NOTES

UNIT 7 HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS

STRUCTURE

7.0. LEARNING OBJECTIVES

After going through this unit you will be able to:

- Define homogeneous linear equation
- General solution of homogeneous linear differential equation
- Find equations reducible to homogeneous linear form

7.1. INTRODUCTION

In the preceding chapter, we learnt the method of solving linear differential equations with constant coefficients. In this chapter, we shall learn the method of solving linear differential equations with a particular type of variable coefficients, namely, the homogeneous linear equations.

7.2. HOMOGENEOUS LINEAR EQUATION

A differential equation of the form

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots + P_n y = Q$$

where $P_0 \neq 0$, P_1, P_2, ..., P_n are all constants and Q is a function of x is called a **homogeneous linear differential equation** of order n.

For example,

$$2x^3 \frac{d^3 y}{dx^3} + 6x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 7y = \sin x$$

is a homogeneous linear differential equation of order 3.

Remark. A homogeneous linear differential equation is also called a **Cauchy linear equation.**

7.3. GENERAL SOLUTION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

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Let $$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = Q \qquad \ldots(1)$$

where $P_0 \neq 0, P_1, P_2, \ldots\ldots, P_n$ are all constants and Q is a function of x be a homogeneous linear differential equation of order n.

Let $z = \log x$ *i.e.*, $x = e^z$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right] = \frac{1}{x^2}\left(\frac{d^3y}{dz^3}\frac{dz}{dx} - \frac{d^2y}{dz^2}\frac{dz}{dx}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)$$

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On putting D for $\frac{d}{dz}$ and clearing the fractions, we get

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)y = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = (D^3 - 3D^2 + 2D)y = D(D-1)(D-2)y$$

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In general, $$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2) \ldots\ldots (D-(n-1))y.$$

$\therefore$ (1) becomes

$$[P_0 D(D-1)(D-2) \ldots\ldots (D-(n-1)) + P_1 D(D-1)(D-2) \ldots\ldots (D-(n-2)) + \ldots\ldots + P_{n-1} D + P_n]y = Z \qquad \ldots(2)$$

where Z is the function of z into which Q is changed.

(2) is a linear differential equation with constant coefficients. This equation can be solved. Let the general solution of equation (2) be $y = f(z)$.

$\therefore$ The general solution of the given equation (1) is $y = f(\log x)$.

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Example 1. *Solve* $x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0.$

Sol. We have $x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0$ ...(1)

Let $z = \log x. \quad \therefore \quad x = e^z$

$\therefore \quad x\frac{d}{dx} = D, \; x^2 \frac{d^2}{dx^2} = D(D-1)$

and $x^3 \frac{d^3}{dx^3} = D(D-1)(D-2)$, where $D = \frac{d}{dz}$.

$\therefore$ (1) becomes, $[D(D-1)(D-2) + 6D(D-1) + 4D - 4]y = 0$

i.e., $(D^3 + 3D^2 - 4)y = 0$ ...(2)

$\therefore$ The A.E. of (2) is

$$D^3 + 3D^2 - 4 = 0$$

$\Rightarrow \quad (D-1)(D^2 + 4D + 4) = 0$

$\Rightarrow \quad (D-1)(D+2)^2 = 0$

$\therefore \quad D = 1, -2, -2$

$\therefore$ The general solution of (2) is $y = c_1e^z + (c_2 + c_3z)\, e^{-2z}$.

$\therefore$ The general solution of (1) is

$$\mathbf{y = c_1x + (c_2 + c_3 \log x)\, x^{-2}}. \qquad (\because \; e^{-2z} = (e^z)^{-2} = x^{-2})$$

Example 2. *Solve* $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}.$

Sol. We have $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$ ...(1)

Let $z = \log x \quad \therefore \quad x = e^z$

$\therefore \quad x\frac{d}{dx} = D$ and $x^2 \frac{d^2}{dx^2} = D(D-1)$, where $D = \frac{d}{dz}$.

