\right]^4$.

Sol. Let $\quad L = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{4n}{n}\right)\right]^{1/n}$

$\therefore \quad \log L = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+......+\log\left(1+\frac{4n}{n}\right)\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{4n}\log\left(1+\frac{r}{n}\right) = \int_0^4 \log(1+x)\,dx$$

$$\left(\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right)$$

[Note that the value of r/n is 0 and 4 for the first and last terms as $n \to \infty$]

$$= \left[\log(1+x)\,.\,x\right]_0^4 - \int_0^4 \frac{1}{1+x}\,.\,x\,dx$$

$$= 4\log 5 - \int_0^4\left(1-\frac{1}{1+x}\right)dx = 4\log 5 - \left[x - \log(1+x)\right]_0^4$$

$$= 4\log 5 - [4 - \log 5] = 5\log 5 - 4 = \log 5^5 - \log e^4 = \log\frac{5^5}{e^4}$$

$\Rightarrow \quad L = \frac{5^5}{e^4} = 5\left[\frac{5}{e}\right]^4.$

1.14. LINEARITY PROPERTIES OF RIEMANN INTEGRAL

NOTES

Theorem 15. *If $f \in R[a, b]$ and $k \in R$, then $kf \in R[a, b]$ and $\int_a^b (kf)(x)\, dx = k \int_a^b f(x)\, dx$.*

Proof. If $k = 0$, then theorem is obvious. So, let $k \neq 0$.

Since $\quad f \in R[a, b], \int_{\underline{a}}^b f(x)\, dx = \int_a^{\bar{b}} f(x)\, dx = \int_a^b f(x)\, dx$

Let $\quad P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of $[a, b]$.

Let m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$

f is bounded on $[a, b] \Rightarrow kf$ is bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supermum of kf on $I_r = [x_{r-1}, x_r]$.

Case 1. Let k > 0.

Then $\quad m_r' = km_r \quad$ and $\quad M_r' = kM_r$

$$\therefore \quad L(P, kf) = \sum_{r=1}^{n} m_r' \delta_r = \sum_{r=1}^{n} (km_r)\, \delta_r = k \sum_{r=1}^{n} m_r \delta_r = kL(P, f)$$

and
$$U(P, kf) = \sum_{r=1}^{n} M_r' \delta_r = \sum_{r=1}^{n} (kM_r)\, \delta_r = k \sum_{r=1}^{n} M_r \delta_r = kU(P, f)$$

$$\therefore \quad \int_{\underline{a}}^b (kf)(x)\, dx = \sup \left\{L(P, kf)\right\}_{P \in P[a,b]} = \sup \left\{kL(P, f)\right\}_{P \in P[a,b]}$$

$$= k \sup \left\{L(P, f)\right\}_{P \in P[a,b]} = k \int_{\underline{a}}^b f(x)\, dx = k \int_a^b f(x)\, dx$$

$$\text{Also } \int_a^{\bar{b}} (kf)(x)\, dx = \inf \left\{U(P, kf)\right\}_{P \in P[a,b]} = \inf \left\{kU(P, f)\right\}_{P \in P[a,b]}$$

$$= k \inf \left\{U(P, f)\right\}_{P \in P[a,b]} = k \int_a^{\bar{b}} f(x)\, dx = k \int_a^b f(x)\, dx$$

$$\therefore \quad \int_{\underline{a}}^b (kf)(x)\, dx = \int_a^{\bar{b}} (kf)(x)\, dx = k \int_a^b f(x)\, dx$$

Hence $\quad kf \in R[a, b] \quad$ and $\quad \int_a^b (kf)(x)\, dx = k \int_a^b f(x)\, dx.$

Case. 2. Let k < 0.

Then $\quad m_r' = kM_r \quad$ and $\quad M_r' = km_r$

$$\therefore \quad L(P, kf) = \sum_{r=1}^{n} m_r' \delta_r = \sum_{r=1}^{n} (kM_r) \delta_r = k \sum_{r=1}^{n} M_r \delta_r = kU(P, f)$$

and
$$U(P, kf) = \sum_{r=1}^{n} M_r' \delta_r = \sum_{r=1}^{n} (km_r) \delta_r = k \sum_{r=1}^{n} m_r \delta_r = kL(P, f)$$

NOTES

$$\therefore \quad \int_{\underline{a}}^{b} (kf)(x)\,dx = \sup\left\{L(P, kf)\right\}_{P\in P[a,b]} = \sup\left\{kU(P, f)\right\}_{P\in P[a,b]}$$

$$= k \inf\left\{U(P, f)\right\}_{P\in P[a,b]} = k\int_{a}^{\bar{b}} f(x)\,dx = k\int_{a}^{b} f(x)\,dx$$

$$\text{Also } \int_{a}^{\bar{b}} (kf)(x)\,dx = \inf\left\{U(P, kf)\right\}_{P\in P[a,b]} = \inf\left\{kL(P, f)\right\}_{P\in P[a,b]}$$

$$= k \sup\left\{L(P, f)\right\}_{P\in P[a,b]} = k\int_{\underline{a}}^{b} f(x)\,dx = k\int_{a}^{b} f(x)\,dx$$

$$\therefore \quad \int_{\underline{a}}^{b} (kf)(x)\,dx = \int_{a}^{\bar{b}} (kf)(x)\,dx = k\int_{a}^{b} f(x)\,dx$$

Hence $\quad kf \in R[a, b] \quad$ and $\quad \int_{a}^{b} (kf)(x)\,dx = k\int_{a}^{b} f(x)\,dx.$

(Second Proof)

Since f is integrable on $[a, b]$, therefore, given $\varepsilon > 0$, there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\} \text{ of } [a, b] \text{ such that } U(P, f) - L(P, f) < \frac{\varepsilon}{|k|} \quad \ldots(1)$$

Let m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$. f is bounded on $[a, b] \;\Rightarrow\; kf$ is bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supermum of kf on $I_r = [x_{r-1}, x_r]$.

$$m_r' = \begin{cases} km_r & \text{if } k > 0 \\ kM_r & \text{if } k < 0 \end{cases} \quad \text{and} \quad M_r' = \begin{cases} kM_r & \text{if } k > 0 \\ km_r & \text{if } k < 0 \end{cases}$$

$$\Rightarrow \quad L(P, kf) = \begin{cases} kL(P, f) & \text{if } k > 0 \\ kU(P, f) & \text{if } k < 0 \end{cases} \quad \text{and} \quad U(P, kf) = \begin{cases} kU(P, f) & \text{if } k > 0 \\ kL(P, f) & \text{if } k < 0 \end{cases}$$

$$\Rightarrow \quad U(P, kf) - L(P, kf) = \begin{cases} k(U(P, f) - L(P, f)) & \text{if } k > 0 \\ -k(U(P, f) - L(P, f)) & \text{if } k < 0 \end{cases}$$

$$= |k|\,(U(P, f) - L(P, f)) < \varepsilon \qquad \text{[from (1)]}$$

$\Rightarrow \quad kf$ is integrable on $[a, b]$.

$$\text{Also} \quad \int_{a}^{b} (kf)(x)\,dx = \int_{a}^{\bar{b}} (kf)(x)\,dx \qquad [\because \; kf \text{ is integrable}]$$

$$= \inf\left\{U(P, kf)\right\}_{P\in P[a,b]} = \begin{cases} \inf\left\{kU(P, f)\right\}_{P\in P[a,b]} & \text{if } k > 0 \\ \inf\left\{kL(P, f)\right\}_{P\in P[a,b]} & \text{if } k < 0 \end{cases}$$

$$= \begin{cases} k\inf\left\{U(P, f)\right\}_{P\in P[a,b]} & \text{if } k > 0 \\ k\sup\left\{L(P, f)\right\}_{P\in P[a,b]} & \text{if } k < 0 \end{cases} = \begin{cases} k\int_{a}^{\bar{b}} f(x)\,dx & \text{if } k > 0 \\ k\int_{\underline{a}}^{b} f(x)\,dx & \text{if } k < 0 \end{cases}$$

$$= k\int_{a}^{b} f(x)\,dx, \text{ since } f \text{ is integrable.}$$

Theorem 16. *If $f \in R[a, b]$, then $|f| \in R[a, b]$ and* $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f|(x)\,dx$.

Proof. Since $f \in R[a, b]$, f is bounded on $[a, b]$

$\therefore$ there exists a positive number k such that

$$|f(x)| \le k \quad \forall \quad x \in [a, b] \quad \Rightarrow \quad |f|(x) \le k \quad \forall \quad x \in [a, b]$$

$\Rightarrow$ $|f|$ is bounded on $[a, b]$.

Since f is integrable on $[a, b]$, therefore, given $\varepsilon > 0$, there exists a partition

$$P = \{a = x_0, x_1, x_2, ..., x_n = b\} \text{ of } [a, b] \text{ such that } U(P, f) - L(P, f) < \varepsilon \quad ...(1)$$

Now, let m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$ and m_r', M_r' be the infimum and supermum of $|f|$ on I_r.

For all $\alpha, \beta \in I_r$, we have

$$||f|(\alpha) - |f|(\beta)| = ||f(\alpha)| - |f(\beta)||$$
$$\le |f(\alpha) - f(\beta)| \le M_r - m_r$$

$\therefore$ $$M_r' - m_r' \le M_r - m_r,\ r = 1, 2, ..., n$$

$$U(P, |f|) - L(P, |f|) = \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r \le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$
$$= U(P, f) - L(P, f) < \varepsilon \qquad \text{[from (1)]}$$

$\therefore$ $$|f| \in R[a, b]$$

Since $$|f| = \max\{f, -f\}$$

$$\left.\begin{aligned} &\therefore & f(x) &\le |f(x)| = |f|(x) \\ &\text{and} & -f(x) &\le |f(x)| = |f|(x) \end{aligned}\right\} \forall\, x \in [a, b]$$

$\Rightarrow$ $$\int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \qquad ...(2)$$

and $$\int_a^b -f(x)\,dx \le \int_a^b |f|(x)\,dx \quad \text{or} \quad -\int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \quad ...(3)$$

Combining (2) and (3), we have $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f|(x)\,dx$.

Remark. The converse of this theorem is not true. Thus, if $|f|$ is integrable on $[a, b]$, then f need not be integrable on $[a, b]$.

Consider a function $f : [a, b] \to R$ defined as $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, then

$$m_r = -1 \quad \text{and} \quad M_r = 1,\ r = 1, 2, ..., n.$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} -\delta_r = -(b - a) = a - b$$

$$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \delta_r = b - a$$

$$\therefore \quad \underline{\int_a^b} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P\in P[a,b]} = \sup\{(a-b)\} = a-b$$

$$\overline{\int_a^b} f(x)\,dx = \inf\left\{U(P, f)\right\}_{P\in P[a,b]} = \inf\{(b-a)\} = b-a$$

NOTES

Since $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$, $f \notin R[a, b]$

But $\quad |f|(x) = |f(x)| = 1 \;\; \forall\; x \in [a, b]$

Since $|f|$ is a constant function, $|f| \in R[a, b]$.

Theorem 17. *If $f, g \in R[a, b]$ then $f + g \in R[a, b]$ and*

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proof. f and g, being integrable, are bounded on $[a, b]$.

$\Rightarrow \quad f + g$ is bounded on $[a, b]$.

Let $\quad P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a, b]$.

Let m_r', M_r' be the infimum and supermum of f on I_r,

m_r'', M_r'' be the infimum and supermum of g on I_r, and

m_r, M_r be the infimum and supermum of $f + g$ on I_r.

Now $\quad M_r', M_r''$ are superma of f, g on I_r

$$\Rightarrow \quad f(x) \le M_r',\; g(x) \le M_r'' \;\forall\; x \in I_r$$

$$\Rightarrow \quad f(x) + g(x) \le M_r' + M_r'' \qquad \forall\; x \in I_r$$

$$\Rightarrow \quad (f+g)(x) \le M_r' + M_r'' \qquad \forall\; x \in I_r$$

$\Rightarrow \quad M_r' + M_r''$ is an upper bound of $f + g$ on I_r.

But M_r is the least upper bound of $f + g$ on I_r.

$$\therefore \quad M_r \le M_r' + M_r'' \text{ on } I_r,\; r = 1, 2, ..., n$$

$$\therefore \quad U(P, f+g) = \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} (M_r' + M_r'')\delta_r$$

$$= \sum_{r=1}^{n} M_r'\delta_r + \sum_{r=1}^{n} M_r''\delta_r = U(P, f) + U(P, g)$$

$$\Rightarrow \quad U(P, f+g) \le U(P, f) + U(P, g)$$

Similarly, we can prove that

$$L(P, f+g) \ge L(P, f) + L(P, g)$$

$$\therefore \quad \omega(P, f+g) = U(P, f+g) - L(P, f+g)$$

$$\le [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)]$$

$$= [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)] = \omega(P, f) + \omega(P, g) \quad ...(1)$$

Let $\varepsilon > 0$ be given.

Since f and g are integrable on $[a, b]$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\omega(P_1, f) < \frac{\varepsilon}{2} \quad \text{and} \quad \omega(P_2, g) < \frac{\varepsilon}{2}$$

Let $\quad P' = P_1 \cup P_2$, then $\quad \omega(P', f) \le \omega(P_1, f) < \frac{\varepsilon}{2}$ ...(2)

and $\quad \omega(P', g) \le \omega(P_2, g) < \frac{\varepsilon}{2}$ ...(3)

Using (1), we have $\quad \omega(P', f+g) \le \omega(P', f) + \omega(P', g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad$ [by (2) and (3)]

$\Rightarrow \quad f + g$ is integrable on $[a, b]$.

Now, let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} = \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(P_1, f)$$

and
$$\int_a^b g(x)\,dx - \frac{\varepsilon}{2} = \underline{\int_a^b} g(x)\,dx - \frac{\varepsilon}{2} < L(P_2, g)$$

If $\quad P = P_1 \cup P_2$, then $\quad L(P_1, f) \le L(P, f)$ and $L(P_2, g) \le L(P, g)$

$$\therefore \quad \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon < L(P_1, f) + L(P_2, g)$$

$$\le L(P, f) + L(P, g) \le L(P, f+g)$$

$$\le \underline{\int_a^b} (f+g)(x)\,dx = \int_a^b (f+g)(x)\,dx$$

($\because \quad f + g$ is integrable)

Since $\quad \varepsilon > 0$ is arbitrary, $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le \int_a^b (f+g)(x)\,dx$ ...(4)

Replacing f by $-f$ and g by $-g$, we have from (4),

$$\int_a^b -f(x)\,dx + \int_a^b -g(x)\,dx \le \int_a^b -(f+g)(x)\,dx$$

$$\Rightarrow \quad \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \ge \int_a^b (f+g)(x)\,dx \qquad ...(5)$$

From (4) and (5), we get

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Cor. 1. *If $f, g \in R[a, b]$, then $f - g \in R[a, b]$*

and
$$\int_a^b (f-g)(x)\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Cor. 2. *If $f, g \in R[a, b]$ and $\alpha, \beta \in R$, then $\alpha f + \beta g \in R[a, b]$.*

Proof.
$f \in R[a, b], \alpha \in R \quad \Rightarrow \quad \alpha f \in R[a, b]$

$g \in R[a, b], \beta \in R \quad \Rightarrow \quad \beta g \in R[a, b]$

$\alpha f \in R[a, b], \beta g \in R[a, b] \quad \Rightarrow \quad \alpha f + \beta g \in R[a, b]$

Also $\quad \int_a^b (\alpha f + \beta g)(x)\,dx = \int_a^b (\alpha f)(x)dx + \int_a^b (\beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$

Theorem 18. *If $f \in R\,[a, b]$ then $f^2 \in R\,[a, b]$.*

Proof. $\quad f \in R[a, b] \quad \Rightarrow \quad |f| \in R[a, b]$

$\therefore \quad f$ is bounded on $[a, b] \quad \Rightarrow \quad |f|$ is bounded on $[a, b] \quad \Rightarrow \quad |f|^2 = f^2$ is bounded on $[a, b]$

Since $f^2 = |f|^2$, without loss of generality, we can assume that $f \geq 0$.

Let sup f in $[a, b] = M$.

Let $\varepsilon > 0$ be given.

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$f \in R[a, b] \Rightarrow$ there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2M+1} \quad \text{or} \quad \sum_{r=1}^{n} (M_r - m_r)\delta_r < \frac{\varepsilon}{2M+1} \qquad ...(1)$$

where m_r, M_r are the infimum and supermum of f on I_r.

In I_r, $\inf(f^2) = (\inf f)^2 = m_r^2$ and $\sup(f^2) = (\sup f)^2 = M_r^2$

$$\therefore \quad U(P, f^2) - L(P, f^2) = \sum_{r=1}^{n} (M_r^2 - m_r^2)\delta_r$$

$$= \sum_{r=1}^{n} (M_r + m_r)(M_r - m_r)\delta_r \leq \sum_{r=1}^{n} (M + M)(M_r - m_r)\delta_r$$

$$= 2M \sum_{r=1}^{n} (M_r - m_r)\delta_r < 2M \cdot \frac{\varepsilon}{2M+1} \qquad \text{[by (1)]}$$

$$< \varepsilon$$

$\Rightarrow$ for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that $U(P, f^2) - L(P, f^2) < \varepsilon$

$\therefore$ f^2 is integrable on $[a, b]$.

Theorem 19. *If $f, g \in R[a, b]$ then $fg \in R[a, b]$.*

Proof. Since $f, g \in R[a, b]$, f and g are bounded on $[a, b]$

$\Rightarrow \exists\, k > 0$ such that $|f(x)| \leq k$ and $|g(x)| \leq k \;\; \forall\, x \in [a, b]$

$\Rightarrow \quad |f(g)(x)| = |f(x)g(x)| = |f(x)|\,|g(x)| \leq k^2 \;\; \forall\, x \in [a, b]$

$\Rightarrow$ fg is bounded on $[a, b]$.

Now $f \in R[a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2k}$$

Also $g \in R[a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exists a partition P_2 of $[a, b]$ such that

$$U(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2k}.$$

Let $P = P_1 \cup P_2$, then

$$\left.\begin{array}{l} U(P, f) - L(P, f) - U(P_1, f) - L(P_1, f) < \dfrac{\varepsilon}{2k} \\ \text{and} \quad U(P, g) - L(P, g) \leq U(P_2, g) \leq L(P_2, g) < \dfrac{\varepsilon}{2k} \end{array}\right] \qquad ...(1)$$

Let m_r, M_r, m_r', M_r' and m_r'', M_r'' be the infimum and supermum of fg, f and g respectively on $I_r = [x_{r-1}, x_r]$.

For all $\alpha, \beta \in I_r$, we have

$$|(fg)(\beta) - (fg)(\alpha)| = |f(\beta)g(\beta) - f(\alpha)g(\alpha)|$$

$$= |f(\beta)\,g(\beta) - f(\alpha)g(\beta) + f(\alpha)\,g(\beta) - f(\alpha)\,g(\alpha)|$$

$$= | g(\beta)(f(\beta) - f(\alpha)) + f(\alpha)(g(\beta) - g(\alpha)) |$$
$$\leq | g(\beta) || f(\beta) - f(\alpha) | + | f(\alpha) || g(\beta) - g(\alpha) |$$
$$\leq k(M_r' - m_r') + k(M_r'' - m_r'')$$
$$\Rightarrow \quad M_r - m_r \leq k(M_r' - m_r') + k(M_r'' - m_r'')$$
$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\delta_r \leq k \sum_{r=1}^{n} (M_r' - m_r')\delta_r + k \sum_{r=1}^{n} (M_r'' - m_r'')\delta_r$$
$$\Rightarrow \quad U(P, fg) - L(P, fg) \leq k[U(P, f) - L(P, f)] + k[U(P, g) - L(P, g)]$$
$$< k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{\varepsilon}{2k} = \varepsilon \qquad \text{[by (1)]}$$

Thus for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that

$$U(P, fg) - L(P, fg) < \varepsilon$$
$$\therefore \quad fg \in R[a, b].$$

(Second Proof)

We may write $\quad fg = \dfrac{1}{4} [(f+g)^2 - (f-g)^2]$

Now $\quad f, g \in R[a, b] \;\Rightarrow\; f+g, f-g \in R[a, b]$ (by Theorem 3 and Cor. 1)

$\Rightarrow \quad (f+g)^2, (f-g)^2 \in R[a, b]$ (by Theorem 4)

$\Rightarrow \quad (f+g)^2 - (f-g)^2 \in R[a, b]$ (by Theorem 3, Cor. 1)

$\Rightarrow \quad \dfrac{1}{4} [(f+g)^2 - (f-g)^2] \in R[a, b]$ (by Theorem 1)

$\Rightarrow \quad fg \in R[a, b].$

Remark. Even though f, g are not integrable on $[a, b]$, fg may be integrable on $[a, b]$.

Consider $f : [a, b] \to R$ and $g : [a, b] \to R$ defined by

$$f(x) = \begin{cases} 0, & x \in Q \\ 1 & x \in R - Q \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x \in Q \\ 0 & x \in R - Q \end{cases}$$

Then f, g are not integrable on $[a, b]$, but

$$(fg)(x) = f(x)g(x) = 0 \quad \forall\, x \in [a, b] \text{ is a constant function.}$$
$$\therefore \quad fg \in R[a, b].$$

Theorem 20. *If $f \in R[a, b]$, and there exists $t > 0$ such that*

$$| f(x) | \geq t \quad \forall\, x \in [a, b], \textit{ then } \frac{1}{f} \in R[a, b].$$

Proof. Since $\quad | f(x) | \geq t \quad \forall\, x \in [a, b].$

$$\therefore \quad \frac{1}{| f(x) |} \leq \frac{1}{t} \quad \forall\, x \in [a, b], t > 0$$

$$\Rightarrow \quad \left| \frac{1}{f(x)} \right| \leq \frac{1}{t} \quad \forall\, x \in [a, b] \;\Rightarrow\; \left| \frac{1}{f}(x) \right| \leq \frac{1}{t} \quad \forall \quad x \in [a, b]$$

$\Rightarrow \quad \dfrac{1}{f}$ is bounded on $[a, b]$.

Since $f \in R[a, b]$, for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < t^2 \varepsilon \qquad \ldots(1)$$

Let m_r', M_r' be the infimum and supermum of f on I_r and m_r, M_r be the infimum and supermum of $\frac{1}{f}$ on I_r.

NOTES

For all $\alpha, \beta \in I_r$, we have

$$\left|\left(\frac{1}{f}\right)(\beta)-\left(\frac{1}{f}\right)(\alpha)\right| = \left|\frac{1}{f(\beta)}-\frac{1}{f(\alpha)}\right| = \frac{|f(\alpha)-f(\beta)|}{|f(\alpha)||f(\beta)|} \leq \frac{M_r'-m_r'}{t^2}$$

$$\therefore \qquad M_r - m_r \leq \frac{1}{t^2}(M_r' - m_r')$$

$$\Rightarrow \qquad \sum_{r=1}^{n}(M_r - m_r)\delta_r \leq \frac{1}{t^2}\sum_{r=1}^{n}(M_r' - m_r')\delta_r$$

$$\Rightarrow \quad U\left(P, \frac{1}{f}\right) - L\left(P, \frac{1}{f}\right) \leq \frac{1}{t^2}[U(P, f) - L(P, f)] < \frac{1}{t^2}\cdot t^2\varepsilon = \varepsilon \qquad \text{[by (1)]}$$

$\therefore$ for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that

$$U\left(P, \frac{1}{f}\right) - L\left(P, \frac{1}{f}\right) < \varepsilon \;\Rightarrow\; \frac{1}{f} \in R[a, b].$$

Theorem 21. *If $f, g \in R[a, b]$ and there exists $t > 0$ such that*

$$|g(x)| \geq t \quad \forall \quad x \in [a, b], \text{ then } \frac{f}{g} \in R[a, b].$$

Proof. $f, g \in R[a, b] \;\Rightarrow\; f, g$ are bounded on $[a, b]$.

$\Rightarrow$ there exists a positive real number k such that

$$|f(x)| \leq k, \; |g(x)| \leq k \;\; \forall \quad x \in [a, b]$$

$$\therefore \qquad \forall\; x \in [a, b], \left|\left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(x)}{g(x)}\right| \leq \frac{k}{t}$$

$\Rightarrow$ $\frac{f}{g}$ is bounded on $[a, b]$.

Since f, g are integrable on $[a, b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\left.\begin{aligned} U(P_1, f) - L(P_1, f) &< \frac{t^2\varepsilon}{2k} \\ \text{and} \qquad U(P_2, g) - L(P_2, g) &< \frac{t^2\varepsilon}{2k} \end{aligned}\right\} \qquad \text{...(1)}$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2, then using (1), we have

$$\left.\begin{aligned} U(P, f) - L(P, f) \leq U(P_1, f) - L(P_1, f) &< \frac{t^2\varepsilon}{2k} \\ \text{and} \qquad U(P, g) - L(P, g) \leq U(P_2, g) - L(P_2, g) &< \frac{t^2\varepsilon}{2k} \end{aligned}\right\} \qquad \text{...(2)}$$

Let m_r', M_r' be the infimum and supermum of f on I_r, m_r'', M_r'' be the infimum and supermum of g on I_r and m_r, M_r be the infimum and supermum of f/g on I_r.