$\therefore$ (1) $\Rightarrow \quad [D(D-1) - 2]y = e^{2z} + e^{-z}$

$\Rightarrow \quad (D^2 - D - 2)y = e^{2z} + e^{-z}$ ...(2)

$\therefore$ The A.E. of (2) is $D^2 - D - 2 = 0. \quad \therefore \quad D = -1, 2$

$\therefore$ C.F. $= c_1e^{-z} + c_2e^{2z} = c_1x^{-1} + c_2x^2$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}(e^{2z} + e^{-z}) = \frac{1}{D^2 - D - 2}e^{2z} + \frac{1}{D^2 - D - 2}e^{-z}$$

$$= z\frac{1}{2D-1}e^{2z} + z\frac{1}{2D-1}e^{-z} \qquad \text{(Case of failure)}$$

$$= z\frac{1}{2(2)-1}e^{2z} + z\frac{1}{2(-1)-1}e^{-z} = \frac{z}{3}e^{2z} + \frac{z}{-3}e^{-z}$$

$$= \frac{z}{3}(e^{2z} - e^{-z}) = \frac{(\log x)}{3}(x^2 - x^{-1}) = \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x.$$

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$$\therefore \quad \mathbf{y = c_1x^{-1} + c_2x^2 + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x}.$$

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Example 3. *Solve* $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$

Sol. We have

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x. \quad ...(1)$$

Let $z = \log x. \quad \therefore \quad x = e^z$

$\therefore \quad x\frac{d}{dx} = D \quad$ and $\quad x^2\frac{d^2}{dx^2} = D(D-1)$, where $D = \frac{d}{dz}$.

$\therefore$ (1) $\Rightarrow$ $[D(D-1) - D + 1]y = 2z$ i.e., $(D^2 - 2D + 1)y = 2z$ ...(2)

$\therefore$ The A.E. of (2) is $(D-1)^2 = 0. \quad \therefore \quad D = 1, 1$

C.F. $= (c_1 + c_2 z)e^z = (c_1 + c_2 \log x)\, x$

$$\text{P.I.} = \frac{1}{(D-1)^2}\, 2z = 2\,\frac{1}{(1-D)^2}\, z = 2(1-D)^{-2} z = 2(1 + 2D + ...)\, z$$

$$= 2[z + 2 \cdot 1] = 2z + 4 = 2\log x + 4$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$\therefore \quad \mathbf{y = (c_1 + c_2 \log x)\, x + 2\log x + 4.}$

Example 4. *Solve* $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$.

Sol. We have $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin\log x.$ ...(1)

Let $z = \log x \quad \therefore \quad x = e^z$

$\therefore \quad x\frac{d}{dx} = D \quad$ and $\quad x^2\frac{d^2}{dx^2} = D(D-1)$, where $D = \frac{d}{dz}$.

$\therefore$ (1) becomes $[D(D-1) - 3D + 5]y = \sin z$

$\Rightarrow \quad (D^2 - 4D + 5)y = \sin z$ ...(2)

$\therefore$ The A.E. of (2) is $D^2 - 4D + 5 = 0. \quad \therefore \quad D = 2 \pm i$

$\therefore$ C.F. $= e^{2z}(c_1\cos z + c_2 \sin z) = x^2[c_1\cos(\log x) + c_2\sin(\log x)]$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5}\sin z = \frac{1}{-1 - 4D + 5}\sin z$$

$$= \frac{1}{4(1-D)}\sin z = \frac{1}{4}\frac{(1+D)}{1-D^2}\sin z = \frac{1}{4}\frac{(1+D)}{1-(-1)}\sin z = \frac{1}{8}(1+D)\sin z$$

$$= \frac{1}{8}[\sin z + \cos z] = \frac{1}{8}[\sin(\log x) + \cos(\log x)]$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$\therefore \quad \mathbf{y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] + \frac{1}{8}[\sin(\log x) + \cos(\log x)].}$

Example 5. *Solve* $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

Sol. We have $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ ...(1)

Let $z = \log x \quad \therefore \quad x = e^z$

$\therefore \quad x\dfrac{d}{dx} = D \quad$ and $\quad x^2 \dfrac{d^2}{dx^2} = D^2$, where $D = \dfrac{d}{dz}$.