For all $\alpha, \beta \in I_r$, we have

$$\left|\left(\frac{f}{g}\right)(\beta)-\left(\frac{f}{g}\right)(\alpha)\right| = \left|\frac{f(\beta)}{g(\beta)}-\frac{f(\alpha)}{g(\alpha)}\right| = \left|\frac{g(\alpha)f(\beta)-f(\alpha)g(\beta)}{g(\alpha)g(\beta)}\right|$$

$$= \frac{|g(\alpha)[f(\beta) - f(\alpha)] - f(\alpha)[g(\beta) - g(\alpha)]|}{|g(\alpha)||g(\beta)|}$$

$$\leq \frac{|g(\alpha)|}{|g(\alpha)||g(\beta)|} \; |f(\beta) - f(\alpha)| + \frac{|f(\alpha)|}{|g(\alpha)||g(\beta)|} \; |g(\beta) - g(\alpha)|$$

$$\leq \frac{k}{t^2} (M_r' - m_r') + \frac{k}{t^2} (M_r'' - m_r'')$$

$$\therefore \quad M_r - m_r \leq \frac{k}{t^2} (M_r' - m_r') + \frac{k}{t^2} (M_r'' - m_r'')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\delta_r \leq \frac{k}{t^2} \sum_{r=1}^{n} (M_r' - m_r')\delta_r + \frac{k}{t^2} \sum_{r=1}^{n} (M_r'' - m_r'')\delta_r$$

$$\Rightarrow \quad U\left(P, \frac{f}{g}\right) - L\left(P, \frac{f}{g}\right) \leq \frac{k}{t^2} [U(P, f) - L(P, f)] + \frac{k}{t^2} [U(P, g) - L(P, g)]$$

$$< \frac{k}{t^2} \cdot \frac{t^2\varepsilon}{2k} + \frac{k}{t^2} \cdot \frac{t^2\varepsilon}{2k} = \varepsilon \qquad \text{[by (2)]}$$

$$\therefore \quad \frac{f}{g} \in R[a, b].$$

Theorem 22. *If $f \in R[a, b]$ and $a < c < b$, then $f \in R[a, c]$, $f \in R[c, b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. $f \in R[a, b] \Rightarrow f$ is bounded on $[a, b]$.

$\Rightarrow$ f is bounded on $[a, c]$ and $[c, b]$. $\qquad (\because \; a < c < b)$

Since $f \in R[a, b]$, for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$

Let $P' = P \cup \{c\}$, then

$$L(P, f) \leq L(P', f) \leq U(P', f) \leq U(P, f)$$

$$\Rightarrow \quad U(P', f) - L(P', f) \leq U(P, f) - L(P, f) < \varepsilon \qquad \ldots(1)$$

Let P_1, P_2 denote the set of points of P' on $[a, c]$, $[c, b]$ respectively, then P_1, P_2 are partitions on $[a, c]$ and $[c, b]$ respectively and $P' = P_1 \cup P_2$.

$$\therefore \quad U(P', f) = U(P_1, f) + U(P_2, f) \quad \text{and} \quad L(P', f) = L(P_1, f) + L(P_2, f)$$

$$\Rightarrow \quad [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] = U(P', f) - L(P', f) < \varepsilon \qquad \text{[by (1)]}$$

Since each of $[U(P_1, f) - L(P_1, f)]$ and $[U(P_2, f) - L(P_2, f)]$ is non-negative, each of these is less than ε

i.e., $\quad U(P_1, f) - L(P_1, f) < \varepsilon$ and $U(P_2, f) - L(P_2, f) < \varepsilon$

for partitions P_1, P_2 of $[a, c]$ and $[c, b]$ respectively.

Hence $f \in R[a, c]$ and $f \in R[c, b]$

Now $\quad U(P', f) = U(P_1, f) + U(P_2, f)$

$$\Rightarrow \quad \inf U(P', f) = \inf U(P_1, f) + \inf U(P_2, f)$$

$$\Rightarrow \quad \int_a^{\bar{b}} f(x)\,dx = \int_a^{\bar{c}} f(x)\,dx + \int_c^{\bar{b}} f(x)\,dx$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

NOTES

Since $f \in R[a, b], f \in R[a, c]$ and $f \in R[c, b]$.

Cor. If f is integrable on $[a, b]$ then f is integrable on any sub-interval of $[a, b]$.

Theorem 23. *If $f \in R[a, c]$, $f \in R[c, b]$ and $a < c < b$ then $f \in R[a, b]$.*

Proof. Since $f \in R[a, c]$ and $f \in R[c, b]$

$\therefore$ given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively such that

$$\left.\begin{array}{l} U(P_1, f) - L(P_1, f) < \dfrac{\varepsilon}{2} \\ U(P_2, f) - L(P_2, f) < \dfrac{\varepsilon}{2} \end{array}\right\} \quad \text{and} \qquad ...(1)$$

If $P = P_1 \cup P_2'$, then P is a partition of $[a, b]$.

Also $U(P, f) - L(P, f)$

$$= [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)]$$
$$= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)]$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{[by (1)]}$$

$\Rightarrow \quad f \in R[a, b]$.

Theorem 24. *If $f \in R[a, b]$ and $f(x) \geq 0 \quad \forall \quad x \in [a, b]$ then $\int_a^b f(x)\,dx \geq 0$.*

Proof. $f \in R[a, b] \quad \Rightarrow \quad f$ is bounded on $[a, b]$.

Let m, M be the infimum and supermum of f on $[a, b]$.

Since $\quad f(x) \geq 0 \quad \forall\, x \in [a, b],\ m \geq 0$

For all partitions P of $[a, b]$, we have $L(P, f) \leq m\,(b - a) \geq 0$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx = \sup\left\{L(P, f)\right\}_{P \in P[a,b]} \geq 0$$

But $\quad \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$ since $f \in R[a, b]$

$$\therefore \quad \int_a^b f(x)\,dx \geq 0.$$

Theorem 25. *If $f, g \in R[a, b]$ and $f(x) \geq g(x)\ \forall\, x \in [a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*

Proof. $\quad f, g \in R[a, b] \quad \Rightarrow \quad f - g \in R[a, b]$

Also $\quad f(x) \geq g(x) \quad \forall\, x \in [a, b]$

$\Rightarrow \quad f(x) - g(x) \geq 0 \quad \forall\, x \in [a, b]$

$\Rightarrow \quad (f - g)(x) \geq 0 \quad \forall\, x \in [a, b]$

$\therefore$ By Theorem 25, $\int_a^b (f - g)(x)\,dx \geq 0$

$$\Rightarrow \quad \int_a^b (f(x) - g(x))\,dx \geq 0 \quad \Rightarrow \quad \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

$$\therefore \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Theorem 26. *If $f \in R[a, b]$ and m, M are the infimum and supermum of f in $[a, b]$, then*

$$\int_a^b f(x)\,dx = \mu(b - a) \text{ where } \mu \in [m, M].$$

NOTES

Proof. For every partition P of $[a, b]$, we have

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a) \quad \ldots(1)$$

Now $\sup \{L(P, f)\}_{P \in P[a, b]} = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \quad [\because f \in R[a, b]]$

$$\Rightarrow \quad L(P, f) \le \int_a^b f(x)\,dx \quad \ldots(2)$$

Also $\inf \{U(P, f)\}_{P \in P[a, b]} = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \quad [\because f \in R[a, b]]$

$$\Rightarrow \quad \int_a^b f(x)\,dx \le U(P, f) \quad \ldots(3)$$

From (1), (2) and (3), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M \text{ for } a \ne b$$

$\Rightarrow \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is a number μ (say) lying between the bounds.

$$\Rightarrow \quad \frac{1}{b-a}\int_a^b f(x)\,dx = \mu \qquad \text{where } m \le \mu \le M$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \mu(b-a) \qquad \text{where } \mu \in [m, M]$$

For $a = b$, the result is trivially true.

Theorem 27. *If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

Proof. f is continuous on $[a, b]$.

$\Rightarrow$ f is bounded on $[a, b]$ and $f \in R[a, b]$. If m, M are the infimum and supermum of f on $[a, b]$, then we know that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$\therefore\ \exists\ \mu \in [m, M]$ such that $\displaystyle\int_a^b f(x)\,dx = \mu(b-a) \quad \ldots(1)$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M.

$\therefore \quad \mu \in [m, M] \quad \Rightarrow \quad \exists$ a number $c \in [a, b]$ such that $f(c) = \mu$.

$\therefore$ From (1), we have $\displaystyle\int_a^b f(x)\,dx = (b-a)\,f(c)$.

1.15. INTEGRATION AND DIFFERENTIATION

In this section we discuss the famous the fundamental theorem of calculus, which stated that integration and differentiation are, in a certain sense, inverse operations. We shall made this study for Riemann integrals.

Integral Function. If f is Riemann integrable function on $[a, b]$ then a function $F(x) = \int_a^x f(t)\,dt$ is called an integral function. Further if $f(x)$ is differentiable on $[a, b]$ and $F'(x) = f(x)$, then $F(x)$ is called the primitive or anit-derivative of f on $[a, b]$.

Here, we note that the primitive of $f(x)$ is not unique also an integrable function is not necessarily continuous but the function associated to f is always continuous as shown in the following theorem.

Theorem 28. Let $f \in R[a, b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$. Furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$. In other word "The integral of a Riemann integrable function is continuous and is differentiable if f is continuous".

Theorem 29. (First Mean Value Theorem)

If $f, g \in R[a, b]$ and g keeps the same sign on $[a, b]$ then there exists a number μ between the infimum and supermum of f on $[a, b]$ such that $\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx$.

Proof. Let g be non-negative on $[a, b]$

Then, $g(x) \ge \forall\ x \in [a, b]$

$$f \in R[a, b] \Rightarrow f \text{ is bounded on } [a, b]$$

$\therefore$ If m, M are the infimum and supermum of f on $[a, b]$, then

$$m \le f(x) \le M \qquad \forall\ x \in [a, b]$$

Since $\qquad g(x) \ge 0 \qquad \forall\ x \in [a, b]$

$$\therefore \qquad mg(x) \le f(x)\,g(x) \le Mg(x)$$

$$\Rightarrow \qquad \int_a^b mg(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \int_a^b Mg(x)\,dx$$

$$\Rightarrow \qquad m\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b f(x)\,dx$$

$$\Rightarrow \quad \exists\, \mu \in [m, M] \text{ such that } \int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)$$

If g be non-positive on $[a, b]$, then $g(x) \le 0 \quad \forall\, x \in [a, b]$

$$\therefore \qquad m \le f(x) \le M \quad \forall\, x \in [a, b]$$

$$\Rightarrow \qquad mg(x) \ge f(x)\,g(x) \ge Mg(x)$$

$$\Rightarrow \qquad m\int_a^b g(x)\,dx \ge \int_a^b f(x)\,g(x)\,dx \ge M\int_a^b g(x)\,dx$$

$$\Rightarrow \qquad \exists\, \mu \in [m, M] \text{ such that } \int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

Note. If we take $g(x) = 1 \quad \forall\, x \in [a, b]$, then $g \in R[a, b]$ and $g(x) > 0 \quad \forall\, x \in [a, b]$. By the mean value theorem, we have

$$\int_a^b f(x)\,dx = \mu\int_a^b 1\,dx, \text{ where } \mu \in [m, M] \quad \text{or} \quad \int_a^b f(x)\,dx = \mu(b - a).$$

Cor. *If f is continuous on $[a, b]$, $g \in R[a, b]$ and g keeps the same sign on $[a, b]$, then there exists $c \in [a, b]$ such that* $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

Proof. f is continuous on $[a, b] \Rightarrow f \in R[a, b]$

By Theorem 14, there exists $\mu \in [m, M]$ such that

$$\int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx \qquad ...(1)$$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M.

$\therefore \qquad \mu \in [m, M] \Rightarrow \exists$ a number $c \in [a, b]$ such that $f(c) = \mu$.

$\therefore$ From (1), we have $\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

Primitive (Def.). *If f and F are two functions defined on $[a, b]$ and $F'(x) = f(x)\ \forall\ x \in [a, b]$, then F is called a primitive of f on $[a, b]$.*

1.16. FUNDAMENTAL THEOREM OF CALCULUS

Theroem 31. *If $f \in R\ [a, b]$ and F is a primitive of f on $[a, b]$, then* $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof. F is a primitive of f on $[a, b]$

$$\Rightarrow \qquad F'(x) = f(x) \quad \forall \quad x \in [a, b] \qquad ...(1)$$

Consider a partition $P = \{a = x_0, x_1, ..., x_n = b\}$ of $[a, b]$.

Since F is differentiable on $[a, b]$, it is differentiable (and hence continuous) on each sub-interval

$$I_r = [x_{r-1}, x_r],\ r = 1, 2, ..., n.$$

Applying Lagrange's Mean Value Theorem to F on each sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$, we have

$$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\,F'(\xi_r) = f(\xi_r)\,\delta_r \qquad [\text{by } (1)]$$

where $\qquad x_{r-1} < \xi_r < x_r,\ r = 1, 2, ..., n$

$$\Rightarrow \qquad \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \sum_{r=1}^{n} [F(x_r) - F(x_{r-1})] = F(x_n) - F(x_0) = F(b) - F(a)$$

$$\therefore \qquad \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \lim_{\|P\| \to 0} [F(b) - F(a)] = F(b) - F(a)$$

But $\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \int_a^b f(x)\,dx$

$$\therefore \qquad \int_a^b f(x)\,dx = F(b) - F(a).$$

Remarks 1. The fundamental theorem does not state that if f is integrable, then f has a primitive on $[a, b]$. It only states that if f has a primitive on $[a, b]$, then this primitive can be used to evaluate $\int_a^b f(x)\,dx$.

2. A function may have a primitive without being integrable. Consider the functions F and f defined on $[-1, 1]$ as follows :

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$$F(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 2x \sin \dfrac{1}{x^2} - \dfrac{2}{x} \cos \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Clearly, $F'(x) = f(x) \;\Rightarrow\;$ F is a primitive of f on $[-1, 1]$.

But f is not integrable on $[-1, 1]$ because f is not bounded on $[-1, 1]$.

Example 15. *Prove that* $\dfrac{1}{\pi} \leq \displaystyle\int_0^1 \frac{\sin \pi x}{1 + x^2}\, dx \leq \frac{2}{\pi}$.

Sol. Let $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = \sin \pi x$, then f, g are continuous on $[0, 1]$ and hence integrable on $[0, 1]$.

Also $\qquad g(x) = \sin \pi x \geq 0$ on $[0, 1]$.

Since f is decreasing on $[0, 1]$, $\inf f = f(1) = \dfrac{1}{2}$ and $\sup f = f(0) = 1$

$\therefore$ By the first Mean Value Theorem, there exists $\mu \in \left[\dfrac{1}{2}, 1\right]$ such that

$$\int_0^1 f(x)\,g(x)\,dx = \mu \int_0^1 g(x)\,dx \quad i.e., \quad \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \int_0^1 \sin \pi x\,dx$$

But $\qquad \displaystyle\int_0^1 \sin \pi x\,dx = -\frac{\cos \pi x}{\pi}\Big]_0^1 = \frac{2}{\pi}$

$\therefore \qquad \displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \cdot \frac{2}{\pi} \qquad \ldots(1)$

Since f is continuous on $[0, 1]$, it attains every value between its bound $\dfrac{1}{2}$ and 1.

$\therefore \qquad \mu \in \left[\dfrac{1}{2}, 1\right] \;\Rightarrow\; \exists$ a number $c \in [0, 1]$ such that $f(c) = \mu$.

From (1), $\qquad f(c) = \mu = \dfrac{\pi}{2} \displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx$

But $\;0 \leq c \leq 1$ and f is decreasing on $[0, 1]$

$\Rightarrow \qquad f(0) \geq f(c) \geq f(1) \;\Rightarrow\; \dfrac{1}{2} \leq \dfrac{\pi}{2} \displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq 1 \quad \therefore \quad \frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{x}.$

Example 16. *Prove that* $\dfrac{\pi^2}{9} \leq \displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \leq \frac{2\pi^2}{9}$.

Sol. Let $f(x) = \dfrac{1}{\sin x}$ and $g(x) = x$, then f, g are continuous on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$ and hence integrable on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$.

Also $\qquad g(x) = x > 0$ on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$.

Since f is decreasing on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$, $\inf f = f\left(\dfrac{\pi}{2}\right) = 1$ and $\sup f = f\left(\dfrac{\pi}{6}\right) = 2$.

$\therefore$ by the first mean value theorem, there exists $\mu \in [1, 2]$ such that

$$\int_{\pi/6}^{\pi/2} f(x)\, g(x)\, dx = \mu \int_{\pi/6}^{\pi/2} g(x)\, dx \quad i.e., \quad \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx = \mu \int_{\pi/6}^{\pi/2} x\, dx$$

But $$\int_{\pi/6}^{\pi/2} x\, dx = \left.\frac{x^2}{2}\right]_{\pi/6}^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right) = \frac{\pi^2}{9}$$

$$\therefore \quad \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx = \mu \,.\, \frac{\pi^2}{9} \qquad \ldots(1)$$

Since f is continuous on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, it attains every value between its bounds 1 and 2.

$$\therefore \quad \mu \in [1, 2] \;\Rightarrow\; \exists \text{ a number } c \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \text{ such that } f(c) = \mu$$

From (1), $$f(c) = \mu = \frac{9}{\pi^2} \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx$$

But $\frac{\pi}{6} \le c \le \frac{\pi}{2}$ and f is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$$\Rightarrow \quad f\left(\frac{\pi}{6}\right) \ge f(c) \ge f\left(\frac{\pi}{2}\right) \;\Rightarrow\; 1 \le \frac{9}{\pi^2} \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx \le 2$$

$$\therefore \quad \frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx \le \frac{2\pi^2}{9}.$$

TEST YOUR KNOWLEDGE 1.2

Prove that:

1. (*i*) $\int_1^2 x\, dx = \frac{3}{2}$ (*ii*) $\int_1^3 (x^2 + 2x + 3)\, dx = \frac{68}{3}$.
2. Evaluate $\int_{-2}^{1} f(x)\, dx$, where $f(x) = |x|$.
3. Show that $\int_0^{a} \cos x\, dx = \sin a$; for a fixed number a.
4. Prove that $\int_0^{\pi/2} \sin x\, dx = 1$.
5. Show that

   (*i*) $\lim_{n \to \infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots\ldots + \left(\frac{n}{n}\right)^2\right] = \frac{1}{3}$

   (*ii*) $\lim_{n \to \infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n}\right] = \log_e 2$

   (*iii*) $\lim_{n \to \infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{3n}\right] = \log_e 3$.

6. Show that $\lim_{n \to \infty} \frac{1}{n}(e^{3/n} + e^{6/n} + e^{9/n} + ... + e^{3n/n}) = \frac{1}{3}(e^3 - 1)$.

NOTES

7. Prove that $\frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1+x^2}} dx \le \frac{1}{3}$.

Answers

2. $\frac{5}{2}$

SUMMARY

- **Upper and Lower Darboux sums:** The sum $M_1\delta_1 + M_2\delta_2 + ... + M_r\delta_r + ... + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$ is called the **upper Darboux sum of f** corresponding to the partition P and is denoted by U(P, f) or U(f, P).

  The sum $m_1\delta_1 + m_2\delta_2 + ... + m_r\delta_r + ... + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is called the **lower Darboux sum of f** corresponding to the partition P and is denoted by L(P, f) or L(f, P).

  Thus, $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r; \quad L(P, f) = \sum_{r=1}^{n} m_r\delta_r$$

- **Lower Riemann Integral of f on $[a, b]$** is defined as $\sup \{L(P, f)\}_{P \in P[a, b]}$ and is denoted by $\underline{\int_a^b} f(x)\,dx$.
- **Upper Riemann Integral of f on $[a, b]$** is defined as $\inf \{U(P, f)\}_{P \in P[a, b]}$ is denoted by $\overline{\int_a^b} f(x)\,dx$.
- A bounded function f is said to be Riemann integrable (or simply R-integrable) on $[a, b]$ if its lower and upper Riemann integrals are equal *i.e.*, if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.
- A bounded function f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.
- If $f \in R[a, b]$ then $f^2 \in R[a, b]$.
- If $f, g \in R[a, b]$ then $fg \in R[a, b]$.
- If $f \in R[a, b]$, and there exists $t > 0$ such that,

  $$|f(x)| \ge t \quad \forall x \in [a, b], \text{ then } \frac{1}{f} \in R[a, b].$$

- If $f, g \in R[a, b]$ and there exists $t > 0$ such that

$$|g(x)| \geq t \quad \forall \quad x \in [a, b], \text{ then } \frac{f}{g} \in R[a, b].$$

- If $f \in R[a, b]$ and $a < c < b$, then $f \in R[a, c]$, $f \in R[c, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

- If $f \in R[a, c]$, $f \in R[c, b]$ and $a < c < b$ then $f \in R[a, b]$.
- If $f \in R[a, b]$ and $f(x) \geq 0 \quad \forall \quad x \in [a, b]$ then $\int_a^b f(x)\,dx \geq 0$.
- If $f, g \in R[a, b]$ and $f(x) \geq g(x) \quad \forall \quad x \in [a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $f \in R[a, b]$ and m, M are the infimum and supermum of f in $[a, b]$, then

$$\int_a^b f(x)\,dx = \mu(b-a) \text{ where } \mu \in [m, M].$$

- If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

❑❑❑

NOTES

UNIT

2

ARBITRARY AND POWER SERIES

STRUCTURE

2.1. INTRODUCTION

So far we have been discussing series of positive terms or alternating series. In this chapter we shall discuss the convergence of series of arbitrary terms *i.e.*, series of terms having any sign. We shall also discuss rearrangement of terms of a series, insertion and removal of brackets, Cauchy product of two series and the convergence of infinite products.

2.2. ABEL'S LEMMA (OR ABEL'S INEQUALITY)

NOTES

If the sequence $< S_n >$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ satisfies $m \le S_n \le M$, $(n \in N)$ and $< b_n >$ is a sequence of non-increasing, non-negative real numbers, then $mb_1 \le \sum_{k=1}^{n} a_k b_k \le Mb_1$.

Proof. Since $S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, ...,$

$$S_n = a_1 + a_2 + ... + a_n$$

$$\therefore \quad a_1 = S_1, a_2 = S_2 - S_1, a_3 = S_3 - S_2, ..., a_n = S_n - S_{n-1}$$

$$\therefore \quad \sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

$$= S_1 b_1 + (S_2 - S_1) b_2 + ... + (S_n - S_{n-1}) b_n$$

$$= S_1(b_1 - b_2) + S_2(b_2 - b_3) + ... + S_{n-1}(b_{n-1} - b_n) + S_n b_n \quad ...(1)$$

Now,
$$mb_1 = m[(b_1 - b_2) + (b_2 - b_3) + ... + (b_{n-1} - b_n) + b_n]$$

$$= m(b_1 - b_2) + m(b_2 - b_3) + ... + m(b_{n-1} - b_n) + mb_n$$

$$\le S_1(b_1 - b_2) + S_2(b_2 - b_3) + ... + S_{n-1}(b_{n-1} - b_n) + S_n b_n$$

$$[\because \quad m \le S_n, n \in N]$$

$$= \sum_{k=1}^{n} a_k b_k \qquad \text{[by (1)]}$$

$$= S_1(b_1 - b_2) + S_2(b_2 - b_3) + ... + S_{n-1}(b_{n-1} - b_n) + S_n b_n$$

$$\le M(b_1 - b_2) + M(b_2 - b_3) + ... + M(b_{n-1} - b_n) + Mb_n$$

$$[\because \quad S_n \le M, n \in N]$$

$$= Mb_1$$

$$\Rightarrow \quad mb_1 \le \sum_{k=1}^{n} a_k b_k \le Mb_1.$$

Remark. If $| S_n | \le M \quad \forall n \in N$, then $\left| \sum_{k=1}^{n} a_k b_k \right| \le Mb_1$

i.e., $| a_1 + a_2 + a_3 + ... + a_n | \le M \Rightarrow | a_1 b_1 + a_2 b_2 + a_3 b_3 + ... + a_n b_n | \le Mb_1$

In particular, if $| a_{m+1} + a_{m+2} + ... + a_n | \le M$

then $| a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + ... + a_n b_n | \le M b_{m+1} \le Mb_1$

[$\because$ the sequence $< b_n >$ is non-increasing]

2.3. ABEL'S TEST

If $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $< b_n >$ is monotonic and bounded, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Proof. Since the sequence $< b_n >$ is monotonic and bounded, it is convergent. Let it converge to b.

Let $$u_n = \begin{cases} b - b_n & \text{if } < b_n > \text{is increasing} \\ b_n - b & \text{if } < b_n > \text{is decreasing} \end{cases}$$

NOTES

When $< b_n >$ *is increasing,* b is the l.u.b. of the sequence so that $b_n \le b \ \forall \ n$

$\Rightarrow$ $$b - b_n \ge 0 \quad \therefore \quad u_n \ge 0 \quad \forall \ n \in N$$

and $$u_n - u_{n+1} = (b - b_n) - (b - b_{n+1}) = b_{n+1} - b_n \ge 0$$

$\Rightarrow$ $$u_n \ge u_{n+1} \quad \forall \ n \in N$$

$\Rightarrow < u_n >$ is a non-increasing sequence of non-negative numbers.

When $< b_n >$ *is decreasing,* b is the g.l.b. of the sequence so that

$$b_n \ge b \ \forall \ n \quad \Rightarrow \quad b_n - b \ge 0 \quad \therefore \quad u_n \ge 0 \quad \forall \ n \in N$$

and $$u_n - u_{n+1} = (b_n - b) - (b_{n+1} - b) = b_n - b_{n+1} \ge 0$$

$\Rightarrow$ $$u_n \ge u_{n+1} \quad \forall \ n \in N$$

$\Rightarrow$ $< u_n >$ is a non-increasing sequence of non-negative numbers.

$\therefore$ $$u_n \ge u_{n+1} \ge 0 \ \forall \ n \in N$$

Now $$b_n = \begin{cases} b - u_n & \text{if } < b_n > \text{is increasing} \\ b + u_n & \text{if } < b_n > \text{is decreasing} \end{cases}$$

$\Rightarrow$ $$a_n b_n = \begin{cases} ba_n - a_n u_n & \text{if } < b_n > \text{is increasing} \\ ba_n + a_n u_n & \text{if } < b_n > \text{is decreasing} \end{cases}$$

Since $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} ba_n$ is convergent.

$\therefore$ $\sum_{n=1}^{\infty} a_n b_n$ will be convergent if $\sum_{n=1}^{\infty} a_n u_n$ is convergent.

Now, since $\sum_{n=1}^{\infty} a_n$ is convergent.

$\therefore$ By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$| a_{m+1} + a_{m+2} + \ldots\ldots + a_n | < \varepsilon \text{ whenever } n > m$$

Applying Abel's Lemma, we have

$$| a_{m+1} u_{m+1} + a_{m+2} u_{m+2} + \ldots\ldots + a_n u_n | \le \varepsilon u_{m+1} \le \varepsilon u_1 \quad [\because \ < u_n > \text{ is decreasing}]$$

$\therefore$ By Cauchy's general principle of convergence, $\sum_{n=1}^{\infty} a_n u_n$ is convergent.

Hence $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

2.4. DIRICHLET'S TEST

If $\sum_{n=1}^{\infty} a_n$ *has bounded partial sums and* $< b_n >$ *is a monotonic sequence converging to zero, then* $\sum_{n=1}^{\infty} a_n b_n$ *is convergent.*

Proof. Replacing b_n by $-b_n$, if necessary, we assume that $< b_n >$ is a monotonically decreasing sequence converging to 0.

Let $$S_n = a_1 + a_2 + ... + a_n,\ n \in N.$$

Since $\sum_{n=1}^{\infty} a_n$ has bounded partial sums, therefore, the sequence $< S_n >$ is bounded.

$\Rightarrow$ $\exists$ a real number $M > 0$ such that $| S_n | \le M\ \forall\ n \in N$

$\therefore$ For $n > m$, we have

$$| a_{m+1} + a_{m+2} + ... + a_n | = | S_n - S_m | \le | S_n | + | S_m | \le M + M = 2M \quad ...(1)$$

Since $< b_n >$ converges to 0, therefore, given $\varepsilon > 0$, $\exists$ a positive integer m_0 such that

$$b_n < \frac{\varepsilon}{2M}\ \forall\ n \ge m_0 \quad ...(2)$$

$\therefore$ From (1), by Abel's Lemma, for $n > m \ge m_0$, we have

$$| a_{m+1} b_{m+1} + a_{m+2} b_{m+2} + ... + a_n b_n | \le 2M b_{m+1} < \varepsilon \quad \text{[by (2)]}$$

$\therefore$ By Cauchy's general principle of convergence, $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Cor. Leibnitz's Test as a particular case of Dirichlet's Test.

The series $\sum_{n=1}^{\infty} (-1)^{n-1}$ has bounded partial sums, since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

If $< a_n >$ is a monotonically decreasing sequence of positive numbers, converging to 0,

i.e., if (*i*) $a_n > 0$ (*ii*) $a_n \ge a_{n+1}\ \forall\ n \in N$ (*iii*) $a_n \to 0$ as $n \to \infty$

then by Dirichlet test, the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, *i.e.*, the alternating series $a_1 - a_2 + a_3 - a_4 + ...$ is convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the series* :

(*i*) $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + ...$ (*ii*) $1 - \frac{1}{4.3} + \frac{1}{4^2.5} - \frac{1}{4^3.7} + ...$

Sol. (*i*) The given series can be considered to have arisen as a result of multiplying the terms of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$ with the terms of the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, ...

Let $a_n = \frac{(-1)^{n-1}}{n^2}$, $b_n = \frac{1}{2n-1}$ then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $| a_n | = \frac{1}{n^2}$, therefore Σa_n is convergent.

Since Σa_n is absolutely convergent, therefore, Σa_n is convergent.
Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

NOTES

(*ii*) The given series can be considered to have arisen as a result of multiplying the terms of the series

$$1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + ... \text{ with the terms of the sequence } 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...$$

Let $a_n = \left(-\frac{1}{4}\right)^{n-1}, b_n = \frac{1}{2n-1}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $| a_n | = \left(\frac{1}{4}\right)^{n-1}$, therefore, $\sum_{n=1}^{\infty} | a_n |$ is a geometric series with common ratio $\frac{1}{4}$. $[| r | = \frac{1}{4} < 1]$

$\Rightarrow$ $\Sigma | a_n |$ is convergent.
$\Rightarrow$ Σa_n is convergent.
Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Example 2. *Show that the series* $\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3} - n}{\log n}$ *is convergent.*

Sol. Let $a_n = (n^3 + 1)^{1/3} - n$, $b_n = \frac{1}{\log n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now
$$a_n = (n^3 + 1)^{1/3} - n = n\left(1 + \frac{1}{n^3}\right)^{1/3} - n$$
$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\cdot\frac{1}{n^6} + ... - 1\right] = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + ...\right]$$

Take
$$c_n = \frac{1}{n^2}, \text{ then } \frac{a_n}{c_n} = \frac{1}{3} - \frac{1}{9n^3} + ...$$

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{1}{3} \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, Σa_n and Σc_n converge or diverge together.

But $\sum c_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore$ Σa_n is convergent.
Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

Example 3. *Show that the convergence of* $\sum_{n=1}^{\infty} a_n$ *implies the convergence of each of the following series :*

NOTES

(i) $\sum_{n=1}^{\infty} \frac{1}{n} a_n$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n^p} a_n, p \geq 0$

(iii) $\sum_{n=1}^{\infty} n^{1/n} a^n$ (iv) $\sum_{n=2}^{\infty} \frac{1}{\log n} a_n$

Sol. (i) Let $b_n = \frac{1}{n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

∴ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(ii) Let $b_n = \frac{1}{n^p}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

∴ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(iii) Let $b_n = n^{1/n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also,
$$\left(1+\frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2) \ldots 2.1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \ldots \left(1-\frac{n-1}{n}\right)$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < e < 3$$

NOTES

$\therefore$ For $n \geq 3$, we have $n > \left(1+\frac{1}{n}\right)^n = \frac{(n+1)^n}{n^n}$ $\Rightarrow$ $n^{n+1} > (n+1)^n$

$\Rightarrow$ $n^{1/n} > (n+1)^{1/n+1}$ $\Rightarrow$ $b_n > b_{n+1}$

$\therefore$ $< b_n >$ is monotonically increasing sequence.

Also, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^{1/n} = 1$

$\Rightarrow$ $< b_n >$ is convergent. $\Rightarrow$ $< b_n >$ is bounded.

Since $< b_n >$ is monotonic and bounded, therefore, by Abel's test, the series $\sum_{n=1}^{\infty}$ is convergent.

(*ii*) Let $b_n = \frac{1}{\log n}$, $n \geq 2$, then $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

2.5. REARRANGEMENT OF TERMS

A series $\sum_{n=1}^{\infty} b_n$ is said to arise from a series $\sum_{n=1}^{\infty} a_n$ by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series so that every a_n is some b_m and conversely.

For example, the series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots..$

is a rearrangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots..$

If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum are arranged. But this is not so when infinite series are involved. An *arrangement* (or equally well *derangement*) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

The following theorem gives the condition under which we may rearrange the terms of the series without altering its sum.

2.6. RIEMANN'S THEOREM

By a suitable rearrangement of the terms, a conditionally convergent series $\sum_{n=1}^{\infty} a_n$ *can be made*

(*i*) *to converge to any pre-assigned under* α, *or*

(*ii*) *to diverge to* ∞ *or* $-\infty$, *or*

(*iii*) *to oscillate finitely or infinitely.*

Proof. Let $u_1, u_2, u_3, ...$ be the positive terms and $-v_1, -v_2, -v_3, ...$ be the negative terms of $\sum_{n=1}^{\infty} a_n$.

For $n \in N$, we define $\quad p_n = \frac{1}{2}(a_n + |a_n|) \quad$ and $\quad q_n = \frac{1}{2}(a_n - |a_n|)$

so that $\quad a_n = p_n$ if $a_n > 0 \quad$ and $\quad a_n = q_n$ if $a_n < 0$

$\Rightarrow p_n$ is the nth positive term and q_n is the nth negative term of $\sum_{n=1}^{\infty} a_n$

$\Rightarrow \quad p_n = u_n$ and $q_n = -v_n$.

Since $\sum_{n=1}^{\infty} a_n$ converges conditionally.

$\therefore \quad \sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent.

$\Rightarrow \sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ are both divergent, $\sum_{n=1}^{\infty} p_n$ diverges to ∞ and $\sum_{n=1}^{\infty} q_n$ diverges to $-\infty$.

Also $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim\limits_{n \to \infty} a_n = 0 \Rightarrow \lim\limits_{n \to \infty} |a_n| = 0$ so that $\lim\limits_{n \to \infty} p_n = 0$

and $\quad \lim\limits_{n \to \infty} q_n = 0.$

Let S_n and S_n' denote the nth partial sums of $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$.

(*i*) We shall now construct strictly increasing sequences $< m_n >$ and $< k_n >$ of positive integers such that

$$p_1 + p_2 + ... + p_{m_1} + q_1 + q_2 + ... + q_{k_1} + p_{m_1+1} + p_{m_1+2} + \\ ...... + p_{m_2} + q_{k_1+1} + q_{k_1+2} + ... + q_{k_2} + ... \quad ...(1)$$

is a rearrangement of $\sum_{n=1}^{\infty} a_n$, converging to α.

Since $\sum_{n=1}^{\infty} p_n$ diverges to ∞, it is always possible to find a partial sum of $\sum_{n=1}^{\infty} p_n$ which exceeds any pre-assigned number.

Also $\sum_{n=1}^{\infty} q_n$ diverges to $-\infty$, therefore, it is possible to find a partial sum of $\sum_{n=1}^{\infty} q_n$ which falls short of any pre-assigned number.

Let m_1 be the smallest positive integer such that the sum of first m_1 terms of $\sum_{n=1}^{\infty} p_n$ exceeds α.

Then $\quad p_1 + p_2 + ... + p_{m_1} > \alpha$

but $\quad p_1 + p_2 + ... + p_{m_1 - 1} < \alpha$ so that $S_{m_1 - 1} < \alpha < S_{m_1}$

Let k_1 be the smallest positive integer such that

$q_1 + q_2 + ... + q_{k_1}$ falls short of $\alpha - p_1 - p_2 ...... - p_{m_1} = \alpha - S_{m_1}$

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Then $q_1 + q_2 + \ldots\ldots + q_{k_1} < \alpha - S_{m_1}$ and $q_1 + q_2 + \ldots\ldots + q_{k_1 - 1} > \alpha - S_{m_1}$

$$\Rightarrow \quad S_{m_1} + S'_{k_1} < \alpha < S_{m_1} + S'_{k_1-1}.$$

Let m_2 be the smallest positive integer such that $m_2 > m_1$ and the sum

$$p_1 + p_2 + \ldots\ldots + p_{m_1} + p_{m_1+1} + \ldots\ldots + p_{m_2} + q_1 + q_2 + \ldots\ldots + q_{k_1} \text{ exceeds } \alpha.$$

Then $\quad S_{m_2 - 1} + S'_{k_1} < \alpha < S_{m_2} + S'_{k_1}$

Let k_2 be the smallest positive integer such that $k_2 > k_1$ and the sum

$$q_1 + q_2 + \ldots\ldots + q_{k_1} + q_{k_1+1} + \ldots\ldots + q_{k_2} < \alpha - S_{m_2}$$

then $\quad S'_{k_2} < \alpha - S_{m_2} < S'_{m_{2-1}} \Rightarrow \quad S_{m_2} + S'_{k_2} < \alpha < S_{m_2} + S'_{k_{2-1}}$

and so on. The process can be continued indefinitely because of the divergence of the two series

$$\sum_{n=1}^{\infty} p_n \quad \text{and} \quad \sum_{n=1}^{\infty} q_n.$$

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed and σ_n be its nth partial sum.

The last term in σ_n will be either p_{m_r} or q_{k_s}. If the last term is p_{m_r}, then $\sigma_n - p_{m_r} < \alpha$, *i.e.*, $\sigma_n - \alpha < p_{m_r}$ and if the last term is q_{k_s}, then $\sigma_n - q_{k_s} > \alpha$ *i.e.*, $\sigma_n - \alpha > q_{k_s}$.

Since p_{m_r} is some positive term of $\sum_{n=1}^{\infty} a_n$ and q_{k_s} is some negative term of $\sum_{n=1}^{\infty} a_n$, therefore, if the last term in σ_n is a_l, then $|\sigma_n - \alpha| < |a_l|$

Since $\sum_{n=1}^{\infty} a_n$ is convergent, $a_l \to 0$ as $l \to \infty$

$$\Rightarrow |a_l| \to 0 \text{ as } l \to \infty \quad \Rightarrow \quad \sigma_n - \alpha \to 0 \text{ as } n \to \infty \quad \Rightarrow \quad \lim_{n\to\infty} \sigma_n = \alpha$$

Hence $\sum_{n=1}^{\infty} b_n$ converges to α.

(*ii*) Now, we show that a rearrangement of $\sum_{n=1}^{\infty} a_n$ can be found which diverges to ∞.

Choose a positive integer m_1 such that

$$p_1 + p_2 + \ldots\ldots + p_{m_1} > 1 - q_1 \quad i.e., \quad p_1 + p_2 + \ldots\ldots + p_{m_1} + q_1 > 1$$

Now choose a positive integer $m_2 > m_1$ such that

$$(p_1 + p_2 + \ldots\ldots + p_{m_1}) + (p_{m_{1+1}} + p_{m_{1+2}} + \ldots\ldots + p_{m_2}) > 2 - q_1 - q_2$$

i.e., $\quad (p_1 + p_2 + \ldots\ldots + p_{m_1} + q_1) + (p_{m_{1+1}} + p_{m_{1+2}} + \ldots\ldots + p_{m_2} + q_2) > 2$

Proceeding in this manner, we have

$$(p_1 + p_2 + \ldots\ldots + p_{m_1} + q_1) + (p_{m_{1+1}} + p_{m_{1+2}} + \ldots\ldots + p_{m_2} + q_2)$$
$$+ \ldots\ldots + (p_{m_{1+1}} + p_{m_{1+2}} + \ldots\ldots + p_{m_k} + q_k) > k$$

where k is a positive integer, however large.

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed. Then the sequence of its partial sums is unbounded above and diverges to ∞.

Hence $\sum_{n=1}^{\infty} b_n$ diverges to ∞.

If we want to make the series diverge to $-\infty$, we choose a positive integer m_1 such that

$$q_1 + q_2 + ... + q_{m_1} < -1 - p_1 \quad i.e., \quad q_1 + q_2 + ... + q_{m_1} + p_1 < -1$$

Now we choose a positive integer $m_2 > m_1$ such that

$$(q_1 + q_2 + ... + q_{m_1}) + (q_{m_1+1} + q_{m_1+2} + ... + q_{m_2}) < -2 - p_1 - p_2$$

i.e., $(q_1 + q_2 + ... + q_{m_1} + p_1) + (q_{m_1+1} + q_{m_1+2} + ... + q_{m_2} + p_2) < -2$

Proceeding in this manner, we have

$$(q_1 + q_2 + ... + q_{m_1} + p_1) + (q_{m_1+1} + q_{m_1+2} + ... + q_{m_2} + p_2)$$

$$+ ...... + (q_{m_{k-1}+1} + q_{m_{k-1}+2} + ... + q_{m_k} + p_k) < -k$$

where k is a positive integer, however large.

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed. Then the sequence of its partial sums is unbounded below and diverges to $-\infty$.

Hence $\sum_{n=1}^{\infty} b_n$ diverges to $-\infty$.

(*iii*) Now we show that a rearrangement of $\sum_{n=1}^{\infty} a_n$ can be found which oscillates between two number α and β.

Take just sufficient number of positive terms so that the sum is greater than α and then take just sufficient number of negative terms so that the sum is less than β. Repeat the process indefinitely. The new series so formed will oscillate between $\beta - 1$ and $\alpha + 1$ finitely or infinitely.

Example 4. *Explain the fallacy in the following :*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + ...$$

$$= 2 - 1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} - \frac{1}{6} + ...$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + ...$$

$$= 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + ...\right] \quad ...(1)$$

Also we know that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ *converges and that its sum S is log 2 which is different from zero. Therefore, from (1), we have S = 2S or 1 = 2.*

Sol. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots..$ is conditionally convergent and, therefore, by Riemann's theorem, rearrangement of the terms may alter the sum of the series.

NOTES

Thus $$2-1-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{5}-\frac{1}{6}+\ldots...$$

may not be equal to $2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\ldots...$

Example 5. *Criticise the following paradox.*

$$\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots...$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots...\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots..\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots...\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots...\right)=0.$$

Sol. The given series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots...=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}$ is conditionally convergent and hence can be made to converge to any limit by a rearrangement of terms (Riemann's Theorem). Hence we are not justified in rearranging the terms of a conditionally convergent series and expecting the same sum.

Example 6. *What is wrong with the following ?*

$$2+2+2+2+2+2+\ldots..=(2+2)+(2+2)+(2+2)+\ldots..$$
$$=4+4+4+\ldots..=2(2+2+2+\ldots...)$$

$$\therefore \qquad 2+2+2+\ldots..=0.$$

Sol. The series $2+2+2+\ldots...$ is divergent and tends to infinity, so that

$$2(2+2+2+\ldots...)-(2+2+2+\ldots..)$$

is of indeterminate form $\infty-\infty$.

On account of this fallacy, we get an absurd result.

☞[**Note.** Riemann's method is of theoretical importance only. For practical applications, the method given by Pringsheim is useful.

Pringsheim's Method (*Without Proof*)

Let $f(n)$ be a positive function decreasing to zero as $n\to\infty$. Then by Leibnitz's test, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ is convergent.

Let the terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ be rearranged by taking alternately α positive and β negative terms.

If $g=mf(m)$ and $k=\frac{\alpha}{\beta}$ then the alteration in the sum due to this rearrangement is $\frac{1}{2}g\log k$.

In particular, if $f(n)=\frac{1}{n}$ so that $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots...$ then we know that the series is conditionally convergent and its sum is log 2.

Also, $g = mf(m) = m \cdot \frac{1}{m} = 1.$

$\therefore$ If the terms are rearranged by taking alternately α positive and β negative terms, then the sum of the new series is

$$\log 2 + \tfrac{1}{2} \log k = \tfrac{1}{2} (2 \log 2 + \log k) = \tfrac{1}{2} \log (4k).]$$

NOTES

Example 7. *Find the sum of the series:*

(i) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \ldots$ (ii) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \ldots$

Sol. (i) The given series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \ldots$ is a rearrangement of the terms of conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \text{ whose sum is } \log 2.$$

Here the rearranged given series is formed by taking alternately one positive and two negative terms so that $k = \frac{\alpha}{\beta} = \frac{1}{2}$.

$\therefore$ Sum of the rearranged given series

$$= \log 2 + \tfrac{1}{2} \log k = \log 2 + \tfrac{1}{2} \log \tfrac{1}{2} = \log 2 - \tfrac{1}{2} \log 2 = \tfrac{1}{2} \log 2.$$

(ii) The given series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \ldots$ is a rearrangement of the terms of conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \text{ whose sum is } \log 2.$$

Here the rearranged series is formed by taking alternately two positive and one negative terms so that

$$k = \frac{\alpha}{\beta} = \frac{2}{1} = 2.$$

$\therefore$ Sum of the rearranged given series

$$= \log 2 + \tfrac{1}{2} \log k = \log 2 + \tfrac{1}{2} \log 2 = \frac{3}{2} \log 2.$$

Example 8. *Investigate what derangement of the series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$ *will reduce its sum to zero.*

Sol. The given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum (-1)^{n-1} \cdot \frac{1}{n}$

It is conditionally convergent with sum log 2.

Let it be deranged by taking alternately α positive and β negative terms so that $k = \frac{\alpha}{\beta}$.

Sum of the deranged series $= \log 2 + \frac{1}{2} \log k$

But it is given to be zero.

$$\therefore \quad \log 2 + \tfrac{1}{2} \log k = 0 \Rightarrow \log k = -2 \log 2 = \log 2^{-2} = \log \tfrac{1}{4} \Rightarrow k = \frac{\alpha}{\beta} = \frac{1}{4}$$

Hence to get the sum zero, one positive term should be followed by four negative terms.

The deranged series is $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \ldots$

Example 9. *What derangement of the series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$ *will reduce its sum to* $\frac{1}{2}\log 2$?

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Sol. Proceeding as in Example 5,

$$\log 2 + \tfrac{1}{2}\log k = \tfrac{1}{2}\log 2 \quad \Rightarrow \quad \tfrac{1}{2}\log k = -\tfrac{1}{2}\log 2$$

$$\Rightarrow \quad \log k = \log 2^{-1} = \log \tfrac{1}{2} \quad \Rightarrow \quad k = \tfrac{1}{2}$$

$\therefore$ The deranged series is $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \ldots\ldots$

Example 10. *Assuming that* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, *find the value of* $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Sol. The series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

of positive terms is convergent and hence we can derange its terms in any order.

Now
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\ldots\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \times \frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

2.7. CAUCHY PRODUCT OF TWO INFINITE SERIES

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series, then their product, called the Cauchy product, is defined as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1$

$= \sum_{n=1}^{\infty} a_r\, b_{n-r+1}$ for each $n \in \mathbf{N}$.

Thus
$$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = (a_1 + a_2 + \ldots\ldots)(b_1 + b_2 + \ldots\ldots)$$

$$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \ldots\ldots = c_1 + c_2 + c_3 + \ldots\ldots$$

The terms in the product are so arranged that all the terms which have the same sum of suffixes are bracketed together.

Remarks 1. The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined as $\sum_{n=0}^{\infty} c_n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_n b_0 = \sum_{r=0}^{\infty} a_r b_{n-r} \text{ for each } n \in \mathbb{N}.$$

2. $$c_n = \sum_{r=1}^{\infty} a_r\, b_{n-r+1} = \sum_{r=1}^{\infty} a_{n-r+1}\, b_r \quad \text{and} \quad c_n = \sum_{r=0}^{\infty} a_r\, b_{n-r} = \sum_{r=0}^{\infty} a_{n-r} b_r$$

3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ convergent, then it is not necessary that

$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ must converge. $\sum_{n=1}^{\infty} c_n$ converges if

(*i*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of non-negative terms, or

(*ii*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, or

(*iii*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and one of them is absolutely convergent.

Now we prove these assertions.

2.8. THEOREM

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of non-negative terms converging to A and B respectively, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) \text{ respectively.}$$

Since $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B.

$\therefore$ $$\lim_{n \to \infty} A_n = A \quad \text{and} \quad \lim_{n \to \infty} B_n = B$$

Now
$$\begin{aligned} C_n = {} & a_1 b_1 \\ & + a_1 b_2 + a_2 b_1 \\ & + a_1 b_3 + a_2 b_2 + a_3 b_1 \\ & \vdots \\ & \vdots \\ & + a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1 \end{aligned}$$

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$$= a_1(b_1 + b_2 + ... + b_n)$$
$$+ a_2(b_1 + b_2 + ... + b_{n-1})$$
$$+ b_3(b_1 + b_2 + ... + b_{n-2})$$
$$\vdots$$
$$\vdots$$
$$+ a_n b_1$$
$$= a_1 B_n + a_2 B_{n-1} + a_3 B_{n-2} + ... + a_n B_1$$
$$A_n B_n = (a_1 + a_2 + a_3 + ... + a_n) B_n$$
$$= a_1 B_n + a_2 B_n + a_3 B_n + ... + a_n B_n$$

and
$$C_{2n} = a_1 b_1$$
$$+ a_1 b_2 + a_2 b_1$$
$$+ a_1 b_3 + a_2 b_2 + a_3 b_1$$
$$\vdots$$
$$\vdots$$
$$+ a_1 b_{2n} + a_2 b_{2n-1} + a_3 b_{2n-2} + ... + a_{2n} b_1$$
$$= a_1(b_1 + b_2 + ... + b_{2n})$$
$$+ a_2(b_1 + b_2 + ... + b_{2n-1})$$
$$+ a_3(b_1 + b_2 + ... + b_{2n-2})$$
$$\vdots$$
$$\vdots$$
$$+ a_{2n} b_1$$
$$= a_1 B_{2n} + a_2 B_{2n-1} + a_3 B_{2n-2} + ... a_{2n} B_1$$

Since $b_n \geq 0 \ \forall \ n \in N$, therefore, $i > j \Rightarrow B_i \geq B_j$

Clearly $$0 \leq C_n \leq A_n B_n \leq C_{2n} \quad ...(1)$$

$\Rightarrow < C_n >$ is a bounded sequence of non-negative numbers.

Also $< C_n >$ is monotonically increasing.

$\therefore < C_n >$ is convergent. Let it converge to C.

Then $\lim\limits_{n \to \infty} C_n = C$ and the series $\sum\limits_{n=1}^{\infty} C_n$ also converges to C.

Now, from (1), we have

$$\lim_{n \to \infty} C_n \leq \lim_{n \to \infty} (A_n B_n) \leq \lim_{n \to \infty} C_{2n}$$

$$\Rightarrow \quad C \leq (\lim_{n \to \infty} A_n)(\lim_{n \to \infty} B_n) \leq C$$

$$\Rightarrow \quad C \leq AB \leq C \quad \Rightarrow \quad C = AB$$

Hence the Cauchy product of $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converges to AB.

Remark. Note that in Theorem 1, $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are convergent series of **non-negative terms.**

NOTES

2.9. CAUCHY'S THEOREM

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then their Cauchy product $\sum_{n=1}^{\infty} c_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} c_n = AB$.

Proof. Let A_n, B_n be the nth partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively.

Since $\qquad \sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$

$\therefore \qquad \lim_{n \to \infty} A_n = A$ and $\lim_{n \to \infty} B_n = B$

Let A_n', B_n' denote the nth partial sums of the series $\sum_{n=1}^{\infty} | a_n |$ and $\sum_{n=1}^{\infty} | b_n |$ respectively.

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, therefore, $\sum_{n=1}^{\infty} | a_n |$ and $\sum_{n=1}^{\infty} | b_n |$ are convergent. Suppose they converge to A′ and B′ respectively.

Then, $\qquad \lim_{n \to \infty} A_n' = A'$ and $\lim_{n \to \infty} B_n' = B'$.

Let the Cauchy product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be $\sum_{n=1}^{\infty} c_n$, then $c_n = a_1 b_n + a_2 b_{n-1} + \ldots.. + a_n b_1$

Let the Cauchy product of $\sum_{n=1}^{\infty} | a_n |$ and $\sum_{n=1}^{\infty} | b_n |$ be $\sum_{n=1}^{\infty} d_n$, then

$$d_n = | a_1 || b_n | + | a_2 || b_{n-1} | + \ldots.. + | a_n || b_1 |.$$

Since $\sum_{n=1}^{\infty} | a_n |$ and $\sum_{n=1}^{\infty} | b_n |$ are series of **non-negative terms,** converging to A′ and B′ respectively, therefore, their Cauchy product $\sum_{n=1}^{\infty} d_n$ converges to A′B′.

Now $\qquad | c_n | = | a_1 b_n + a_2 b_{n-1} + \ldots.. + a_n b_1 |$

$$\leq | a_1 || b_n | + | a_2 || b_{n-1} | + \ldots.. + | a_n || b_1 | = d_n \; \forall \; n$$

and $\quad \sum_{n=1}^{\infty} d_n$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} | c_n |$ is convergent (by comparison test) $\Rightarrow \sum_{n=1}^{\infty} c_n$ is absolutely convergent.

Now we shall prove that $\sum_{n=1}^{\infty} c_n = AB$.

NOTES

Let C_n and D_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ respectively.

$$\begin{aligned}
C_n - A_nB_n = (&a_1b_1 \\
&+ a_1b_2 + a_2b_1 \\
&+ a_1b_3 + a_2b_2 + a_3b_1 \\
&\vdots \\
&\vdots \\
&+ a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \ldots + a_nb_1) \\
&- (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) \\
= &\, a_1(b_1 + b_2 + \ldots + b_n) \\
&+ a_2(b_1 + b_2 + \ldots + b_{n-1}) \\
&+ a_3(b_1 + b_2 + \ldots + b_{n-2}) \\
&\vdots \\
&\vdots \\
&+ a_nb_1 \\
&- a_1(b_1 + b_2 + \ldots + b_n) \\
&- a_2(b_1 + b_2 + \ldots + b_n) \\
&- a_3(b_1 + b_2 + \ldots + b_n) \\
&\vdots \\
&\vdots \\
&- a_n(b_1 + b_2 + \ldots + b_n) \\
= &- a_2b_n - a_3(b_{n-1} + b_n) - \ldots \\
&- a_n(b_2 + b_3 + \ldots + b_n)
\end{aligned}$$

$$\Rightarrow \quad |C_n - A_nB_n| \le |a_2|\,|b_n| + |a_3|\,(|b_{n-1}| + |b_n|) + \ldots + |a_n|\,(|b_2| + |b_3| + \ldots + |b_n|)$$

$$= A_n'B_n' - D_n \qquad \ldots(1)$$

Now $\lim_{n\to\infty} A_n' = A'$, $\lim_{n\to\infty} B_n' = B'$ and $\lim_{n\to\infty} D_n = A'B'$

so that $A_n'B_n' - D_n \to 0$ as $n \to \infty$.

$\therefore$ From (1), $|C_n - A_nB_n| \to 0$ as $n \to \infty$

Also $|A_nB_n - AB| \to 0$ as $n \to \infty$

$\therefore |C_n - AB| = |(C_n - A_nB_n) + (A_nB_n - AB)| \le |C_n - A_nB_n| + |A_nB_n - AB|$

$\Rightarrow |C_n - AB| \to 0$ as $n \to \infty$ $\Rightarrow$ $C_n - AB \to 0$ as $n \to \infty$ $\Rightarrow$ $\lim_{n\to\infty} C_n = AB$

Hence $\sum_{n=1}^{\infty} c_n = AB$.

Remark. Note that in Cauchy's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both absolutely convergent.

2.10. MERTEN'S THEOREM

NOTES

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $\sum_{n=1}^{\infty} a_n$ converge absolutely. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ respectively.

Then $\lim_{n \to \infty} A_n = A$ and $\lim_{n \to \infty} B_n = B$

Let $\beta_n = B_n - B \; \forall \; n$ so that $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} B_n - B = B - B = 0$

Now,

$$\begin{aligned} C_n &= a_1 b_1 \\ &\quad + a_1 b_2 + a_2 b_1 \\ &\quad + a_1 b_3 + a_2 b_2 + a_3 b_1 \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1 \\ &= a_1(b_1 + b_2 + \ldots\ldots + b_n) \\ &\quad + a_2(b_1 + b_2 + \ldots\ldots + b_{n-1}) \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + a_n \, b_1 \\ &= a_1 B_n + a_2 B_{n-1} + \ldots.. + a_n B_1 \\ &= a_1(B + \beta_n) + a_2(B + \beta_{n-1}) + \ldots.. + a_n(B + \beta_1) \quad [\because \; B_n = B + \beta_n \; \forall \; n] \\ &= B(a_1 + a_2 + \ldots\ldots + a_n) + (a_1\beta_n + a_2\beta_{n-1} + \ldots.. + a_n\beta_1) \\ &= BA_n + \gamma_n \text{ where } \gamma_n = a_1\beta_n + a_2\beta_{n-1} + \ldots. + a_n\beta_1 \end{aligned}$$

$\therefore$ To prove that $\lim_{n \to \infty} C_n = AB$, it is sufficient to prove that $\lim_{n \to \infty} \gamma_n = 0$.

Now $\sum_{n=1}^{\infty} a_n$ converges absolutely $\Rightarrow \sum_{n=1}^{\infty} | a_n |$ converges.

Let $\sum_{n=1}^{\infty} | a_n | = \alpha$.

Since $\lim_{n \to \infty} \beta_n = 0$, $< \beta_n >$ converges.

NOTES

$\Rightarrow\ <\beta_n>$ is bounded. $\Rightarrow$ There exists a real number $K > 0$ such that $|\beta_n| < K\ \forall\ n$ ...(1)

Also $\lim_{n\to\infty} \beta_n = 0 \Rightarrow$ given $\varepsilon > 0$, there exists a positive integer p such that

$$|\beta_n| < \frac{\varepsilon}{2\alpha+1}\ \forall\ n > p.$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy's general principle of convergence, there exists a positive integer q such that

$$||a_{q+1}| + |a_{q+2}| + ... + |a_n|| < \frac{\varepsilon}{2K+1}\ \forall\ n > q$$

or $$|a_{q+1}| + |a_{q+2}| + ... + |a_n| < \frac{\varepsilon}{2K+1}\ \forall\ n > q$$

If $m = \max\{p, q\}$, then for $n > m$, we have $|\beta_n| < \frac{\varepsilon}{2\alpha+1}$ ...(2)

and $$|a_{m+1}| + |a_{m+2}| + ... + |a_n| < \frac{\varepsilon}{2K+1} \quad ...(3)$$

$\therefore$ For $n > 2m$, we have $n - m > m$ and

$$\begin{aligned}|\gamma_n| &= |a_1\beta_n + a_2\beta_{n-1} + ... + a_n\beta_1| \\ &= |\beta_1 a_n + \beta_n a_{n-1} + ... + \beta_{m+1}\, a_{n-m} + \beta_{m+2}\, a_{n-m-1} + ... + \beta_n a_1| \\ &\le (|\beta_1||a_n| + |\beta_2||a_{n-1}| + ... + |\beta_{m+1}||a_{n-m}|) \\ &\qquad + (|\beta_{m+2}||a_{n-m-1}| + ... + |\beta_n||a_1|) \\ &< K(|a_{n-m}| + ... + |a_{n-1}| + |a_n|) \\ &\qquad + \frac{\varepsilon}{2\alpha+1}(|a_1| + |a_2| + ... + |a_{n-m+1}|)\end{aligned}$$

[Using (1) and (2)]

$$< K\,.\frac{\varepsilon}{2K+1} + \frac{\varepsilon}{2\alpha+1}\,.\,\alpha \qquad \text{[Using (3)]}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \left[\begin{aligned}&\because \sum_{n=1}^{\infty} |a_n| = \alpha \\ &\therefore \sum_{n=1}^{n-m+1} |a_n| < \alpha\end{aligned}\right.$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists a positive integer $2m$ such that

$$|\gamma_n| < \varepsilon\ \forall\ n > 2m \Rightarrow \lim_{n\to\infty} \gamma_n = 0$$

Now $$C_n = BA_n + \gamma_n$$

$$\therefore \quad \lim_{n\to\infty} C_n = \lim_{n\to\infty} (BA_n + \gamma_n) = B\lim_{n\to\infty} A_n + \lim_{n\to\infty} \gamma_n = BA + 0 = AB.$$

Hence $\sum_{n=1}^{\infty} C_n$ converges to AB.

Remark. Note the in Merten's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and one of them converges absolutely.

2.11. CESARO'S THEOREM

If two sequences $<a_n>$ and $<b_n>$ converge to a and b respectively, then the sequence $<x_n>$ where $x_n = \dfrac{a_1b_n + a_2b_{n-1} + ... + a_nb_1}{n}$ converges to ab.

Proof. Let $a_n = a + \alpha_n \ \forall \ n \in \mathbb{N}$

Then $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} a_n - a = a - a = 0 \ \Rightarrow \ \lim\limits_{n\to\infty} | \alpha_n | = 0$

Now $x_n = \dfrac{a_1b_n + a_2b_{n-1} + ... + a_nb_1}{n} = \dfrac{(a+\alpha_1)b_n + (a+\alpha_2)b_{n-1} + ... + (a+\alpha_n)b_1}{n}$

$$= \frac{a(b_n + b_{n-1} + ... + b_1) + (\alpha_1 b_n + \alpha_2 b_{n-1} + ... + \alpha_n b_1)}{n}$$

$$= a\left(\frac{b_1 + b_2 + ... + b_n}{n}\right) + \left(\frac{b_n\alpha_1 + b_{n-1}\alpha_2 + ... + b_1\alpha_n}{n}\right) \quad ...(1)$$

By Cauchy's first theorem on limits

$$\lim_{n\to\infty} b_n = b \quad \Rightarrow \quad \lim_{n\to\infty} \frac{b_1 + b_2 + ... + b_n}{n} = b \quad ...(2)$$

and $\lim\limits_{n\to\infty} |\alpha_n| = 0 \quad \Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{n} = 0$

Since $<b_n>$ converges, $<b_n>$ is bounded.

$\therefore$ $\exists$ a real number $K > 0$ such that $| b_n | < K \ \forall \ n \in \mathbb{N}$ ...(3)

Now, $0 \le \left| \dfrac{b_n\alpha_1 + b_{n-1}\alpha_2 + ... + b_1\alpha_n}{n} \right|$

$\le \dfrac{|b_n||\alpha_1| + |b_{n-1}||\alpha_2| + ... + |b_1||\alpha_n|}{n}$ [Using (3)]

$\le K\left(\dfrac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{n}\right) \to 0$ as $n \to \infty$ [Using (2)]

$\therefore$ By Squeeze principle, $\lim\limits_{n\to\infty} \dfrac{b_n\alpha_1 + b_{n-1}\alpha_2 + ... + b_1\alpha_n}{n} = 0$

From (1), $\lim\limits_{n\to\infty} x_n = a \lim\limits_{n\to\infty} \left(\dfrac{b_1 + b_2 + ... + b_n}{n}\right) + \lim\limits_{n\to\infty} \dfrac{b_n\alpha_1 + b_{n-1}\alpha_2 + ... + b_1\alpha_n}{n}$

$= a(b) + 0 = ab$

Hence the sequence $<x_n>$ converges to ab.

2.12. ABEL'S TEST

Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be two convergent series such that $\sum\limits_{n=1}^{\infty} a_n = A$ and $\sum\limits_{n=1}^{\infty} b_n = B$. If their Cauchy product $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\sum\limits_{n=1}^{\infty} c_n = AB$.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ and $\sum\limits_{n=1}^{\infty} c_n$ respectively.

NOTES

Now
$$C_n = a_1 b_1$$
$$+ a_1 b_2 + a_2 b_1$$
$$+ a_1 b_3 + a_2 b_2 + a_3 b_1$$
$$\vdots$$
$$\vdots$$
$$+ a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots + a_n b_1$$
$$= a_1 (a_1 + b_2 + \ldots + b_n)$$
$$+ a_2 (b_1 + b_2 + \ldots + b_{n-1})$$
$$+ a_3 (b_1 + b_2 + \ldots + b_{n-2})$$
$$\vdots$$
$$\vdots$$
$$+ a_n b_1$$
$$= a_1 B_n + a_2 B_{n-1} + \ldots + a_n B_1$$

$$\Rightarrow \quad C_n = a_1 B_n + a_2 B_{n-1} + a_3 B_{n-2} + \ldots + a_n B_1$$

Changing n to $n-1, n-2, \ldots, 1$ successively, we get

$$C_{n-1} = a_1 B_{n-1} + a_2 B_{n-2} + \ldots + a_{n-1} B_1$$
$$C_{n-2} = a_1 B_{n-2} + a_2 B_{n-3} + \ldots + a_{n-2} B_1$$
$$\vdots$$
$$\vdots$$
$$C_1 = a_1 B_1$$

Adding, we have

$$C_1 + C_2 + \ldots + C_n = a_1 B_n + (a_1 + a_2) B_{n-1} + (a_1 + a_2 + a_3) B_{n-2} + \ldots + (a_1 + a_2 + \ldots + a_n) B_1$$
$$= A_1 B_n + A_2 B_{n-1} + A_3 B_{n-2} + \ldots + A_n B_1 \qquad (\because \; a_1 = A_1)$$

$$\Rightarrow \quad \frac{C_1 + C_2 + \ldots + C_n}{n} = \frac{A_1 B_n + A_2 B_{n-1} + \ldots + A_n B_1}{n} \qquad \ldots(1)$$

Suppose $\sum_{n=1}^{\infty} c_n$ converges and $\sum_{n=1}^{\infty} c_n = C$, then $\lim_{n \to \infty} C_n = C$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{C_1 + C_2 + \ldots + C_n}{n} = C \quad \text{(By Cauchy's first theorem on limits)}$$

Also $< A_n >$ converges to A and $< B_n >$ converges to B

$$\Rightarrow \lim_{n \to \infty} \frac{A_1 B_n + A_2 B_{n-1} + \ldots + A_n B_1}{n} = AB \qquad \text{(By Cesaro's theorem)}$$

$\therefore$ From (1), C = AB.

Remark. Note that in Abel's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B respectively. Abel's theorem does not confirm (under the conditions) the convergence of $\sum_{n=1}^{\infty} c_n$. However, if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = AB$.

NOTES

2.13. TAUBER'S THEOREM

Let $f(x) = \Sigma c_n x^n$ $(-1 < x < 1)$ and suppose that $\lim_{n \to \infty} nc_n = 0$. If $f(x) \to s$ as $x \to 1$, then Σc_n converges and has sum s.

Proof. Let $s_n = c_1 + c_2 + c_3 + ... + c_n$ and $\{\sigma_n\}$ is the sequence of arithmetic means defined by $\sigma_n = (s_1 + s_2 + s_3 + ... + s_n)/n$.

or
$$n\sigma_n = (s_1 + s_2 + s_3 + ... + s_n) + c_1 + c_1 + c_2 + c_1 + c_2 + c_3 + ... + c_1 + c_2 + c_3 + ... + c_n)$$

$$= c_1 + 2c_2 + 3c_3 + ... + nc_n = \sum_{k=1}^{n} kc_k \text{ and since } nc_n \to 0 \text{ as } n \to \infty.$$

Then $\sigma_n \to 0$ as $n \to \infty$. Consider $x_n = 1 - 1/n$ then as $n \to \infty$, $x_n \to 1$, therefore $\lim_{n \to \infty} f(x_n) = s$. Hence for given $\varepsilon > 0$ there exists positive integer N such that

$$|f(x_n) - s| < \frac{\varepsilon}{3}, |\sigma_n - 0| < \frac{\varepsilon}{3} \quad \text{and} \quad |nc_n - 0| < \frac{\varepsilon}{3}.$$

Now for $-1 < x < 1$, we can write

$$s_n - s = \sum_{k=0}^{n} c_k x^k - s = \sum_{k=0}^{n} c_k x^k - s + \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= f(x) - s + \sum_{k=0}^{n} c_k (1 - x^k) - \sum_{k=n+1}^{\infty} c_k x^k.$$

Now for $x \in (0, 1)$, we have

$$(1 - x^k) = (1 - x)(1 + x + x^2 + ... + x^{k-1}) \le k(1 - x) \text{ for each } k.$$

Hence for $n \ge N$ and $x \in (0, 1)$, we have

$$|s_n - s| \le |f(x) - s| + (1 - x) \sum_{k=0}^{n} k|c_k| + \left\{\frac{\varepsilon}{n}(1 - x)\right\}.$$

Taking $x = x_0 = 1 - \frac{1}{n}$, we have $|s_n - s| < \left(\frac{\varepsilon}{3}\right) + \left(\frac{\varepsilon}{3}\right) + \left(\frac{\varepsilon}{3}\right) = \varepsilon$. Hence the series Σc_n converges and has sum s.

Example 11. *Show that the Cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ with itself is not convergent.*

Sol. Let $\quad a_n = b_n = \frac{(-1)^{n-1}}{n}, n \in \text{N}.$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

NOTES

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$$

$$= \frac{(-1)^0}{1} \cdot \frac{(-1)^{n-1}}{n} + \frac{(-1)^1}{2} \cdot \frac{(-1)^{n-2}}{n-1} + \dots\dots + \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^0}{1}$$

$$= (-1)^{n-1}\left[\frac{1}{1.n} + \frac{1}{2(n-1)} + \dots\dots + \frac{1}{n.1}\right]$$

$$\geq (-1)^{n-1}\left[\frac{1}{n.n} + \frac{1}{n.n} + \dots\dots + \frac{1}{n.n}\right] \qquad \left[\because\ r \leq n \ \Rightarrow\ \frac{1}{r} \geq \frac{1}{n}\right]$$

$$= (-1)^{n-1} \cdot \frac{n}{n^2} = \frac{(-1)^{n-1}}{n} \quad \Rightarrow \quad |\, c_n \,| \geq \frac{1}{n}\ \forall\, n \in \mathrm{N}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} |\, c_n \,|$ is divergent. $\Rightarrow \lim_{n \to 1} c_n \neq 0$.

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Note 1. The condition $\lim_{n \to \infty} a_n = 0$ is absolutely essential for convergence of any series $\sum_{n=1}^{\infty} a_n$.

Note 2. The above example illustrates that the Cauchy product of two conditionally convergent series need not be necessarily convergent.

Example 12. *Show that the Cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself is not convergent.*

Sol. Let $$a_n = b_n = \frac{(-1)^{n-1}}{\sqrt{n}},\ n \in \mathrm{N}.$$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$$

$$= \frac{(-1)^0}{\sqrt{1}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^{n-2}}{\sqrt{n-1}} + \dots\dots + \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{(-1)^0}{1}$$

$$= (-1)^{n-1} \cdot \left[\frac{1}{\sqrt{1.n}} + \frac{1}{\sqrt{2(n-1)}} + \dots\dots + \frac{1}{\sqrt{n.1}}\right]$$

$$\geq (-1)^{n-1} \cdot \left[\frac{1}{\sqrt{n.n}} + \frac{1}{\sqrt{n.n}} + \dots\dots + \frac{1}{\sqrt{n.n}}\right]$$

$$= (-1)^{n-1} . \left[\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}\right] = (-1)^{n-1} . \frac{n}{n} = (-1)^{n-1}$$

$$\Rightarrow \quad | c_n | \geq 1 \ \forall \ n \in N \ \Rightarrow \ \lim_{n \to \infty} c_n \neq 0$$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Example 13. *Show that the Cauchy product of the convergent series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ *with itself is not convergent.*

Sol. Let $\quad a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, n \in N$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + ... + a_n b_1$$

$$= \frac{(-1)^1}{\sqrt{2}} . \frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}} . \frac{(-1)^{n-1}}{\sqrt{n}} + ... + \frac{(-1)^n}{\sqrt{n+1}} . \frac{(-1)^1}{\sqrt{2}}$$

$$= (-1)^{n-1} . \left[\frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{3n}} + ... + \frac{1}{\sqrt{(n+1) . 2}}\right]$$

$$\geq (-1)^{n-1} . \left[\frac{1}{\sqrt{(n+1)(n+1)}} + \frac{1}{\sqrt{(n+1)(n+1)}} + ... + \frac{1}{\sqrt{(n+1)(n+1)}}\right]$$

$$= (-1)^{n-1} \frac{n}{n+1} \Rightarrow | c_n | \geq \frac{n}{n+1} \ \forall \ n \in N$$

Since $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent. $\quad \left(\because \lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0\right)$

$\Sigma \mid c_n \mid$ is divergent. $\Rightarrow \lim_{n \to \infty} c_n \neq 0$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Example 14. *Show that the Cauchy product of the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *with itself converges to* $\frac{\pi^4}{36}$.

Sol. Let $\quad a_n = b_n = \frac{1}{n^2}, n \in N$ then $A_n = \sum_{n=1}^{\infty} a_r = \frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} = \frac{\pi^2}{6}$

and $\quad B_n = A_n = \frac{\pi^2}{6} \ \Rightarrow \ \lim_{n \to \infty} A_n = \frac{\pi^2}{6} = \lim_{n \to \infty} B_n$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = \frac{\pi^2}{6}$$

NOTES

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive terms, their Cauchy product $\sum_{n=1}^{\infty} c_n$ is also convergent and $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \frac{\pi^2}{6} \times \frac{\pi^2}{6} = \frac{\pi^4}{36}$

Hence the Cauchy product of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with itself converges to $\frac{\pi^4}{36}$.

Example 15. *Show that:*

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)^2 = 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\ldots\right].$$

Sol. Proceeding as in example 20, we have $\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)^2$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1.n}+\frac{1}{2(n-1)}+\ldots\ldots+\frac{1}{n.1}\right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}.\frac{2}{n+1}\left[1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right]$$

$$= 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\ldots\ldots\right].$$

Example 16. *Show that:*

$$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\ldots\ldots$$

Sol. Let $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \sum_{n=1}^{\infty} a_n,$

then $\sum_{n=1}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the Cauchy product $\sum_{n=1}^{\infty} c_n$ of $\sum_{n=1}^{\infty} a_n$ with itself converges, then

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} c_n \qquad \ldots(1)$$

Now $c_n = 1\,.\,\frac{(-1)^{n-1}}{2n-1} - \frac{1}{3}.\frac{(-1)^{n-2}}{2n-3} + \ldots\ldots + \frac{(-1)^{n-2}}{2n-3}\left(-\frac{1}{3}\right) + \frac{(-1)^{n-1}}{2n-1}\,.\,1$

$$= (-1)^{n-1}\left[\frac{1}{1(2n-1)}+\frac{1}{3(2n-3)}+\ldots\ldots+\frac{1}{(2n-3).3}+\frac{1}{(2n-1).1}\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[\frac{(2n-1)+1}{1.(2n-1)}+\frac{(2n-3)+3}{3(2n-3)}+\ldots\ldots+\frac{(2n-3)+3}{(2n-3).3}+\frac{(2n-1)+1}{(2n-1).1}\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[\left(1+\frac{1}{2n-1}\right)+\left(\frac{1}{3}+\frac{1}{2n-3}\right)+ \ldots\ldots + \left(\frac{1}{2n-3}+\frac{1}{3}\right)+\left(\frac{1}{2n-1}+1\right)\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[2+\frac{2}{3}+\ldots\ldots+\frac{2}{2n-3}+\frac{2}{2n-1}\right]$$

$$= \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right] \qquad \ldots(2)$$

$$\therefore \; |c_n| = \frac{1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}}{n} \to 0 \text{ as } n \to \infty \text{ (by Cauchy's first theorem on limits)}$$

Also $|c_{n+1}| - |c_n|$

$$= \frac{1}{n+1}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n+1}\right)-\frac{1}{n}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)$$

$$= \left(\frac{1}{n+1}-\frac{1}{n}\right)\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)+\frac{1}{(n+1)(2n+1)}$$

$$= \frac{-1}{n(n+1)}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)+\frac{1}{(n+1)(2n+1)}$$

$$< \frac{-1}{n(n+1)}+\frac{1}{(n+1)(2n+1)} \qquad \left[\because\; 1+\frac{1}{3}+\ldots..+\frac{1}{2n-1}>1\right]$$

$$= \frac{-(2n+1)+n}{n(n+1)(2n+1)} = \frac{-(n+1)}{n(n+1)(2n+1)} = \frac{-1}{n(2n+1)} < 0$$

$\Rightarrow \quad |c_n| > |c_{n+1}|$

By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\ldots..+\frac{1}{2n-1}\right] \text{ [by (2)] converges.}$$

Hence, from (1), we have

$$\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right]$$

$$= \frac{1}{1}(1)-\frac{1}{2}\left(1+\frac{1}{3}\right)+\frac{1}{3}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\ldots\ldots$$

$$\Rightarrow \; \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots\right).$$

Example 17. *Show that* $\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots\ldots\right)^2$

$$= \sum_{n=2}^{\infty} (-1)^n \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right)\frac{x^n}{n}$$

when (i) $|x| < 1$ *and* (ii) $x = 1$.

Sol. (*i*) Let $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - ... = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} x^{n-1} = \sum_{n=2}^{\infty} a_n$

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then
$$|a_n| = \frac{|x|^{n+1}}{n+1}$$
$$|a_{n+1}| = \frac{|x|^{n+2}}{n+2}$$

$\therefore$
$$\frac{|a_n|}{|a_{n+1}|} = \frac{n+2}{n+1} \cdot \frac{1}{|x|} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \frac{1}{|x|} \to \frac{1}{|x|} \text{ as } n \to \infty$$

Since, $|x| < 1$, $\frac{1}{|x|} > 1$

$\therefore$ By ratio test, the series $\sum_{n=2}^{\infty} |a_n|$ converges.

$\Rightarrow$ the series $\sum_{n=2}^{\infty} a_n$ converges absolutely.

By Cauchy's theorem, the Cauchy product $\sum_{n=2}^{\infty} c_n$ of $\sum_{n=2}^{\infty} a_n$ with itself also converges absolutely and

$$\left(\sum_{n=2}^{\infty} a_n\right)^2 = \sum_{n=2}^{\infty} c_n \quad ...(1)$$

Now $c_n = x \cdot \frac{(-1)^n}{n-1} x^{n-1} - \frac{1}{2}x^2 \cdot \frac{(-1)^{n-1}}{n-2} x^{n-2} + ...$

$$+ \frac{(-1)^{n-1}}{n-2} x^{n-2}\left(-\frac{1}{2}x^2\right) + \frac{(-1)^n}{n-1} x^{n-1} \cdot x \text{ for } n \geq 2$$

$$= \frac{(-1)^n}{n} x^n \left[\frac{(n-1)+1}{1.(n-1)} + \frac{(n-2)+2}{2(n-2)} + ... + \frac{(n-2)+2}{(n-1).2} + \frac{(n-1)+1}{(n-1).1}\right]$$

$$= (-1)^n \cdot \frac{x^n}{n}\left[\left(1+\frac{1}{n-1}\right) + \left(\frac{1}{2}+\frac{1}{n-2}\right) + ... + \left(\frac{1}{n-2}+\frac{1}{2}\right) + \left(\frac{1}{n-1}+1\right)\right]$$

$$= (-1)^n \cdot \frac{x^n}{n}\left[2 + \frac{2}{2} + ... + \frac{2}{n-2} + \frac{2}{n-1}\right] = 2(-1)^n \cdot \frac{x^n}{n}\left(1+\frac{1}{2}+......+\frac{1}{n-1}\right)$$

$\therefore$ From (1), we have

$$\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - ......\right)^2 = \sum_{n=2}^{\infty} 2(-1)^n \cdot \frac{x^n}{n}\left(1+\frac{1}{2}+......+\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - ......\right)^2 = \sum_{n=2}^{\infty} (-1)^n \left(1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{n-1}\right) \cdot \frac{x^n}{n},$$

(*ii*) Here $x = 1$

$\therefore$ We have to show that

$$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots\right)^2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right)$$

Let $1 - \frac{1}{2}+\frac{1}{3} - \ldots\ldots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} = \sum_{n=2}^{\infty} a_n$ then $\sum_{n=2}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the Cauchy product $\sum_{n=2}^{\infty} c_n$ of $\sum_{n=2}^{\infty} a_n$ with itself converges, then

$$\left(\sum_{n=2}^{\infty} a_n\right)^2 = \sum_{n=2}^{\infty} c_n \qquad \ldots(2)$$

Now $c_n = 1 \cdot \frac{(-1)^n}{n-1} - \frac{1}{2} \cdot \frac{(-1)^{n-1}}{n-2} + \ldots\ldots + \frac{(-1)^{n-1}}{n-2} \cdot \left(-\frac{1}{2}\right) + \frac{(-1)^n}{n-1} \cdot 1$

$$= \frac{(-1)^n}{n}\left[\frac{(n-1)+1}{1.(n-1)} + \frac{(n-2)+2}{2(n-2)} + \ldots\ldots + \frac{(n-2)+2}{(n-2).2} + \frac{(n-1)+1}{(n-1).1}\right]$$

$$= \frac{(-1)^n}{n}\left[\left(1+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\ldots\ldots+\left(\frac{1}{n-2}+\frac{1}{2}\right)+\left(\frac{1}{n-1}+1\right)\right]$$

$$= \frac{(-1)^n}{n}\left(2+\frac{2}{2}+\ldots\ldots+\frac{2}{n-2}+\frac{2}{n-1}\right] = (-1)^n \cdot \frac{2}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right) \qquad \ldots(3)$$

$$\therefore \; |c_n| = \frac{2}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right) = \frac{2(n-1)}{n} \cdot \left(\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}}{n-1}\right)$$

$$= 2\left(1-\frac{1}{n}\right)\left(\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}}{n-1}\right) \to 0 \text{ as } n \to \infty$$

(By Cauchy's first theorem on limits)

Also $|c_{n+1}| - |c_n| = \frac{2}{n+1}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}+\frac{1}{n}\right) - \frac{2}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right)$

$$= \left(\frac{2}{n+1}-\frac{2}{n}\right)\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right)+\frac{2}{n(n+1)}$$

$$= \frac{-2}{n(n+1)}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}-1\right)$$

$$= \frac{-2}{n(n+1)}\left(\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right) < 0$$

$\Rightarrow \qquad |c_n| > |c_{n+1}|$

NOTES

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=2}^{\infty} c_n = \sum_{n=2}^{\infty} (-1)^n \cdot \frac{2}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right) \text{ [by (3)] converges.}$$

Hence, from (2), we have

$$\left(1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots\right)^2 = \sum_{n=2}^{\infty} (-1)^n \cdot \frac{2}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots\right)^2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right).$$

Example 18. *Prove that:*

$$\left(1+x+\frac{x^2}{2!}+\ldots\ldots\right)\left(1+y+\frac{y^2}{2!}+\ldots\ldots\right) = 1+\frac{x+y}{1!}+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}+\ldots\ldots$$

for all values of x and y.

Sol. We know that $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ for all $x \in \mathbb{R}$

and $e^y = 1 + y + \frac{y^2}{2!} + \ldots$ for all $y \in \mathbb{R}$

$\therefore$ The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots\ldots$

converges to e^x, and the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2!} + \ldots$ converges to e^y.

Let $\sum_{n=0}^{\infty} c_n$ denote the Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, then

$c_0 = a_0 b_0 = 1 \times 1 = 1$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_n b_0$$

$$= 1.\frac{y^n}{n!} + x \cdot \frac{y^{n-1}}{(n-1)!} + \frac{x^2}{2!} \cdot \frac{y^{n-2}}{(n-2)!} + \ldots\ldots + \frac{x^n}{n!} \cdot 1$$

$$= \frac{1}{n!}\left[y^n + ny^{n-1} + \frac{n(n-1)}{2!} y^{n-2} x^2 + \ldots\ldots + x^n\right] = \frac{(y+x)^n}{n!}$$

$$\therefore \quad \sum_{n=0}^{\infty} c_n = 1 + \frac{x+y}{1!} + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \ldots\ldots \text{ which converges to } e^{x+y}.$$

$\therefore$ By Abel's test, $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$

$$\Rightarrow \left(1+x+\frac{x^2}{2!}+\ldots\ldots\right)\left(1+y+\frac{y^2}{2!}+\ldots\ldots\right) = 1 + \frac{x+y}{1!} + \frac{(y+y)^2}{2!} + \frac{(x+y)^3}{3!} + \ldots\ldots$$

Example 19. *Show that the Cauchy product of the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$, $(p > 0)$ *with itself*

(i) *converges if* $p > \frac{1}{2}$ (ii) *does not converge if* $p \le \frac{1}{2}$.

Sol. Let $a_n = b_n = \dfrac{(-1)^{n-1}}{n^p}$.

The Cauchy product of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is $\sum_{n=1}^{\infty} c_n$

where $c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1$

$$= \frac{1}{1^p} \cdot \frac{(-1)^{n-1}}{n^p} - \frac{1}{2^p} \cdot \frac{(-1)^{n-2}}{(n-1)^p} + \frac{1}{3^p} \cdot \frac{(-1)^{n-3}}{(n-2)^p} - \ldots\ldots + \frac{(-1)^{n-1}}{n^p} \cdot \frac{1}{1^p}$$

$$= (-1)^{n-1} \left[\frac{1}{1^p \cdot n^p} + \frac{1}{2^p \cdot (n-1)^p} + \frac{1}{3^p (n-2)^p} + \ldots\ldots + \frac{1}{n^p \cdot 1^p} \right]$$

$$= (-1)^{n-1} \cdot d_n$$

where $d_n = \dfrac{1}{1^p \cdot n^p} + \dfrac{1}{2^p \cdot (n-1)^p} + \dfrac{1}{3^p (n-2)^p} + \ldots\ldots + \dfrac{1}{n^p \cdot 1^p}$

$$= \sum_{r=1}^{n} \frac{1}{r^p (n-r+1)^p} = \sum_{r=1}^{n} \frac{1}{[r(n-r+1)]^p}$$

Now $r(n-r+1) = -r^2 + (n+1)r$

$$= -\left[r^2 - (n+1)r + \left(\frac{n+1}{2}\right)^2 \right] + \left(\frac{n+1}{2}\right)^2 = \left(\frac{n+1}{2}\right)^2 - \left(r - \frac{n+1}{2}\right)^2 \le \left(\frac{n+1}{2}\right)^2$$

$\Rightarrow$ $r(n-r+1)$ is maximum when $r - \dfrac{n+1}{2} = 0$ *i.e.*, when $r = \dfrac{n+1}{2}$ and the max. value is $\left(\dfrac{n+1}{2}\right)^2$.

$$\Rightarrow \quad r(n-r+1) \le \left(\frac{n+1}{2}\right)^2 \Rightarrow r^p (n-r+1)^p \le \left(\frac{n+1}{2}\right)^{2p}$$

$$\Rightarrow \quad \frac{1}{[r(n-r+1)]^p} \ge \left(\frac{2}{n+1}\right)^{2p}$$

$$\Rightarrow \quad d_n \ge \sum_{r=1}^{n} \left(\frac{2}{n+1}\right)^{2p} = \left(\frac{2}{n+1}\right)^{2p} \cdot n = \left(\frac{2n}{n+1}\right)^{2p} \cdot n^{1-2p}$$

$$= \left(\frac{2}{1+\frac{1}{n}}\right)^{2p} \cdot n^{1-2p} \ge n^{1-2p}$$

$$\left[\because \forall\, n \in \mathrm{N}, \frac{1}{n} \le 1 \Rightarrow 1 + \frac{1}{n} \le 2 \Rightarrow \frac{2}{1+\frac{1}{n}} \ge 1 \right]$$

If $1 - 2p < 0$ *i.e.*, $p > \frac{1}{2}$, $d_n \to 0$ as $n \to \infty$ and by Leibnitz's test, the series $\sum_{n=1}^{\infty} c_n$ converges.

NOTES

If $1 - 2p \geq 0$ *i.e.*, $p \leq \frac{1}{2}$, $d_n \geq 1$ and the series $\sum_{n=1}^{\infty} c_n$ does not converge.

Example 20. *Assuming* $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$, *prove that*

$$(\tan^{-1} x)^2 = 2 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots.. + \frac{1}{2n+1}\right)$$

Also, show that the series is absolutely convergent if $|x| < 1$ *and convergent if* $x = 1$.

Or

For $-1 < x \leq 1$, *show that* $\frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\right)^2$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots.. + \frac{1}{2n+1}\right).$$

Sol. Let $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.. = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} a_n$

then $$|a_n| = \frac{|x|^{2n+1}}{2n+1} \text{ and } |a_{n+1}| = \frac{|x|^{2n+3}}{2n+3}$$

$$\therefore \quad \frac{|a_n|}{|a_{n+1}|} = \frac{2n+3}{2n+1} \cdot \frac{1}{|x|^2} = \frac{1 + \frac{3}{2n}}{1 + \frac{1}{2n}} \cdot \frac{1}{x^2}$$

$$\lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \frac{1}{x^2}.$$

$\therefore$ By ratio test, the series $\sum_{n=0}^{\infty} |a_n|$ is convergent if $\frac{1}{x^2} > 1$ *i.e.*, if $x^2 < 1$ *i.e.*, if $|x| < 1$

$\Rightarrow$ the series $\sum_{n=0}^{\infty} a_n$ converges absolutely for $|x| < 1$.

By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ with itself converges absolutely for $|x| < 1$ and $\left(\sum_{n=0}^{\infty} a_n\right)^2 = \sum_{n=0}^{\infty} c_n$.

Now $c_0 = x \cdot x = x^2$ and for $n \geq 1$.

$$c_n = x \cdot (-1)^n \frac{x^{2n+1}}{2n+1} - \frac{x^3}{3} \cdot (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \frac{x^5}{5} \cdot (-1)^{n-2} \frac{x^{2n-3}}{2n-3} - \ldots.$$
$$+ (-1)^n \frac{x^{2n+1}}{2n+1} \cdot x$$

$$= (-1)^n x^{2n+2}\left[\frac{1}{1.(2n+1)}+\frac{1}{3(n-1)}+\frac{1}{5(2n-3)}+......+\frac{1}{(2n+1).1}\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[\frac{(2n+1)+1}{1.(2n+1)}+\frac{(2n-1)+3}{3(2n-1)}+\frac{(2n-3)+5}{5(2n-3)}+......+\frac{(2n+1)+1}{(2n+1).1}\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[\left(1+\frac{1}{2n+1}\right)+\left(\frac{1}{3}+\frac{1}{2n-1}\right)+\left(\frac{1}{5}+\frac{1}{2n-3}\right)+......+\left(\frac{1}{2n+1}+1\right)\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[2+\frac{2}{3}+\frac{2}{5}+......+\frac{2}{2n+1}\right]$$

$$= 2(-1)^n . \frac{x^{2n+2}}{2n+2}\left(1+\frac{1}{3}+......+\frac{1}{2n+1}\right)$$

For $|x| < 1$, $\left(\sum_{n=0}^{\infty} a_n\right)^2 = \sum_{n=0}^{\infty} c_n.$

$$\Rightarrow \quad \left(x-\frac{x^3}{3}+\frac{x^5}{5}-......\right)^2 = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{2n+2}\left(1+\frac{1}{3}+......+\frac{1}{2n+1}\right)$$

But $x-\frac{x^3}{3}+\frac{x^5}{5}-...... = \tan^{-1} x.$

$$\therefore \quad (\tan^{-1} x)^2 = 2\sum_{n=0}^{\infty}(-1)^n . \frac{x^{2n+2}}{2n+2}\left(1+\frac{1}{3}+......+\frac{1}{2n+1}\right)$$

The result holds good for $x = 1$ also. (See example 22)

Example 21. *For all $x \in R$, show that*

$$\left(1+\frac{x}{1^2}+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+......\right)\left(1-\frac{x}{1^2}+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+......\right)$$

$$= 1-\frac{x^2}{1^2 . 2!}+\frac{x^4}{(2!)^2 4!}-\frac{x^6}{(3!)^2 6!}+.....$$

Sol. Let $1+\frac{x}{1^2}+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+..... = \sum_{n=0}^{\infty}\frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} a_n$

and $1-\frac{x}{1^2}+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+..... = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} b_n$

Then $|a_n| = \frac{|x|^n}{(n!)^2}$ and $|a_{n+1}| = \frac{|x|^{n+1}}{(x+1!)^2}$

$$\therefore \quad \frac{|a_n|}{|a_{n+1}|} = \frac{(n+1)^2}{|x|} \to \infty \text{ as } n \to \infty \text{ for all } x \neq 0.$$

$\Rightarrow$ The series $\sum_{n=0}^{\infty} |a_n|$ converges for all $x \neq 0$

For $x = 0$, the series becomes $1 + 0 + 0 + 0 + ...$

$\therefore$ the series $\sum_{n=0}^{\infty} a_n$ converges absolutely for all $x \in$ R.

NOTES

Similarly, the series $\sum_{n=0}^{\infty} b_n$ converges absolutely for all $x \in$ R.

$\therefore$ By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ also converges absolutely and $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$.

Now $c_0 = 1 \times 1 = 1$ and for $n \geq 1$,

$$c_n = 1 \cdot (-1)^n \frac{x^n}{(n!)^2} + \frac{x}{1^2} \cdot (-1)^{n-1} \frac{x^{n-1}}{[(n-1)!]^2} + \frac{x^2}{(2!)^2} \cdot (-1)^{n-2} \frac{x^{n-2}}{[(n-2)!]^2} + \ldots + \frac{x^n}{(n!)^2} \cdot 1$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - \frac{(n!)^2}{[1^2 \cdot (n-1)!]^2} + \frac{(n!)^2}{(2!)^2 (n-2!)^2} - \ldots + (-1)^n \frac{(n!)^2}{(n!)^2}\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - \left(\frac{n}{1}\right)^2 + \left(\frac{n(n-1)^2}{2!}\right)^2 + \ldots + (-1)^n (1)^2\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - ({}^nC_1)^2 + ({}^nC_2)^2 - \ldots + (-1)^n \cdot ({}^nC_n)^2\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2} \times \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{3n/2}}{(n!)^2} \cdot {}^nC_{n/2} x^n, & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{3n/2}}{n!\left(\frac{n}{2}!\right)^2} \cdot x^n, & \text{if } n \text{ is even} \end{cases} \qquad \left[\because \; {}^nC_{n/2} = \frac{n!}{\left(\frac{n}{2}\right)^2}\right]$$

$\therefore$ For all $x \in$ R, $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$.

$$\Rightarrow \left(1 + \frac{x}{1^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \ldots\right)\left(1 - \frac{x}{1^2} + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \ldots\right)$$

$$= 1 - \frac{x^2}{1^2 \cdot 2!} + \frac{x^4}{(2!)^2\, 4!} - \frac{x^6}{(3!)^2\, 6!} + \ldots$$

Note. $(1-x^2)^n = (1+x)^n (1-x)^n = (x+1)^n(1-x)^n$
$= [C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n]\,[C_0 - C_1x + C_2x^2 - ..... + (-1)^nC_nx^n]$

$\therefore \quad C_0{}^2 - C_1{}^2 + C_2{}^2 - ..... + (-1)^nC_n{}^2$ is the co-efficient of x^n on R.H.S.

Also $x^n = (x^2)^{n/2}$, so that the term containing x^n in $(1-x^2)^n$ will be $\left(\frac{n}{2}+1\right)$ th

$$T_{\frac{n}{2}+1} \text{ of } (1-x^2)^n = {}^nC_{n/2}\,.\,(-x^2)^{n/2} = (-1)^{n/2}\,.\,{}^nC_{n/2}\,.\,x^n$$

$\therefore \quad C_0{}^2 - C_1{}^2 + C_2{}^2 - ..... + (-1)^nC_n{}^2 = (-1)^{n/2}\,.\,{}^nC_{n/2}$.

Moreover, the term containing x^n occurs in the expansion of $(1-x^2)^n$ only when $n/2 \in N$ *i.e.*, only when n is even.

Example 22. *If for* $|x| < 1$, *the series* $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ *is absolutely convergent to* $A(x)$, *then show that*

$$(1-x)^{-1} A(x) = \sum_{n=0}^{\infty} S_nx^n, \quad \textit{where } S_n = a_0 + a_1 + ... + a_n.$$

Hence, show that $\sum_{n=0}^{\infty} (n+1)\,x^n = (1-x)^{-2}$.

Sol. The geometric series $\sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$ and has sum $= \frac{1}{1-x} = (1-x)^{-1}$.

Thus, $$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

Also, for $|x| < 1$, the series $a_0 = a_1x + a_2x^2 + ... = \sum_{n=0}^{\infty} a_nx^n$ converges absolutely and has sum $A(x)$.

$$\therefore \quad \sum_{n=0}^{\infty} a_nx^n = A(x).$$

By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} a_nx^n$ converges absolutely for $|x| < 1$ and

$$= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} a_nx^n\right) = \sum_{n=0}^{\infty} c_n \quad i.e., \quad (1-x)^{-1}\,.\,A(x) = \sum_{n=0}^{\infty} c_n \quad ...(1)$$

Now $\quad c_0 = 1 \times a_0 = a_0 \quad$ and $\quad$ for $n \geq 1$,
$c_n = 1\,.\,a_nx^n + x\,.\,a_{n-1}x^{n-1} + x^2\,.\,a_{n-2}x^{n-2} + ... + x^n.\,a_0$
$= (a_0 + a_1 + ... + a_n)\,x^n = S_nx^n$

$$\therefore \quad \text{From (1), for } |x| < 1, \ (1-x)^{-1}\,.\,A(x) = \sum_{n=0}^{\infty} S_nx^n \quad ...(2)$$

If $a_n = 1\ \forall\ n \geq 0$, then $\quad S_n = 1 + 1 + ... + 1 = n + 1$

and $$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}$$

$\therefore \quad$ From (2), we have $\sum_{n=0}^{\infty} (n+1)x^n = (1-x)^{-1}\,(1-x)^{-1} = (1-x)^{-2}$.

Example 23. *If* $|x| < 1$, *show that* $\frac{1}{1-x}\log\frac{1}{1-x} = \sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)x^n$.

Sol. For $|x| < 1$, we know that

NOTES

$$x + \frac{x^2}{2} + \frac{x^3}{3} + ... = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{and} \quad 1 + x + x^2 + ... = \sum_{n=1}^{\infty} x^{n-1}$$

converge absolutely to $-\log(1-x) = \log\frac{1}{1-x}$ and $\frac{1}{1-x}$ respectively.

$\therefore$ By Cauchy's theorem, the Cauchy product $\sum_{n=1}^{\infty} c_n$ of $\sum_{n=1}^{\infty}\frac{x^n}{n}$ and $\sum_{n=1}^{\infty} x^{n-1}$

converges absolutely for $|x| < 1$ and $\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty} x^n\right) = \sum_{n=1}^{\infty} c_n$. ...(1)

Now $c_n = x \,.\, x^{n-1} + \frac{x^2}{2} \,.\, x^{n-2} + \frac{x^3}{3} \,.\, x^{n-3} + ... + \frac{x^n}{n} \,.\, 1 = \left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)x^n$

$\therefore$ From (1), we have $\frac{1}{1-x}\log\frac{1}{1-x} = \sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)x^n$.

2.14. POWER SERIES

An infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ...$$

is called a power series in $(x - x_0)$.

Here the co-efficients $a_0, a_1, a_2, ...$ are constants and x is a variable. The fixed number x_0 is called the **centre of the power series**.

In particular, if $x_0 = 0$, then the power series in x is

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + ...$$

For example, $1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + ...$ is a power series in $(x - 2)$. The centre of this power series is 2.

2.15. CONVERGENCE OF POWER SERIES

Let $$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ... \quad ...(1)$$

be a power series with centre x_0.

Let $$S_n(x) = \sum_{m=0}^{n} a_m (x - x_0)^m$$
$$= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots\ldots + a_n (x - x_0)^n$$

The power series (1) is said to be **convergent at $x = c$** if $\lim_{n \to \infty} S_n(c)$ where $S_n(c) = a_0 + a_1 (c - x_0) + a_2(c - x_0)^2 + \ldots\ldots + a_n (c - x_0)^n$, exists finitely. The finite value of the limit is called the sum of the power series $\sum_{m=0}^{\infty} a_m (c - x_0)^m$.

Clearly, the power series (1) is always convergent at $x = x_0$ because in this case
$$S_n(x_0) = a_0 + 0 + 0 + \ldots\ldots + 0 = a_0$$
and $$\lim_{n \to \infty} S_n(x_0) = \lim_{n \to \infty} a_0 = a_0 \text{ which is finite.}$$

The set of all points (*i.e.*, values of x) for which (1) is convergent is called the **interval of convergence.** If the interval of convergence is finite then it is of the form $|x - x_0| < R$, *i.e.*, $x_0 - R < x < x_0 + R$ or $(x_0 - R, x_0 + R)$. The constant R is called the **radius of convergence.**

Clearly, if $x_0 = 0$, then the interval of convergence of the power series
$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots \text{ is } (-R, R).$$

The radius of convergence can be determined by the formula
$$\frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| \quad \text{or} \quad \frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{|a_m|}$$

If $R = \infty$, then the interval of convergence is $(-\infty, \infty)$ and the power series converges for all x.

If $R = 0$, then the power series converges only at $x = x_0$.

2.16. WORKING RULE FOR FINDING RADIUS OF CONVERGENCE AND INTERVAL OF CONVERGENCE

Let the given power series be $\sum_{m=k}^{\infty} a_m (x - x_0)^m$ where x_0 may or may not be zero and k is a non-negative integer.

(*a*) Find $\left| \dfrac{a_{m+1}}{a_m} \right|$ (*b*) Let $\lim_{m \to \infty} \left| \dfrac{a_{m+1}}{a_m} \right| = l$

(*i*) If $l = 0$, then $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

(*ii*) If $l = \infty$, then $R = 0$ and the power series converges only at $x = x_0$.

(*iii*) If l is non-zero and finite, then $R = \dfrac{1}{l}$ and the interval of convergence is $(x_0 - R, x_0 + R)$.

Remark. If a_m involves m in the index, then use

$$\frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{|a_m|}.$$

NOTES

Note. Consider $\sum_{m=0}^{\infty} m(m+3)^2 x^{m+2}$

Suppose we want to express it in terms of x^m.

Put $m + 2 = k$ so that $m = k - 2$.

When $m = 0$, $k = 2$. As $m \to \infty$, $k \to \infty$

$$\therefore \quad \sum_{m=0}^{\infty} m(m+3)^2 x^{m+2} = \sum_{k=2}^{\infty} (k-2)(k+1)^2 x^k, \text{ Replacing } k \text{ by } m$$

$$= \sum_{m=2}^{\infty} (m-2)(m+1)^2 x^m$$

which follows directly by changing m to $(m - 2)$.

2.17. POWER SERIES AS FUNCTIONS

2.17.1. Uniqueness Theorem for Power Series

1. *Theorem* 1. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence $R > 0$. Then f is *continuous* on $\{z : |z - z_0| < R\}$. That is, we can take the limit under the sum.

2. *Theorem* 2. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, both with radius of convergence $R > 0$, then $a_n = b_n$ for all n.

2.17.2. Derivatives and Integrals of Power Series

1. *Theorem* 3. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence $R > 0$. Then

(*a*) $f(z)$ is differentiable.

(*b*) $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$, and

(*c*) The radius of convergence of $f'(z)$ is also R.

2. *Theorem* 4. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence $R > 0$. If $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$ then

(*a*) $F'(z) = f(z)$.

(*b*) The radius of convergence of $F(z)$ is also R.

3. *Theorem* 5. A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with radius of convergence $R > 0$ represents a function analytic on the disk $\{z : |z - z_0| < R\}$. $f'(z)$ is obtained by term-by-term integration, and the integral of f is obtained by term-by-term integration.

Example 24. *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} \frac{x^m}{(m+2)!}$ (*ii*) $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^m$.

Sol. (*i*) Comparing $\sum_{m=0}^{\infty} \frac{x^m}{(m+2)!}$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{1}{(m+2)!} \quad \therefore \quad a_{m+1} = \frac{1}{(m+3)!}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(m+2)!}{(m+3)!}\right| = \frac{1}{m+3}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\frac{1}{m+3} = 0$$

$\Rightarrow$ Radius of convergence $R = \infty$.

(*ii*) Comparing $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{(-1)^m}{8^m} \quad \therefore \quad a_{m+1} = \frac{(-1)^{m+1}}{8^{m+1}}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{-8^m}{8^{m+1}}\right| = \frac{1}{8}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\frac{1}{8} = \frac{1}{8}$$

$\Rightarrow$ Radius of convergence $R = 8$.

Example 25. *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (m+1)^2 x^m$ (*ii*) $\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m$

Sol. (*i*) Comparing $\sum_{m=0}^{\infty} (m+1)^2 x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = (m+1)^2 \quad \therefore \quad a_{m+1} = (m+2)^2$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+2)^2}{(m+1)^2} = \left(\frac{1+\frac{2}{m}}{1+\frac{1}{m}}\right)^2$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left(\frac{1+\frac{2}{m}}{1+\frac{1}{m}}\right)^2 = \left(\frac{1+0}{1+0}\right)^2 = 1$$

$\Rightarrow$ Radius of convergence $R = 1$.

NOTES

(*ii*) Comparing $\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{(3m)!}{(m!)^3} \quad \therefore \quad a_{m+1} = \frac{(3m+3)!}{((m+1)!)^3}$$

and

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(3m+3)!}{((m+1)!)^3} \times \frac{(m!)^3}{(3m)!}$$

$$= \frac{(3m+3)(3m+2)(3m+1)}{(m+1)^3} = \frac{3(3m+2)(3m+1)}{(m+1)^2}$$

$$= \frac{3\left(3+\frac{2}{m}\right)\left(3+\frac{1}{m}\right)}{\left(1+\frac{1}{m}\right)^2}$$

$$\therefore \quad \frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \frac{3\left(3+\frac{2}{m}\right)\left(3+\frac{1}{m}\right)}{\left(1+\frac{1}{m}\right)^2} = \frac{3(3)(3)}{(1)^2} = 27$$

$\Rightarrow$ Radius of convergence $R = \frac{1}{27}$.

Example 26. *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (m+2)^m x^m$ (*ii*) $\sum_{m=0}^{\infty} (m+2)(m+3)x^{m+1}$.

Sol. (*i*) Comparing $\sum_{m=0}^{\infty} (m+2)^m x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = (m+2)^m \qquad \text{(involves } m \text{ in the index)}$$

$$\therefore \quad \sqrt[m]{|a_m|} = [(m+2)^m]^{1/m} = m+2$$

$$\frac{1}{R} = \lim_{m\to\infty} \sqrt[m]{|a_m|} = \lim_{m\to\infty} (m+2) = \infty$$

$\Rightarrow$ Radius of convergence R = 0.

(*ii*) The given power series $\sum_{m=0}^{\infty} (m+2)(m+3)\,x^{m+1}$ is in terms of x^{m+1}.

To express it in terms of x^m, we replace by $(m-1)$.

$$\therefore \quad \sum_{m=0}^{\infty} (m+2)(m+3)\,x^{m+1} = \sum_{m=1}^{\infty} (m+1)(m+2)x^m$$

Comparing $\sum_{m=1}^{\infty} (m+1)(m+2)x^m$ with $\sum_{m=1}^{\infty} a_m x^m$, we get

$$a_m = (m+1)(m+2) \quad \therefore \quad a_{m+1} = (m+2)(m+3)$$

and $$\left|\frac{a_{m+1}}{a_m}\right| = \frac{m+3}{m+1} = \frac{1+\frac{3}{m}}{1+\frac{1}{m}}$$

$\therefore$ $$\frac{1}{R} = \lim_{m \to \infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m \to \infty}\frac{1+\frac{3}{m}}{1+\frac{1}{m}} = \frac{1+0}{1+0} = 1$$

$\Rightarrow$ Radius of convergence R = 1.

Example 27. *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (-1)^m x^{2m}$ (*ii*) $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$

(*iii*) $\sum_{m=0}^{\infty} \frac{(x-1)^{2m}}{2^m}$

Sol. (*i*) The given power series is

$$\sum_{m=0}^{\infty} (-1)^m x^{2m} = \sum_{m=0}^{\infty} (-1)^m (x^2)^m = \sum_{m=0}^{\infty} (-1)^m y^m, \quad \text{where } y = x^2.$$

Comparing it with $\sum_{m=0}^{\infty} a_m y^m$, we get

$$a_m = (-1)^m \qquad \therefore \quad a_{m+1} = (-1)^{m+1}$$

and $$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(-1)^{m+1}}{(-1)^m}\right| = |(-1)| = 1$$

$\therefore$ $$\frac{1}{R} = \lim_{m \to \infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m \to \infty} 1 = 1$$

$\Rightarrow$ Radius of convergence for the power series $\sum_{m=0}^{\infty} (-1)^m y^m$ is 1

$\Rightarrow$ The power series $\sum_{m=0}^{\infty} (-1)^m y^m$ converges for $|y| < 1$

i.e., $$|x^2| < 1 \quad \text{or} \quad |x|^2 < 1 \quad \text{or} \quad |x| < 1$$

$\Rightarrow$ Radius of convergence for the given series is 1.

Remark. Here $a_m = \frac{1}{3^m}$ involves m in the index. Therefore,

we can also use $$\frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{|a_m|}.$$

NOTES

(*ii*) The given power series is

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} (x^2)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} y^m, \quad \text{where } y = x^2$$

Comparing it with $\sum_{m=0}^{\infty} a_m y^m$, we get

$$a_m = \frac{(-1)^m}{k^m} \quad \therefore \quad a_{m+1} = \frac{(-1)^{m+1}}{k^{m+1}}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|-\frac{k^m}{k^{m+1}}\right| = \frac{1}{|k|}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \frac{1}{|k|} = \frac{1}{|k|}$$

$\Rightarrow$ Radius of convergence for the power series $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} y^m$ is $|k|$

$\Rightarrow$ The power series $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} y^m$ converges for $|y| < |k|$

i.e.,
$$|x^2| < |k| \quad \text{or} \quad |x|^2 < |k| \quad \text{or} \quad |x| < \sqrt{|k|}$$

$\Rightarrow$ Radius of convergence for the given series is $\sqrt{|k|}$.

(*iii*) The given power series is

$$\sum_{m=0}^{\infty} \frac{1}{2^m} (x-1)^{2m} = \sum_{m=0}^{\infty} \frac{y^m}{2^m}, \text{ where } \quad y = (x-1)^2$$

Comparing it with $\sum_{m=0}^{\infty} a_m y^m$, we get

$$a_m = \frac{1}{2^m} \quad \therefore \quad a_{m+1} = \frac{1}{2^{m+1}}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{2^m}{2^{m+1}}\right| = \frac{1}{2}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$ Radius of convergence for $\sum_{m=0}^{\infty} \frac{y^m}{2^m}$ is 2

$\Rightarrow$ The power series $\sum_{m=0}^{\infty} \frac{y^m}{2^m}$ converges for $|y| < 2$

i.e.,
$$|(x-1)^2| < 2 \quad \text{or} \quad |x-1|^2 < 2 \quad \text{or} \quad |x-1| < \sqrt{2}$$

$\Rightarrow$ Radius of convergence for the given series is $\sqrt{2}$.

Remark. Here centre is 1 and interval of convergence is $(1-\sqrt{2}, 1+\sqrt{2})$.

TEST YOUR KNOWLEDGE

NOTES

1. Test the convergence of series:

   (i) $1-\frac{1}{5\sqrt{2}}+\frac{1}{9\sqrt{3}}-\frac{1}{13\sqrt{4}}+\ldots$

   (ii) $\frac{1}{\log 2}-\frac{1}{2^3\log 3}+\frac{1}{3^3\log 4}-\frac{1}{4^3\log 5}+\ldots$

2. Show that the convergence of $\sum_{n=1}^{\infty} a_n$ an implies the convergence of each of the following series:

   (i) $\sum_{n=1}^{\infty}\frac{n+1}{n}\,a_n$ (ii) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n$.

3. Find the sum of the series

   (i) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\ldots$

   (ii) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+\ldots$

4. Find how the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\ldots$ should be deranged so that the sum is doubled.

5. Show that the Cauchy product of the two divergent series

$$\sum_{n=1}^{\infty} a_n = 1-\frac{3}{2}-\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^3-\ldots\ldots$$

and $\sum_{n=1}^{\infty} b_n = 1+\left(2+\frac{1}{2^2}\right)+\frac{3}{2}\left(2^2+\frac{1}{2^3}\right)+\left(\frac{3}{2}\right)^2\left(2^3+\frac{1}{2^4}\right)+\ldots$ is convergent.

6. Show that

$$\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots\ldots\right)^2=\sum_{n=1}^{\infty}(-1)^{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right).\frac{x^{n+1}}{n+1}.$$

   when (i) $|x|<1$ and (ii) $x=1$.

   $\left[\textbf{Hint: Here } a_n=\frac{(-1)^{n+1}}{n}x^n\right]$

7. Assuming that $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots$, prove that:

$$(\tan^{-1}x)^2=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right).$$

8. Find the radius of convergence of the following power series:

   (i) $\sum_{m=0}^{\infty}(m+1)!\,x^m$ (ii) $\sum_{m=0}^{\infty}\frac{x^m}{5^m}$

   (iii) $\sum_{m=0}^{\infty}\frac{x^m}{m!}$ (iv) $\sum_{m=1}^{\infty}(2m-1)!\,x^m$

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(v) $\sum_{m=2}^{\infty} m^m \cdot x^m$ (vi) $\sum_{m=0}^{\infty} \frac{x^{2m}}{3^m}$

(vii) $\sum_{m=0}^{\infty} \frac{(-1)^m}{5^m} x^{3m}$ (viii) $\sum_{m=0}^{\infty} \left(\frac{5}{4}\right)^m x^{4m}$

(ix) $\sum_{m=0}^{\infty} \frac{(x-2)^{2m}}{m!}$

Answers

1. (i) Convergent (ii) Convergent 3. (i) $\frac{1}{2}\log 12$ (ii) $\log\sqrt{6}$

4. $k = 4$, Deranged series is $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{2}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}+\frac{1}{17}-\ldots\ldots$

8. (i) 0 (ii) 5 (iii) ∞ (iv) 0 (v) 0

(vi) $\sqrt{3}$ (vii) $\sqrt[3]{5}$ (viii) $\left(\frac{4}{5}\right)^{1/4}$ (ix) ∞

SUMMARY

- If the sequence $< S_n >$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ satisfies $m \le S_n \le M$, $(n \in N)$ and $< b_n >$ is a sequence of non-increasing, non-negative real numbers, then $mb_1 \le \sum_{k=1}^{n} a_k b_k \le Mb_1$.
- If $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $< b_n >$ is monotonic and bounded, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.
- If $\sum_{n=1}^{\infty} a_n$ has bounded partial sums and $< b_n >$ is a monotonic sequence converging to zero, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.
- A series $\sum_{n=1}^{\infty} b_n$ is said to arise from a series $\sum_{n=1}^{\infty} a_n$ by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series so that every a_n is some b_m and conversely.
- If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum are arranged. But this is not so when infinite series are

involved. An *arrangement* (or equally well *derangement*) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

- By a suitable rearrangement of the terms, a conditionally convergent series $\sum_{n=1}^{\infty} a_n$ can be made
  (*i*) to converge to any pre-assigned under α, or
  (*ii*) to diverge to ∞ or $-\infty$, or
  (*iii*) to oscillate finitely or infinitely.
- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of non-negative terms converging to A and B respectively, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.
- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series such that $\sum_{n=1}^{\infty} a_n = \text{A}$ and $\sum_{n=1}^{\infty} b_n = \text{B}$. If their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = \text{AB}$.
- Let $f(x) = \Sigma c_n x^n$ $(-1 < x < 1)$ and suppose that $\lim_{n \to \infty} {}^n\text{C}_n = 0$. If $f(x) \to s$ as $x \to 1$, then Σc_n converges and has sum s.
- An infinite series of the form

  $$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2 (x - x_0)^2 + \ldots..$$

  is called a power series in $(x - x_0)$.

  Here the co-efficients $a_0, a_1, a_2, \ldots\ldots$ are constants and x is a variable. The fixed number x_0 is called the **centre of the power series**.
- A power series is said to be **convergent at $x = c$** if $\lim_{n \to \infty} \text{S}_n(c)$ where $\text{S}_n(c) = a_0 + a_1 (c - x_0) + a_2(c - x_0)^2 + \ldots\ldots + a_n (c - x_0)^n$, exists finitely. The finite value of the limit is called the sum of the power series $\sum_{m=0}^{\infty} a_m (c - x_0)^m$.
- Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with radius of convergence $\text{R} > 0$. Then f is *continuous* on $\{z : |z - z_0| < \text{R}\}$. That is, we can take the limit under the sum.
- If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$, both with radius of convergence $\text{R} > 0$, then $a_n = b_n$ for all n.

□□□

NOTES

UNIT

3

SEQUENCES AND SERIES OF FUNCTIONS

STRUCTURE

3.1. INTRODUCTION

In this unit, we will discuss the convergence of sequences and series *of real-valued functions* defined on an interval.

3.2. SEQUENCES OF REAL-VALUED FUNCTIONS

Let f_n be a real-valued function defined on an interval I (or on a subset D of R) and for each $n \in$ N. Then

$$< f_1, f_2, f_3, \ldots\ldots, f_n, \ldots\ldots >$$

is called a sequence of real-valued functions on I. It is denoted by $\{f_n : I \to R, n \in N\}$ or briefly by $\{f_n\}$ or $< f_n >$.

For example :

(*i*) If f_n is a real-valued function defined by $f_n(x) = x^n$, $0 \le x \le 1$

then $$< f_1(x), f_2(x), f_3(x), ... > = < x^1, x^2, x^3, ... >$$

is a sequence of real-valued functions on [0, 1].

(*ii*) If f_n is a real-valued function defined by $f_n(x) = \dfrac{\sin nx}{n}$, $0 \le x \le 1$

then $$< f_1(x), f_2(x), f_3(x), ...> = < \sin x, \frac{\sin 2x}{2}, \frac{\sin 3x}{3}, ... >$$

is a sequence of real-valued functions on [0, 1].

If $< f_n >$ is a sequence of functions defined on I, then for $c \in$ I,

$< f_n(c) > = < f_1(c), f_2(c), ..., f_n(c), ... >$ is a sequence of real numbers.

For example, if $< f_n >$ is a sequence of functions defined by $f_n(x) = x^n$, $0 \le x \le 1$, then

$$< f_n(\tfrac{1}{2}) > = < f_1(\tfrac{1}{2}), f_2(\tfrac{1}{2}), f_3(\tfrac{1}{2}), ..., f_n(\tfrac{1}{2}), ... > = < \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ..., \frac{1}{2^n}, ... >$$

is a sequence of real numbers corresponding to $\frac{1}{2} \in [0, 1]$.

Thus to each $x \in$ I, we have a sequence of real numbers.

3.3. POINTWISE CONVERGENCE OF A SEQUENCE OF FUNCTIONS

Let $< f_n >$ be a sequence of functions on I and $c \in$ I. Then the sequence of real numbers $< f_n(c) >$ may be convergent. In fact for each $c \in$ I, the corresponding sequence of real numbers may be convergent.

If $< f_n >$ is a sequence of real-valued functions on I and for each $x \in$ I, the corresponding sequence of real numbers is convergent, then we say the sequence $< f_n >$ converges pointwise. The limiting values of the sequences of real numbers corresponding to $x \in$ I define a function f called the limit function or simply the limit of the sequence $< f_n >$ of functions on I.

Definition. Let $< f_n >$ be a sequence of functions on I. If to each $x \in$ I and to each $\varepsilon > 0$, there corresponds to positive integer m such that $| f_n(x) - f(x) | < \varepsilon \ \forall n \ge m$ then we say that $< f_n >$ converges pointwise to the function f on I.

Note 1. $< f_n >$ converges pointwise to the function f on I.

$\Leftrightarrow \lim_{n \to \infty} f_n(x) = f(x) \ \forall x \in$ I. $f(x)$ is called the limit function or simply the limit or the pointwise limit of $< f_n(x) >$ on I.

Note 2. The positive integer m depends on $x \in$ I and given $\varepsilon > 0$, *i.e.*, $m = m(x, \varepsilon)$.

Let us consider a few examples :

(*i*) Let $f_n(x) = x^n$, $x \in [0, 1]$

Since $$\lim_{n \to \infty} x^n = 0 \text{ for } 0 \le x < 1$$

we have $$\lim_{n \to \infty} f_n(x) = 0 \text{ for } 0 \le x < 1$$

When $x = 1$, the corresponding sequence $< f_n(1) > = < 1, 1, 1, ... >$ is a constant sequence converging to 1.

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$$\therefore \qquad \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$

Hence $< f_n >$ converges pointwise on [0, 1].

$$f(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$ is the limit function of $< f_n(x) >$ on [0, 1].

(*ii*) Let $\qquad f_n(x) = \dfrac{x}{1+nx}, x \ge 0$

Then for $x > 0$, $\quad \lim_{n \to \infty} f_n(x) = 0$

Also $f_n(0) = 0 \ \forall \ n \in$ N so that $< f_n(0) >$ converges to 0.

$$\therefore \qquad \lim_{n \to \infty} f_n(x) = 0 \ \forall \ x \ge 0$$

Hence $< f_n >$ converges to zero pointwise on $[0, \infty)$ and $f(x) = 0$ is the limit function of $< f_n(x) >$ on $[0, \infty)$.

(*iii*) Let $\qquad f_n(x) = \dfrac{nx}{1+n^2x^2}, x \in$ R

For $x \ne 0$, $\qquad f_n(x) = \dfrac{\frac{1}{nx}}{\frac{1}{n^2x^2}+1} \to 0$ as $n \to \infty$

Also $\qquad f_n(0) = 0 \ \forall \ n \in$ N

$$\therefore \qquad \lim_{n \to \infty} f_n(x) = 0 \ \forall \ x \in \text{R}.$$

Hence $< f_n >$ converges to zero pointwise on R and $f(x) = 0$ is the limit function of $< f_n(x) >$ on R.

Note 3. For a sequence $< f_n >$ of functions, an important question is :

If each function of a sequence $< f_n >$ has a certain property such as continuity, differentiability or integrability, then to what extent is this property transferred to the limit function f? In fact, pointwise convergence is not strong enough to transfer any of the properties mentioned above from the terms f_n of $< f_n >$ to the limit function.

Let us consider a few examples :

(*i*) *A sequence of continuous functions with a discontinuous limit function.*

Consider the sequence $< f_n >$ where $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}, x \in$ R

Then $$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \frac{1}{2} & \text{if } |x| = 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

Here, each f_n is continuous on R but f is discontinuous at $x = \pm 1$.

(*ii*) *A sequence of differentiable functions in which the limit of the derivatives is not equal to the derivative of the limit function.*

Consider the sequence $< f_n >$ where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}, x \in$ R.

Then $$f(x) = \lim_{n \to \infty} \frac{\sin nx}{\sqrt{n}} = 0 \quad \forall \ x \in \text{R}$$

$$\therefore \qquad f'(x) = 0 \quad \forall \ x \in \text{R} \ \Rightarrow \ f'(0) = 0$$

But $\qquad f_n'(x) = \sqrt{n} \cos nx$

$\Rightarrow \qquad f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$

Thus, at $x = 0$, $\lim_{n\to\infty} f_n'(x) \neq f'(0)$.

(iii) A sequence of functions in which the limit of integrals is not equal to the integral of the limit function.

Consider the sequence $< f_n >$ where $f_n(x) = nx(1-x^2)^n$, $x \in [0, 1]$

Then $\qquad f_n(x) = 0$ when $x = 0$ or 1

Also, if $0 < x < 1$, then $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n \qquad |$ Form $\infty \times 0$

$$= \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \qquad \Big| \text{ Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x^2)^{-n}\log(1-x^2)} = \lim_{n\to\infty} \frac{-(x)(1-x^2)^n}{\log(1-x^2)} = 0$$

$\therefore \qquad f(x) = 0 \ \forall\ x \in [0, 1]$

Now $\qquad \int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx$

$$= -\frac{n}{2}\int_0^1 (1-x^2)^n \cdot (-2x)\,dx$$

$$= -\frac{n}{2}\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1 = \frac{n}{2(n+1)}$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$$

Also $\qquad \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ so that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$.

The above few examples show that we need to investigate under what supplementary conditions these or other properties of the terms f_n of $< f_n >$ are transferred to the limit function f. A concept of great importance in this respect is that known as *uniform convergence.*

3.4. UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

We know that a sequence $< f_n >$ of function on I converges pointwise to a function f if to each $x \in$ I and to each $\varepsilon > 0$, there corresponds a positive integer m such that $| f_n(x) - f(x) | < \varepsilon \ \forall\ n \geq m$

The positive integer m depends on $x \in$ I and given $\varepsilon > 0$, *i.e.*, $m = m(x, \varepsilon)$. It is not always possible to find an m which works for each $x \in$ I.

For example, consider the sequence $< f_n >$ defined by $f_n(x) = x^n$, $x \in [0, 1]$.

It converges pointwise to the function f on $[0, 1]$ where $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

Let $\varepsilon = \frac{1}{2}$ be given.

Then for each $x \in [0, 1]$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \tfrac{1}{2} \ \ \forall\ \ n \geq m \qquad \ldots(1)$$

NOTES

If $x = 0$, $f(x) = 0$ and $f_n(x) = 0 \quad \forall\, n \in N$

$$\therefore \quad |\, f_n(x) - f(x) \,| = |\, 0 - 0 \,| = 0 < \tfrac{1}{2} \;\forall\, n \geq 1$$

Thus (1) is true for $m = 1$

Similarly, (1) is true for $m = 1$ when $x = 1$.

If $$x = \frac{3}{4},\ f(x) = 0 \quad \text{and} \quad f_n(x) = \left(\frac{3}{4}\right)^n$$

$$\therefore \quad |\, f_n(x) - f(x) \,| = \left|\left(\frac{3}{4}\right)^n - 0\right| = \left(\frac{3}{4}\right)^n < \frac{1}{2} \quad \forall\, n \geq 3$$

Thus (1) is true for $m = 3$.

Similarly, (1) is true for $m = 7$ when $x = \dfrac{9}{10}$.

Hence there is no single value of m for which (1) holds for all $x \in [0, 1]$. That is m depends both on x and ε.

Now consider the sequence $< f_n >$ defined by $f_n(x) = \dfrac{x}{1 + nx}$, $x \geq 0$

It converges pointwise to zero, *i.e.*, $f(x) = 0$ for all $x \geq 0$.

Now $$0 \leq f_n(x) = \frac{x}{1 + nx} \leq \frac{x}{nx} = \frac{1}{n}$$

$\therefore$ For any $\varepsilon > 0$. $|\, f_n(x) - f(x) \,| = |\, f_n(x) \,| \leq \dfrac{1}{n} < \varepsilon$

for all $x \in [0, \infty)$ provided $\dfrac{1}{n} < \varepsilon$ *i.e.*, $n > \dfrac{1}{\varepsilon}$

If m is a positive integer $> \dfrac{1}{\varepsilon}$, then $|\, f_n(x) - f(x) \,| < \varepsilon \;\forall\, n \geq m$ and $\forall\, x \in [0, \infty)$.

Thus, in this example, we can find an m which depends only on ε and not on $x \in [0, \infty)$. We say that the sequence $< f_n >$ is uniformly convergent to f on $[0, \infty)$.

Definition. *Let $< f_n >$ be a sequence of functions on I. Then $< f_n >$ is said to be uniformly convergent to a function f on I if to each $\varepsilon > 0$, there exists a positive integer m (depending only on ε) such that*

$$|\, f_n(x) - f(x) \,| < \varepsilon \;\forall\, x \geq m \text{ and } \forall\, x \in I.$$

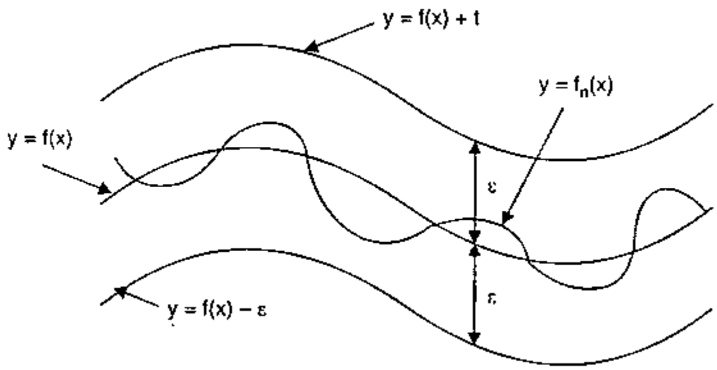
The function f is called **uniform limit** of the sequence $< f_n >$ on I.

Geometrical Interpretation of Uniform Convergence

A sequence $< f_n(x) >$ of functions defined on I is said to be uniformly convergent to a function f on I if for each $\varepsilon > 0$, there exists a positive integer m (depending only on ε) such that

$$|\, f_n(x) - f(x) \,| < \varepsilon \quad \forall\; n \geq m \quad \text{and} \quad \forall\, x \in I$$

i.e., $$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in I.$$



This shows that the graph of $f_n(x)$ for all $n \geq m$ and for all $x \in$ I lies between the graphs of $f(x) - \varepsilon$ and $f(x) + \varepsilon$, *i.e.*, within a band of height 2ε situated symmetrically about the graph of f.

Note 1. In the definition of uniform convergence, $m \in$ N is the same for every $x \in$ I and depends only on given $\varepsilon > 0$.

Note. 2. If a sequence $< f_n >$ of functions defined on I converges uniformly to a function f on I, then the sequence $< f_n >$ converges pointwise to f also.

Thus uniform convergence $\Rightarrow$ pointwise convergence.

However, the converse is not true. For example, if $f_n(x) = x^n$, $x \in [0, 1]$, then the sequence $< f_n >$ converges pointwise to the function f on [0, 1], where $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$ but $< f_n >$ does not converge uniformly on [0, 1].

Note. 3. A sequence $< f_n >$ of functions defined on I does not converge uniformly to f on I iff there exists some $\varepsilon > 0$ such that there is no positive integer m for which the statement

"$| f_n(x) - f(x) | < \varepsilon \; \forall \; n \geq m$ and $\forall \; x \in$ I" holds.

Note 4. Uniform convergence is a property associated with an interval (or an infinite subset S of R) and not with a single point. On no account we speak of uniform convergence at a single point.

3.5. UNIFORMLY BOUNDED SEQUENCE OF FUNCTIONS

A sequence $< f_n >$ of functions defined on I is said to be uniformly bounded on I if there exists a positive real number K such that

$$| f_n(x) | < \text{K} \quad \forall \; n \in \text{N} \quad \text{and} \quad \forall \; x \in \text{I}.$$

The number K is called a **uniform bound** for $< f_n >$ on I.

For example, if $f_n(x) = \sin nx$, $x \in$ R, then $| f_n(x) | = | \sin nx | \leq 1 \; \forall \; n \in$ N and $\forall$ $x \in$ R

$\therefore$ The sequence $< f_n >$ is uniformly bounded on R.

3.6. POINT OF NON-UNIFORM CONVERGENCE

Let $< f_n >$ be a sequence of functions defined on I. A point $x \in$ I said to be a point of non-uniform convergence if $< f_n >$ does not converge uniformly in any neighbourhood (however small) of x.

For example, if $f_n(x) = x^n$, $x \in [0, 1]$, then 1 is a point of non-uniform convergence of $< f_n >$.

3.7. THEOREM (CAUCHY'S CRITERION FOR UNIFORM CONVERGENCE)

NOTES

A sequence $< f_n >$ of functions defined on I is uniformly convergent on I if and only if for each $\varepsilon > 0$ and for all $x \in I$, there exists a positive integer m such that

$$| f_{n_1}(x) - f_{n_2}(x) | < \varepsilon \ \forall \ n_1, n_2 \geq m.$$

Proof. Necessary part. Let a sequence $< f_n >$ of functions defined on I be uniformly convergent on I.

Let $< f_n >$ converge uniformly to f on I.

Then for each $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{2} \ \forall \ n \geq m \quad \text{and} \quad \forall \ x \in I.$$

If $n_1, n_2 \in N$ are such that $n_1, n_2 \geq m$, then

$$| f_{n_1}(x) - f(x) | < \frac{\varepsilon}{2} \ \forall \ x \in I \qquad ...(1)$$

and
$$| f_{n_2}(x) - f(x) | < \frac{\varepsilon}{2} \ \forall \ x \in I \qquad ...(2)$$

$$\therefore \ | f_{n_1}(x) - f_{n_2}(x) | = | f_{n_1}(x) - f(x) + f(x) - f_{n_2}(x) | = | (f_{n_1}(x) - f(x)) - (f_{n_2}(x) - f(x)) |$$

$$\leq | f_{n_1}(x) - f(x) | + | f_{n_2}(x) - f(x) |$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall \ n_1, n_2 \geq m \text{ and } \forall \ x \in I \qquad \text{[by (1) and (2)]}$$

Sufficiency part. Let $< f_n >$ be a sequence of functions defined on I such that for each $\varepsilon > 0$, there exists a positive integer m such that

$$| f_{n_1}(x) - f_{n_2}(x) | < \varepsilon \quad \forall \ n_1, n_2 \geq m \quad \text{and} \quad \forall \ x \in I \qquad ...(3)$$

From (3), we find that for each $x \in I$, the sequence $< f_n(x) >$ of real numbers is a Cauchy sequence and hence $< f_n(x) >$ is convergent. Thus the sequence $< f_n >$ is pointwise convergent.

Let the sequence $< f_n >$ converge pointwise to the function f on I.

Then
$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall \ x \in I \qquad ...(4)$$

Putting $n_1 = n$ and keeping n fixed, from (3), we have $| f_n(x) - f_{n_2}(x) | < \varepsilon \ \forall \ n, n_2 \geq m$ and $\forall \ x \in I$.

Also, from (4), as $n_2 \to \infty$, we have $f_{n_2}(x) \to f(x)$

$$\therefore \qquad | f_n(x) - f(x) | < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in I$$

$\Rightarrow \ < f_n >$ converges uniformly to f on I.

Note. The above theorem can also be stated as follows :

'A sequence $< f_n >$ of functions defined on I is uniformly convergent on I if and only if for each $\varepsilon > 0$ and for all $x \in I$, there exists a positive integer m such that for any integer $p \geq 1$,

$$| f_{n+p}(x) - f_n(x) | < \varepsilon \ \forall \ n \geq m.$$

NOTES

ILLUSTRATIVE EXAMPLES—A

Example 1. *Show that the sequence* $<f_n>$ *where* $f_n(x) = x^n$ *is uniformly convergent on* $[0, k]$, $k < 1$ *but only pointwise convergent on* $[0, 1]$.

Sol. Here, $$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Thus, the sequence $<f_n>$ converges pointwise to f on $[0, 1]$.

To see whether the sequence $<f_n>$ is uniformly convergent, let $\varepsilon > 0$ be given.

For $0 < x < 1$, $\quad | f_n(x) - f(x) | = | x^n - 0 | = x^n < \varepsilon$

if $$\frac{1}{x^n} > \frac{1}{\varepsilon} \quad \text{or} \quad \text{if} \quad n \log \frac{1}{x} > \log \frac{1}{\varepsilon}$$

or if $$n > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}}$$ | Note that $0 < x < 1$ $\Rightarrow \frac{1}{x} > 1$ so that $\log \frac{1}{x} > 0$

The number $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}}$ increases with x having maximum value $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{k}}$ on $(0, k]$, $k < 1$.

Choose a positive integer m just $\ge \dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{k}}$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad 0 < x < 1.$$

At $x = 0$, $\quad | f_n(x) - f(x) | = | 0 - 0 | = 0 < \varepsilon \quad \forall\, n \ge 1$

Thus, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad \forall\, x \in [0, k],\ k < 1.$$

$\Rightarrow$ $<f_n>$ is uniformly convergent on $[0, k]$, $k < 1$.

When $x \to 1$, the number $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} \to \infty$. Thus it is not possible to find a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \ \ \forall\, n \ge m \quad \text{and} \quad \forall\, x \in [0, 1].$$

Hence, the sequence $<f_n>$ is not uniformly convergent on any interval containing 1 and in particular on $[0, 1]$.

Example 2. *Show that the sequence* $<f_n>$ *defined by* $f_n(x) = x^n$, $x \in [0, 1]$ *is not uniformly convergent.*

Sol. *Please try yourself.*

Example 3. *Show that the sequence of functions* $<f_n>$, *where* $f_n(x) = \dfrac{1}{n + x}$, *is uniformly convergent in any interval* $[0, k]$, $k > 0$.

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = 0 \quad \forall\, x \ge 0$$

Let $\varepsilon > 0$ be given.

$$|f_n(x) - f(x)| = \left|\frac{1}{n+x} - 0\right| = \frac{1}{n+x} < \varepsilon \text{ if } n + x > \frac{1}{\varepsilon}$$

Now $\forall x \in [0, k]$, $0 \le x \le k$

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$$\therefore \quad n + k \ge n + x > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon} - k$$

Choose a positive integer m just $\ge \frac{1}{\varepsilon} - k$.

Then $|f_n(x) - f(x)| < \varepsilon \ \forall n \ge m$ and $\forall x \in [0, k]$.

$\Rightarrow$ $< f_n >$ is uniformly convergent on $[0, k]$.

Example 4. *Show that the sequence $< f_n >$ where $f_n(x) = \frac{n}{x+n}$, $x \ge 0$ is uniformly convergent in any finite interval.*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{x+n} = \lim_{n\to\infty} \frac{1}{\frac{x}{n}+1} = 1 \ \forall x \in R$$

Let $\varepsilon > 0$ be given.

For $x = 0$, $\quad |f_n(x) - f(x)| = |1 - 1| = 0 < \varepsilon \ \forall n \ge 1$

For $x > 0$, we have $|f_n(x) - f(x)| = \left|\frac{n}{x+n} - 1\right| = \left|\frac{-x}{x+n}\right| = \frac{x}{x+n} < \varepsilon$

if $$\frac{x+n}{x} > \frac{1}{\varepsilon} \text{ or if } 1 + \frac{n}{x} > \frac{1}{\varepsilon} \text{ or if } n > x\left(\frac{1}{\varepsilon} - 1\right)$$

Now $x\left(\frac{1}{\varepsilon} - 1\right)$ increases with x and tends to infinity as $x \to \infty$ so that it is not possible to choose a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \ge m \text{ and } \forall x \ge 0.$$

However, if we consider a finite interval $[0, k]$ where k is any fixed positive number, however large, then the maximum value of $x\left(\frac{1}{\varepsilon} - 1\right)$ is $k\left(\frac{1}{\varepsilon} - 1\right)$.

If we choose a positive integer m just $\ge k\left(\frac{1}{\varepsilon} - 1\right)$, then

$$|f_n(x) - f(x)| < \varepsilon \ \forall n \ge m \text{ and } \forall x \in [0, k].$$

Hence $< f_n >$ is uniformly convergent on any finite interval.

Example 5. *Test for uniform convergence the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ for $x \ge 0$.*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have $|f_n(x) - f(x)| = |e^{-nx} - 0| = e^{-nx} < \varepsilon$

if $$e^{nx} > \frac{1}{\varepsilon} \quad i.e., \text{ if } nx > \log\frac{1}{\varepsilon}$$

or if $$n > \frac{\log\frac{1}{\varepsilon}}{x} \qquad (\because x > 0)$$

Now $\frac{\log\frac{1}{\varepsilon}}{x}$ decreases as x increases.

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Choose a positive integer m just $\geq \dfrac{\log \frac{1}{\varepsilon}}{x}$, then

$| f_n(x) - f(x) | < \varepsilon \ \forall\ n \geq m$ and $x > 0$.

$\Rightarrow$ $< f_n >$ is uniformly convergent on $[a, b]$, $a > 0$.

However, when $x \to 0$, $\dfrac{\log \frac{1}{\varepsilon}}{x} \to \infty$ so that it is not possible to choose a positive integer m such that

$| f_n(x) - f(x) | < \varepsilon \quad \forall\ n \geq m$ and $\forall\ x \geq 0$.

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, b]$.

Example 6. *Show that the sequence of functions $< f_n >$ defined as $f_n(x) = \dfrac{x^n}{n}$ on $(-\infty, \infty)$ is not uniformly convergent.*

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{n} = 0 \ \forall\ x$$

Let $\varepsilon > 0$ be given.

$$| f_n(x) - f(x) | = \left| \frac{x^n}{n} - 0 \right| = \frac{|x|^n}{n} < \varepsilon \text{ if } n > \frac{|x|^n}{\varepsilon}$$

When $| x | > 1$, $\dfrac{|x|^n}{\varepsilon} \to \infty$ as $n \to \infty$ so that it is not possible to choose a positive integer m such that

$| f_n(x) - f(x) | < \varepsilon \quad \forall\ n \geq m$ and $\forall\ x \in \mathbb{R}$.

$\Rightarrow$ $< f_n >$ is not uniformly convergent on $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$.

Example 7. *Show that the sequence $< f_n >$ where $f_n(x) = \dfrac{x^n}{n}$, $0 \leq x \leq 1$ converges uniformly to 0.*

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{n} = 0 \ \forall\ x$$

Let $\varepsilon > 0$ be given.

$$| f_n(x) - f(x) | = \left| \frac{x^n}{n} - 0 \right| = \frac{x^n}{n} < \varepsilon \text{ if } n > \frac{x^n}{\varepsilon}$$

Since $0 \leq x \leq 1 \Rightarrow 0 \leq x^n \leq 1$

$\therefore$ If we choose a positive integer m just $\geq \dfrac{x^n}{\varepsilon}$, then

$| f_n(x) - f(x) | < \varepsilon \quad \forall\ n \geq m$ and $\forall\ x \in [0, 1]$.

$\Rightarrow$ $< f_n >$ converges uniformly to 0.

Example 8. *Show that $x = 0$ is a point of non-uniform convergence of the sequence of functions $< f_n >$ where $f_n(x) = \dfrac{nx}{1 + n^2x^2}$.*

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + n^2x^2} = \lim_{n \to \infty} \frac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0 \ \forall\ x \in \mathbb{R}$$

Let $\varepsilon > 0$ be given. Then $| f_n(x) - f(x) | = \left| \frac{nx}{1+n^2x^2} - 0 \right| = \frac{n|x|}{1+n^2x^2} < \varepsilon$

if $\quad n\,|x| < \varepsilon + n^2x^2\varepsilon \quad$ *i.e.,* if $\varepsilon x^2n^2 - |x|\,n + \varepsilon > 0$

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i.e., if $\quad n > \dfrac{|x| + \sqrt{x^2 - 4\varepsilon^2x^2}}{2\varepsilon x^2} \quad$ *i.e.,* if $\quad n > \dfrac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon|x|}$

If we choose a positive integer m just $\geq \dfrac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon|x|}$, $x \neq 0$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad x \neq 0$$

Thus the sequence $< f_n >$ is uniformly convergent in every interval which does not contain 0.

But, when $x \to 0$, $\dfrac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon|x|} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in R.$$

Hence $x = 0$ is a point of non-uniform convergence.

Example 9. *Show that the sequence of functions* $< f_n >$ *where* $f_n(x) = \dfrac{n^2x}{1+n^2x^2}$ *is non-uniformly convergent on* $[0, 1]$.

Sol. When $\quad x = 0, \quad f_n(x) = 0 \quad \forall\, n$

When $0 < x \leq 1$, $\displaystyle\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n^2} + x^2} = \frac{1}{x}$

$$\therefore \qquad f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given. Then for $0 < x \leq 1$, we have

$$| f_n(x) - f(x) | = \left| \frac{n^2x}{1+n^2x^2} - \frac{1}{x} \right| = \left| \frac{-1}{x(1+n^2x^2)} \right| = \frac{-1}{x(1+n^2x^2)} < \varepsilon$$

if $\quad x(1 + n^2x^2) > \dfrac{1}{\varepsilon} \quad$ *i.e.,* if $\quad n > \dfrac{1}{x}\sqrt{\dfrac{1}{\varepsilon x} - 1}$

Since $\quad 0 < x \leq 1. \quad \therefore \quad 0 < \dfrac{1}{x} \leq 1$

If we choose a positive integer m just $\geq \dfrac{1}{x}\sqrt{\dfrac{1}{\varepsilon x} - 1}$, then $| f_n(x) - f(x) | < \varepsilon\ \forall\, n \geq m$ and $0 < x \leq 1$.

But, when $x \to 0$. $\dfrac{1}{x}\sqrt{\dfrac{1}{\varepsilon x} - 1} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, 1].$$

Hence $< f_n >$ is non-uniformly convergent on $[0, 1]$ and $x = 0$ is a point of non-uniform convergence.

Example 10. *Show that the sequence $< \tan^{-1} nx >$, $x \geq 0$, is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.*

Sol. Here $\quad f_n(x) = \tan^{-1} nx, x \geq 0$

$$\therefore \quad f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \tan^{-1} nx = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$, $\quad | f_n(x) - f(x) | = \left| \tan^{-1} nx - \frac{\pi}{2} \right|$

$$= | \cot^{-1} nx | \qquad \left[\because \tan^{-1} nx + \cot^{-1} nx = \frac{\pi}{2} \right]$$

$$= \cot^{-1} nx < \varepsilon$$

if $\quad nx > \cot \varepsilon \quad$ *i.e.*, if $\quad n > \frac{\cot \varepsilon}{x}$

Now $\frac{\cot \varepsilon}{x}$ decreases as x increases, the maximum value of $\frac{\cot \varepsilon}{x}$ being $\frac{\cot \varepsilon}{a}$ in $[a, b]$, $a > 0$.

If we choose a positive integer m just $\geq \frac{\cot \varepsilon}{a}$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall n \geq m \quad \text{and} \quad \forall x \in [a, b], a > 0.$$

But as $x \to 0$, $\frac{\cot \varepsilon}{x} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \;\; \forall n \geq m \quad \text{and} \quad \forall x \geq 0.$$

Hence $< f_n >$ is not uniformly convergent on $[0, b]$ but is only pointwise convergent on $[0, b]$.

Example 11. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{nx}{nx + 1}$ is uniformly convergent on $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.*

Sol. When $x = 0$, $\quad f_n(x) = 0 \;\forall n$

When $x > 0$, $\quad \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{nx + 1} = \lim_{n \to \infty} \frac{x}{x + \frac{1}{n}} = 1$

$$\therefore \quad f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have

$$| f_n(x) - f(x) | = \left| \frac{nx}{nx + 1} - 1 \right| = \left| \frac{-1}{nx + 1} \right| = \frac{1}{nx + 1} < \varepsilon$$

if $\quad nx + 1 > \frac{1}{\varepsilon} \quad$ *i.e.*, if $\quad n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$

Now $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ decreases as x increases. The maximum value of $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ on $[a, b]$, $a > 0$ is $\frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$.

$\therefore$ If we choose a positive integer m just $\geq \frac{1}{a}\left(\frac{1}{\varepsilon}-1\right)$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [a, b],\, a > 0.$$

However, when $x \to 0$, $\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right) \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \;\; \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, b].$$

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, b]$ but is only pointwise convergent on $[0, b]$.

Example 12. *Show that the sequence* $< f_n >$ *where* $f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$ *is non-uniformly convergent on* $[0, 1]$.

Sol. Here, $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{1 + n^4 x^2} = \lim_{n \to \infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4} + x^2} = 0 \;\; \forall\, x \in [0, 1]$$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have

$$| f_n(x) - f(x) | = \left| \frac{n^2 x}{1 + n^4 x^2} - 0 \right| = \frac{n^2 |x|}{1 + n^4 x^2} < \varepsilon$$

if $\qquad n^2 \,|\, x \,| < \varepsilon + \varepsilon x^2 n^4 \qquad$ *i.e.*, if $\;\; \varepsilon x^2 n^4 - |\, x \,|\, n^2 + \varepsilon > 0$

i.e., if $\qquad n^2 > \frac{|x| + \sqrt{x^2 - 4\varepsilon^2 x^2}}{2\varepsilon x^2} \qquad$ *i.e.*, if $\;\; n > \left[1 + \frac{\sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2}$

If we choose a positive integer m just $\geq \left[1 + \frac{\sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2}$,

then $\qquad | f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad$ and $\quad x \neq 0$.

$\Rightarrow \;\; < f_n >$ is uniformly convergent on $[k, 1]$ where $0 < k < 1$.

As $x \to 0$, $\left[\frac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, 1].$$

Hence $< f_n >$ is non-uniformly convergent on $[0, 1]$ and $x = 0$ is a point of non-uniform convergence.

Example 13. *Show that* $x = 0$ *is a point of non-uniform convergence of the sequence* $< f_n >$ *where* $f_n(x) = nxe^{-nx^2}$.

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} nxe^{-nx^2} = \lim_{n \to \infty} \frac{nx}{e^{nx^2}} = 0 \text{ for all } x.$$

Thus the sequence $< f_n >$ converges pointwise to 0 on any interval $[0, k]$, $k > 0$.

Let us suppose, if possible, the sequence $< f_n >$ converges uniformly on $[0, k]$, so that for any $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | = nxe^{-nx^2} < \varepsilon \quad \forall\, n \geq m \text{ and } x \geq 0 \qquad \ldots(1)$$

Let m_0 be an integer greater than m and $e^2\varepsilon^2$, then for $x = \dfrac{1}{\sqrt{m_0}}$ and $n = m_0$, (1) gives

$$m_0 \cdot \frac{1}{\sqrt{m_0}} \cdot e^{-m_0 \cdot \frac{1}{m_0}} < \varepsilon \quad \text{or} \quad \frac{\sqrt{m_0}}{e} < \varepsilon \quad \text{or} \quad m_0 < e^2\varepsilon^2$$

Thus we arrive at a contradiction.

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, k]$.

3.8. A TEST FOR UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

To determine whether a given sequence $< f_n >$ is uniformly convergent or not in a given interval, we have been using the definition of uniform convergence. Thus, we find a positive integer m, independent of x which is not easy in most of the cases. The following test is more convenient in practice and does not involve the computation of m.

Theorem. (M_n Test)

Let $< f_n >$ be a sequence of functions on I such that

$$\lim_{n \to \infty} f_n(x) = f(x), \ \forall \ x \in I$$

and let $\quad M_n = \sup \{ \mid f_n(x) - f(x) \mid : x \in I\}$

Then $< f_n >$ converges uniformly on I if and only if $\lim_{n \to \infty} M_n = 0$.

Proof. Necessary part

Let $< f_n >$ converge uniformly to f on I, so that for a given $\varepsilon > 0$, there exists a positive integer m such that

$$\mid f_n(x) - f(x) \mid < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in \text{I}$$

$$\Rightarrow \quad M_n = \sup \{\mid f_n(x) - f(x) \mid : x \in \text{I}\} < \varepsilon \quad \forall \ n \geq m$$

$$\Rightarrow \quad M_n < \varepsilon \ \ \forall \ n \geq m$$

Since $\varepsilon > 0$ is arbitrary, $M_n \to 0$ as $n \to \infty$ *i.e.*, $\lim_{n \to \infty} M_n = 0$.

Sufficiency part

Let $\lim_{n \to \infty} M_n = 0$ then for each $\varepsilon > 0$, there exists a positive integer m such that

$$M_n < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in \text{I}$$

$$\Rightarrow \quad \sup \{\mid f_n(x) - f(x) \mid : x \in \text{I}\} < \varepsilon \quad \forall \ n \geq m$$

$$\Rightarrow \quad \mid f_n(x) - f(x) \mid < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in \text{I}$$

$\Rightarrow$ the sequence $\varepsilon < f_n >$ converges uniformly to f on I.

Note 1. M_n = the maximum value of $\mid f_n(x) - f(x) \mid$ for fixed n and $x \in$ I.

Note 2. If M_n does not tend to 0, then the sequence $< f_n >$ is not uniformly convergent.

Note 3. $F(x)$ is maximum at $x = c \in$ I if

(*i*) $F'(c) = 0$ and (*ii*) $F''(c) < 0$.

ILLUSTRATIVE EXAMPLES—B

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Example 1. *Show that the sequence of functions* $< f_n >$, *where* $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in R$ *converges uniformly on any closed interval* $[a, b]$.

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \ \forall\ x \in \mathrm{R}$$

$$| f_n(x) - f(x) | = \left| \frac{x}{1+nx^2} - 0 \right| = \left| \frac{x}{1+nx^2} \right|$$

Let $$y = \frac{x}{1+nx^2} \text{ then } \frac{dy}{dx} = \frac{(1+nx)^2 \,.\, 1 - x \,.\, 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

For max. or min. $\dfrac{dy}{dx} = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = \dfrac{1}{\sqrt{n}}$

Also. $$\frac{d^2y}{dx^2} = \frac{(1+nx^2)^2\,(-2nx) - (1-nx^2)\,.\,2(1+nx^2)\,.\,2nx}{(1+nx^2)^4}$$

$$= \frac{-2nx\,(1+nx^2) - 4nx\,(1-nx^2)}{(1+nx^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}(1+1)}{(1+1)^3} = -\frac{\sqrt{n}}{2} < 0$$

$\Rightarrow$ y is maximum when $x = \dfrac{1}{\sqrt{n}}$ and maximum value of $y = \dfrac{\frac{1}{\sqrt{n}}}{1+1} = \dfrac{1}{2\sqrt{n}}$

$$\therefore \quad \mathrm{M}_n = \max_{x\in[a,b]} | f_n(x) - f(x) | = \max_{x\in[a,b]} \left| \frac{x}{1+nx^2} \right| = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Hence $< f_n >$ converges uniformly to f on $[a, b]$.

Example 2. *Show that if* $f_n(x) = \dfrac{n^2x}{1+n^4x^2}$, *then* $< f_n >$ *converges non-uniformly on* $[0, 1]$.

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2} = 0 \ \forall\ x \in [0, 1]$$

$$| f_n(x) - f(x) | = \left| \frac{n^2x}{1+n^4x^2} - 0 \right| = \left| \frac{n^2x}{1+n^4x^2} \right|$$

Let $$y = \frac{n^2x}{1+n^4x^2}$$

then $$\frac{dy}{dx} = \frac{(1+n^4x^2)\,.\,n^2 - n^2x\,.\,2n^4x}{(1+n^4x^2)^2} = \frac{n^2[1+n^4x^2 - 2n^4x^2]}{(1+n^4x^2)^2} = \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$$

For max. or min. $\dfrac{dy}{dx} = 0 \Rightarrow 1 - n^4x^2 = 0 \Rightarrow x = \dfrac{1}{n^2}$

Also, $$\frac{d^2y}{dx^2} = n^2 \cdot \frac{(1+n^4x^2)^2(-2n^4x) - (1-n^4x^2).2(1+n^4x^2).2n^4x}{(1+n^4x^2)^4}$$

$$= \frac{n^2[-2n^4x(1+n^4x^2) - 4n^4x(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$= \frac{-2n^6x[(1+n^4x^2) + 2(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^2}} = \frac{-2n^6 \cdot \frac{1}{n^2}\left(1+n^4 \cdot \frac{1}{n^4}\right)}{\left(1+n^4 \cdot \frac{1}{n^4}\right)^3} = \frac{-4n^4}{8} = -\frac{n^4}{2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{1}{n^2}$ and maximum value of y

$$= \frac{n^2 \cdot \frac{1}{n^2}}{1+n^4 \cdot \frac{1}{n^4}} = \frac{1}{2}. \text{ Also } x = \frac{1}{n^2} \to 0 \text{ as } n \to \infty.$$

$\therefore$ $$M_n = \max_{x\in[0,1]} |f_n(x) - f(x)| = \max_{x\in[0,1]} \left|\frac{n^2x}{1+n^4x^2}\right| = \frac{1}{2}$$

which does not tend to 0 as $n \to \infty$.

Hence $< f_n >$ converges non-uniformly on [0, 1].

Example 3. *Show that the sequence of functions $< f_n >$, where $f_n(x) = \frac{n^2x}{1+n^2x^2}$ is non-uniformly convergent on [0, 1].*

Sol. When $x = 0$, $f_n(x) = 0 \quad \forall\, n$

When $0 < x \le 1$, $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n^2}+x^2} = \frac{1}{x}$$

$\therefore$ $$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

When $0 < x \le 1$, $|f_n(x) - f(x)| = \left|\frac{n^2x}{1+n^2x^2} - \frac{1}{x}\right| = \frac{1}{x(1+n^2x^2)}$

Let $y = \frac{1}{x(1+n^2x^2)}$, then y is maximum when $x = \frac{1}{n}$ and maximum value of y is $\frac{n}{2}$. **[Prove it yourself]**

Also $x = \frac{1}{n} \to 0$ as $n \to \infty$., $M_n = \max_{x\in[0,1]} |f_n(x) - f(x)| = \max_{x\in[0,1]} \left[\frac{1}{x(1+n^2x^2)}\right] = \frac{n}{2}$

which does not tend to zero as $n \to \infty$.

Hence the sequence $< f_n >$ is not uniformly convergent on [0, 1].

NOTES

Example 4. *Show that the sequence of functions $< f_n >$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on [0, 1].*

Sol. For $0 < x < 1$, $f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} nx(1-x)^n$

$$= \lim_{n \to \infty} \frac{nx}{(1-x)^{-n}} \qquad \Big| \text{Form } \frac{\infty}{\infty}$$

$$= \lim_{n \to \infty} \frac{x}{-(1-x)^{-n} \log(1-x)} = \lim_{n \to \infty} \frac{-x(1-x)^n}{\log(1-x)}$$

$= 0$ since $(1-x)^n \to 0$ as $n \to \infty$

Also, when $x = 0$, $f_n(x) = 0 \quad \forall\, n$; when $x = 1$, $f_n(x) = 0 \;\forall\, n$

$\therefore$ $f(x) = 0 \quad \forall\, x \in [0, 1]$

$$| f_n(x) - f(x) | = | nx(1-x)^n - 0 | = nx(1-x)^n$$

Let $y = nx(1-x)^n$

then $$\frac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$$

$$= n(1-x)^{n-1}[(1-x) - nx] = n(1-x)^{n-1}[1-(n+1)x]$$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{n+1}$

Also, $$\frac{d^2y}{dx^2} = -n(n-1)(1-x)^{n-2}[1-(n+1)x] - n(n+1)(1-x)^{n-1}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n+1}} = -n(n+1).\left(\frac{n}{n+1}\right)^{n-1} < 0$$

$\Rightarrow$ y is maximum at $x = \frac{1}{n+1}$ and the maximum value of y is

$$\frac{n}{n+1}\left(1-\frac{1}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$$

Also, $x = \frac{1}{n+1} \to 0$ as $n \to \infty$.

$$\therefore \quad M_n = \max_{x \in [0,1]} | f_n(x) - f(x) | = \left(1-\frac{1}{n+1}\right)^{n+1} \to \frac{1}{e} \text{ as } n \to \infty.$$

Since M_n does not tend to 0 as $n \to \infty$, the sequence $< f_n >$ is not uniformly convergent on [0, 1].

Here 0 is a point of non-uniform convergence since $x \to 0$ as $n \to \infty$.

Example 5. *Show that the sequence $< x^{n-1}(1-x) >$ is uniformly convergent on [0, 1].*

Sol. Here $f_n(x) = x^{n-1}(1-x)$

For $0 < x < 1$, $f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^{n-1}(1-x) = 0$

Also when $x = 0$, $f_n(x) = 0 \quad \forall\, n$; when $x = 1$, $f_n(x) = 0 \;\forall\, n$

$\therefore$ $f(x) = 0 \quad \forall\, x \in [0, 1]$

$$| f_n(x) - f(x) | = | x^{n-1}(1-x) - 0 | = x^{n-1}(1-x)$$

Let $\qquad y = x^{n-1}(1-x)$

then $\qquad \dfrac{dy}{dx} = (n-1)\,x^{n-2}(1-x) - x^{n-1}$

$$= x^{n-2}[(n-1)(1-x) - x] = x^{n-2}[(n-1) - nx]$$

For max. or min. $\qquad \dfrac{dy}{dx} = 0 \Rightarrow \quad x = \dfrac{n-1}{n}$

Also $\qquad \dfrac{d^2y}{dx^2} = (n-2)\,x^{n-3}[(n-1) - nx] - nx^{n-2}$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{n-1}{n}} = -n\left(\frac{n-1}{n}\right)^{n-2} < 0$$

$\Rightarrow \quad y$ is maximum when $x = \dfrac{n-1}{n}$ and the maximum value of y is

$$\left(\frac{n-1}{n}\right)^{n-1}\left(1-\frac{n-1}{n}\right) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

$\therefore$
$$M_n = \max_{x\in[0,1]} |\,f_n(x) - f(x)\,| = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

$$= \frac{1}{n}\left(1-\frac{1}{n}\right)^{n}\left(1-\frac{1}{n}\right)^{-1} \to 0 \times \frac{1}{e} \times 1 = 0 \text{ as } n \to \infty$$

Hence the sequence $< f_n >$ is uniformly convergent on [0, 1].

Example 6. *Show that the sequence* $< f_n >$, *where* $f_n(x) = nx\,e^{-nx^2}$, $x \ge 0$ *is not uniformly convergent on* $[0, k]$, $k > 0$.

Sol. Here, $\qquad f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} nxe^{-nx^2} = \lim\limits_{n\to\infty} \dfrac{nx}{e^{nx^2}} \qquad \Big|$ Form $\dfrac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = 0 \;\forall\; x \in [0, k]$$

$$|\,f_n(x) - f(x)\,| = |\,nxe^{-nx^2} - 0\,| = nxe^{-nx^2}$$

Let $\qquad y = nxe^{-nx^2}$

then $\qquad \dfrac{dy}{dx} = ne^{-nx^2} + nx\,.\,e^{-nx^2}\,.\,(-2nx) = n\,e^{-nx^2}(1-2nx^2)$

For max. or min. $\qquad \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad x = \dfrac{1}{\sqrt{2n}}$

Also, $\qquad \dfrac{d^2y}{dx^2} = ne^{-nx^2}(-2nx)(1-2nx^2) + n\,e^{-nx^2}(-4\,nx)$

$$= -2n^2xe^{-nx^2}[(1-2nx^2) + 2]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{2n}}} = -\frac{2n^2}{\sqrt{2n}}e^{-1/2}\,.\,2 < 0$$

$\Rightarrow \quad y$ is maximum when $x = \dfrac{1}{\sqrt{2n}}$ and the maximum value of y is $n\,.\,\dfrac{1}{\sqrt{2n}}\,.\,e^{-1/2}$

$= \sqrt{\dfrac{n}{2e}}\,.$

NOTES

Also $$x = \frac{1}{\sqrt{2n}} \to 0 \text{ as } n \to \infty$$

$$\therefore \qquad M_n = \max_{x \in [0, 1]} \mid f_n(x) - f(x) \mid = \sqrt{\frac{n}{2e}} \to \infty \text{ as } n \to \infty$$

Since M_n does not tend to zero as $n \to \infty$, the sequence $< f_n >$ is not uniformly convergent on $[0, k]$, $k > 0$. Here 0 is a point of non-uniform convergence.

Example 7. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0, \pi]$.*

Sol. Here $$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\sin nx}{\sqrt{n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n}} \cdot \sin nx = 0 \ \forall \ x \in [0, \pi]$$

$$\mid f_n(x) - f(x) \mid = \left| \frac{\sin nx}{\sqrt{n}} - 0 \right| = \left| \frac{\sin nx}{\sqrt{n}} \right|$$

Let $$y = \frac{\sin nx}{\sqrt{n}} \text{ then } \frac{dy}{dx} = \sqrt{n} \cos nx$$

For max. or min., $$\frac{dy}{dx} = 0 \ \Rightarrow \ nx = \frac{\pi}{2} \ \text{ or } \ x = \frac{\pi}{2n}$$

Also $$\frac{d^2y}{dx^2} = - n^{3/2} \sin nx$$

$$\left.\frac{d^2y}{dx^2}\right]_{x = \frac{\pi}{2n}} = - n^{3/2} \sin \frac{\pi}{2} = - n^{3/2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{\pi}{2n}$ and the maximum value of y is $\frac{\sin \frac{\pi}{2}}{\sqrt{n}} = \frac{1}{\sqrt{n}}$.

Also $x = \frac{\pi}{2n} \to 0$ as $n \to \infty$.

$$\therefore \qquad M_n = \max_{x \in [0, \pi]} \mid f_n(x) - f(x) \mid = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

Hence the sequence $< f_n >$ converges uniformly to 0 on $[0, \pi]$.

3.9. SERIES OF REAL-VALUED FUNCTIONS

Def. *If $< f_n >$ is a sequence of real-valued functions on an interval I, then $f_1 + f_2 + \ldots\ldots + f_n + \ldots$ is called a series of real-valued functions defined on I.*

This series is denoted by $\sum_{n=1}^{\infty} f_n$ or simply by Σf_n.

For example :

(*i*) If $f_n : [0, \infty) \to R$ is defined by $f_n(x) = \dfrac{1}{n+x}$, then the series is

$$\Sigma f_n = f_1 + f_2 + f_3 + \ldots = \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \ldots$$

(*ii*) If $f_n : R \to R$ is defined by $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, then the series is

$$\Sigma f_n = f_1 + f_2 + f_3 + \ldots = \sin x + \frac{\sin 2x}{\sqrt{2}} + \frac{\sin 3x}{\sqrt{3}} + \ldots$$

3.10. CONVERGENCE (OR POINTWISE CONVERGENCE) OF A SERIES OF FUNCTIONS

Let Σf_n be a series of a functions defined on an interval I.

Let $S_1 = f_1, S_2 = f_1 + f_2, \ldots$

$S_n = f_1 + f_2 + \ldots + f_n$

then the sequence $< S_n >$ is a sequence of partial sums of the series Σf_n.

If the sequence $< S_n >$ converges pointwise on I, then the series Σf_n is said to converge pointwise on I. The limit function f of $< S_n >$ is called the pointwise sum or simply the sum of the series Σf_n and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \forall\, x \in I \quad \text{or} \quad \text{simply} \quad \Sigma f_n = f.$$

For example, consider the series Σf_n defined by $f_n(x) = x^n, -1 < x < 1$

then $\Sigma f_n(x) = x + x^2 + x^3 + \ldots + x^n + \ldots$ where $-1 < x < 1$

$$S_n(x) = x + x^2 + \ldots + x^n = \frac{x(1-x^n)}{1-x} \to \frac{x}{1+x} \text{ as } n \to \infty$$

[since $-1 < x < 1$, $x^n \to 0$ as $n \to \infty$]

$\Rightarrow$ The sequence $< S_n >$ of partial sums converges pointwise to $\dfrac{x}{1-x}$ on $(-1, 1)$.

$\Rightarrow$ The series Σf_n converges pointwise to $f(x) = \dfrac{x}{1-x}$ on $(-1, 1)$.

$\Rightarrow$ $\Sigma f_n(x) = \dfrac{x}{1-x}$ on $(-1, 1)$.

3.11. UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

Def. Let Σf_n be a series of functions defined on an interval I and $S_n = f_1 + f_2 + \ldots + f_n$. If the sequence $< S_n >$ of partial sums converges uniformly on I, then the series Σf_n is said to be uniformly convergent on I.

NOTES

Thus, *a series of functions* Σf_n *converges uniformly to a function f on an interval I if for each* $\varepsilon > 0$ *and for each* $x \in I$, *there exists a positive integer m (depending only on* ε *and not on x) such that*

$$| S_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m.$$

The uniform limit function f of $< S_n >$ is called the sum of the series Σf_n and we write $\Sigma f_n = f \ \forall \ x \in I$.

3.12. THEOREM (Cauchy's Criterion for uniform convergence of a series of functions)

A series of functions Σf_n *is uniformly convergent on an interval I if and only if for each* $\varepsilon > 0$ *and for all* $x \in I$, *there exists a positive integer m (depending only on* ε*) such that*

$$| f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) \ | < \varepsilon \ \forall \ n \geq m, \ p \in N.$$

Proof. Σf_n is uniformly convergent on I.

$\Leftrightarrow$ The sequence $< S_n >$ of its partial sums is uniformly convergent on I.

$\Leftrightarrow$ By Cauchy's general principle of convergence of a sequence, for each $\varepsilon > 0$ and for all $x \in I$, there exists a positive integer m (depending only on ε and not on x) such that

$$| S_{n+p}(x) - S_n(x) | < \varepsilon \ \forall \ n \geq m, \ p \in N$$

i.e., $\quad | f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) \ | < \varepsilon \ \forall \ n \geq m, \ p \in N.$

Note 1. By definition, uniform convergence of a series implies pointwise convergence.

Note 2. The method of testing the uniform convergence of a series Σf_n, by definition, involves finding S_n which is not always easy. The following test avoids S_n.

3.13. THEOREM (Weierstrass's M-Test)

A series of functions $\sum_{n=1}^{\infty} f_n$ *converges uniformly (and absolutely) on an interval I if there exists a convergent series* $\sum_{n=1}^{\infty} M_n$ *of non-negative terms (i.e.,* $M_n \geq 0 \quad \forall \ n \in N$*) such that*

$$| f_n(x) | \leq M_n \quad \forall \ n \in N \quad \text{and} \quad \forall \ x \in I.$$

Proof. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's criterion, for each $\varepsilon > 0$, there exists a positive integer m such that

$$| M_{n+1} + M_{n+2} + \ldots\ldots + M_{n+p} | < \varepsilon \ \forall \ n \geq m, \ p \in N \quad \ldots(1)$$

Now, for all $x \in I$, $| f_n(x) | \leq M_n$ $\quad \ldots(2)$

$$\therefore \quad | f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) \ | \leq | f_{n+1}(x) | + | f_{n+2}(x) | + \ldots\ldots + | f_{n+p}(x) |$$
$$\leq M_{n+1} + M_{n+2} + \ldots\ldots + M_{n+p} \quad \text{[by (2)]}$$
$$< \varepsilon \ \forall \ n \geq m, \ p \in N \quad \text{[by (1)]}$$

$\Rightarrow$ By Cauchy's criterion, the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on I.

Also, $\quad |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots\ldots + |f_{n+p}(x)| < \varepsilon \ \forall\ n \geq m,\ p \in \mathrm{N}$ and $x \in \mathrm{I}$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent on I.

Hence the series $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely on I.

ILLUSTRATIVE EXAMPLES—C

Example 1. *Show that the series* $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ *is uniformly convergent in (0, b) b > 0 but is not so in (0, ∞).*

Sol. The given series is $\sum_{n=1}^{\infty} \frac{x}{n(n+1)} = \sum_{n=1}^{\infty} f_n(x)$

so that
$$f_n(x) = \frac{x}{n(n+1)} = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$\Rightarrow$
$$f_1(x) = x\,[1 - \tfrac{1}{2}]$$
$$f_2(x) = x\,[\tfrac{1}{2} - \tfrac{1}{3}]$$
$$f_3(x) = x\,[\tfrac{1}{3} - \tfrac{1}{4}]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_n(x) = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$\therefore$
$$S_n(x) = f_1(x) + f_2(x) + \ldots\ldots + f_n(x) = x\left[1 - \frac{1}{n+1}\right] = \frac{nx}{n+1}$$

$$S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{nx}{n+1} = \begin{cases} 0, & \text{if } x = 0 \\ x, & \text{if } x > 0 \end{cases}$$

$\therefore$ For $x > 0$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{nx}{n+1} - x\right| = \left|\frac{-x}{n+1}\right| = \frac{x}{n+1} < \varepsilon$$

if
$$n + 1 > \frac{x}{\varepsilon} \quad \text{or} \quad \text{if } n > \frac{x}{\varepsilon} - 1$$

If we choose a positive integer m just $\geq \frac{x}{\varepsilon} - 1$, then $|S_n(x) - S(x)| < \varepsilon\ \forall\ n \geq m$ and $x > 0$

Also if $x = 0$, $|S_n(x) - S(x)| = 0 < \varepsilon \quad \forall\ n \geq 1$ so that $m = 1$ works in this case.

But when $x \to \infty$, $n \to \infty$.

This shows that the same value of m cannot be found which serves uniformly for every x in $(0, \infty)$.

But if the interval is $(0, b)$ where b is any positive number, then the maximum value of $\frac{x}{\varepsilon} - 1$ is $\frac{b}{\varepsilon} - 1$ on $(0, b)$.

$\therefore$ If we choose a positive integer m just $\geq \frac{b}{\varepsilon} - 1$, then the same value of m serves equally for every value of x in $(0, b)$, $b > 0$.

Thus, the sequence $< S_n >$ converges uniformly in $(0, b)$ but not in $(0, \infty)$.

NOTES

Hence the series $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergent in $(0, b)$, $b > 0$ but not so in $(0, \infty)$.

Example 2. *Show that $x = 0$ is a point of non-uniform convergence of the series*

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots$$

Sol. Here $S_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots + \frac{x^2}{(1+x^2)^{n-1}}$ (which is a G.P.)

$$= \frac{x^2\left[1 - \frac{1}{(1+x^2)^n}\right]}{1 - \frac{1}{1+x^2}} = (1+x^2)\left[1 - \frac{1}{(1+x^2)^n}\right] = (1+x^2) - \frac{1}{(1+x^2)^{n-1}}$$

$$\therefore \qquad S(x) = \lim_{n \to \infty} S_n(x) = \begin{cases} 1+x^2, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Now for $x \neq 0$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| (1+x^2) - \frac{1}{(1+x^2)^{n-1}} - (1+x^2) \right| = \frac{1}{(1+x^2)^{n-1}} < \varepsilon$$

if $\qquad (1+x^2)^{n-1} > \frac{1}{\varepsilon}$ or if $n - 1 > \frac{\log \frac{1}{\varepsilon}}{\log(1+x^2)}$ or if $n > 1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+x^2)}$

This shows that if $x \to 0$, then $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of $< S_n >$ and hence of the given series.

Note. However, if we consider the interval $[a, \infty)$, $a > 0$, then the maximum value of

$$1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+x^2)} \text{ is } 1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+a^2)}.$$

If we choose a positive integer m just $\geq 1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+a^2)}$, then $| S_n(x) - S(x) | < \varepsilon \ \forall\ n \geq m$ and $\forall\ x \in [a, \infty)$.

Thus the series is uniformly convergent in $[a, \infty)$, $a > 0$ and non-uniformly convergent in $[0, \infty)$.

Example 3. *Show that the series* $\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots\ldots$ *is uniformly convergent on $[a, \infty)$, $a > 0$. Show that the series is non-uniformly convergent near $x = 0$.*

Sol. The given series is $\sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1](nx+1)} = \sum_{n=1}^{\infty} f_n(x)$

so that $\qquad f_n(x) = \frac{x}{[(n-1)x+1](nx+1)} = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$

$\Rightarrow$ $$f_1(x) = 1 - \frac{1}{x+1}$$

$$f_2(x) = \frac{1}{x+1} - \frac{1}{2x+1}$$

$$f_3(x) = \frac{1}{2x+1} - \frac{1}{3x+1}$$

....

$$f_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$\therefore$ $$S_n(x) = 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}$$

$$S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{nx}{nx+1} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

For $x > 0$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{nx}{nx+1} - 1\right| = \left|\frac{-1}{nx+1}\right| = \frac{1}{nx+1} < \varepsilon$$

if $$nx + 1 > \frac{1}{\varepsilon} \quad \text{or if} \quad n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$$

This shows that if $x \to 0$, $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in (0, \infty).$$

Thus the convergence is non-uniform near $x = 0$.

Since $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ increases as x decreases, if we consider the interval $[a, \infty)$, $a > 0$, then the maximum value of $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ is $\frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$. If we choose a positive integer m just $\geq \frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$ then

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [a, \infty).$$

Hence the series is uniformly convergent on $[a, \infty)$.

Example 4. *Show that the series* $\sum_{n=1}^{\infty}\left(\frac{n}{x+n} - \frac{n-1}{x+n-1}\right)$ *is uniformly convergent on any finite interval.*

Sol. Here $$f_n(x) = \frac{n}{x+n} - \frac{n-1}{x+n-1}$$

$\therefore$ $$f_1(x) = \frac{1}{x+1} - 0$$

$$f_2(x) = \frac{2}{x+2} - \frac{1}{x+1}$$

$$f_3(x) = \frac{3}{x+3} - \frac{2}{x+2}$$

....

$$f_n(x) = \frac{n}{x+n} - \frac{n-1}{x+n-1}$$

$$\Rightarrow \qquad S_n(x) = \frac{n}{x+n}$$

Now proceed as in Example 4, Illustrative Examples—A.

NOTES

Example 5. *Show that x = 0 is a point of non-uniform convergence of the series*

$$\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right].$$

Sol. Here $\qquad f_n(x) = \frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}$

$$\therefore \qquad f_1(x) = \frac{x}{1+x^2} - 0$$

$$f_2(x) = \frac{2x}{1+2^2x^2}-\frac{x}{1+x^2}$$

$$f_3(x) = \frac{3x}{1+3^2x^2}-\frac{2x}{1+2^2x^2}$$

....

$$f_n(x) = \frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}$$

$$\Rightarrow \qquad S_n(x) = \frac{nx}{1+n^2x^2}$$

Now proceed as in Example 8, Illustrative Examples—A.

Example 6. *Test the series* $\sum_{n=1}^{\infty} x\left[\frac{n}{1+n^2x^2}-\frac{n+1}{1+(n+1)^2x^2}\right]$ *for uniform convergence in [0, 1].*

Sol. Here $\qquad f_n(x) = \frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}$

$$\therefore \qquad f_1(x) = \frac{x}{1+x^2}-\frac{2x}{1+2^2x^2}$$

$$f_2(x) = \frac{2x}{1+2^2x^2}-\frac{3x}{1+3^2x^2}$$

$$f_3(x) = \frac{3x}{1+3^2x^2}-\frac{4x}{1+4^2x^2}$$

....

$$f_n(x) = \frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\Rightarrow \qquad S_n(x) = \frac{x^2}{1+x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\therefore \qquad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} \frac{x}{1+x^2}, & \text{if } 0 < x \le 1 \\ 0 \quad , & \text{if } x = 0 \end{cases}$$

For $0 < x \le 1$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| \frac{x}{1+x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}-\frac{x}{1+x^2}\right|$$