$\therefore \quad (1) \Rightarrow \quad [D(D-1)+3D+1]y = \dfrac{1}{(1-e^z)^2} \quad$ *i.e.*, $\quad (D^2+2D+1)y = \dfrac{1}{(1-e^z)^2}$

or $$(D+1)^2 y = (1-e^z)^{-2} \qquad ...(2)$$

$\therefore$ The A.E. of (2) is $\quad (D+1)^2 = 0. \quad \therefore \quad D = -1, -1$

$\therefore$ C.F. $= (c_1 + c_2 z)e^{-z} = (c_1 + c_2 \log x)\, x^{-1}$

$$\text{P.I.} = \frac{1}{(D+1)^2}(1-e^z)^{-2} = \frac{1}{(D+1)}\frac{1}{(D+1)}\frac{1}{(1-e^z)^2}$$

$$= \frac{1}{(D+1)}e^{-z}\int \frac{1}{(1-e^z)^2}e^z\,dz \qquad \left[\because \frac{1}{D-\alpha}Q = e^{\alpha x}\int Qe^{-\alpha x}\,dx\right]$$

$$= \frac{1}{D+1}e^{-z}\int \frac{1}{(1-t)^2}\,dt, \qquad \text{where } e^z = t$$

$$= \frac{1}{D+1}e^{-z}\left[\frac{-(1-t)^{-2+1}}{-2+1}\right] = \frac{1}{D+1}e^{-z}(1-t)^{-1} = \frac{1}{D+1}e^{-z}(1-e^z)^{-1}$$

$$= e^{-z}\int e^{-z}(1-e^z)^{-1}e^z\,dz \qquad \left[\because \frac{1}{D-\alpha}Q = e^{\alpha x}\int Qe^{-\alpha x}\,dx\right]$$

$$= e^{-z}\int \frac{1}{1-e^z}\,dz = e^{-z}\int \frac{e^{-z}\,dz}{e^{-z}-1} = e^{-z}[-\log(e^{-z}-1)]$$

$$= -e^{-z}\log(e^{-z}-1) = -x^{-1}\log(x^{-1}-1) = -\frac{1}{x}\log\left(\frac{1}{x}-1\right)$$

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$$\therefore \qquad y = (c_1 + c_2\log x)\,x^{-1} - \frac{1}{x}\log\left(\frac{1}{x}-1\right)$$

or $$\mathbf{y = \frac{1}{x}\left[c_1 + c_2 \log x - \log\left(\frac{1}{x}-1\right)\right].}$$

EXERCISE 1

Solve the following differential equations :

1. $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} + 25y = 5$

2. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$

3. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

4. $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x^3$

5. $(x^2D^2 + xD - 1)y = x^m$

6. $x^2\dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

7. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

8. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

9. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

10. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$

11. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2\log x$

12. $x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + 13y = \log x.$

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13. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x$ 14. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x + \sin x.$

15. $(x^3D^3 + 3x^2D^2 + xD + 8)\, y = 65 \cos (\log x)$

16. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\, y = (1 + \log x)^2$

17. $(x^2D^2 - xD + 4)y = \cos (\log x) + x \sin (\log x)$

Answers

1. $y = \frac{1}{5} + x^{-4} [c_1 \cos (\log x^3) + c_2 \sin (\log x^3)]$ 2. $y = x^2 (c_1 + c_2 \log x) + x^2 (\log x)^2$

3. $y = c_1x^4 + c_2x^{-1} + \frac{1}{5} x^4 \log x$ 4. $y = (c_1 - c_2 \log x)\, x^{-2} + \frac{x^3}{25}$

5. $y = c_1x + c_2x^{-1} + \frac{x^m}{m^2 - 1}$ 6. $y = c_1 + c_2x^3 + c_3x^4 + \frac{2}{3} x$

7. $y = c_1x^{-5} + c_2x^4 - \frac{1}{14} x^2 - \frac{1}{9} x - \frac{1}{20}$ 8. $y = c_1x^2 + x^{5/2}\left[c_2x^{\frac{1}{2}\sqrt{21}} + c_3x^{-\frac{1}{2}\sqrt{21}}\right] - \frac{x^3}{5}$

9. $y = c_1x^{-1} + x \{c_2 \cos (\log x) + c_3 \sin (\log x)\} + 5x + \frac{2}{x} \log x$

10. $y = x[c_1 \cos (\log x) + c_2 \sin (\log x)] + x \log x$

11. $y = c_1x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left(\frac{2}{3} + \log x\right)$

12. $y = x^{-7/2} \left[c_1 \cos\left(\frac{\sqrt{3}}{2} \log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} \log x\right)\right] + \frac{1}{169} (13 \log x - 7)$

13. $y = c_1x + c_2x^{-1} + \left(1 - \frac{1}{x}\right) e^x$ 14. $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{x}{6} - \frac{1}{x^2} \sin x$

15. $y = c_1x^{-2} + x[c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) - \sin (\log x)$

16. $y = (c_1 + c_2 \log x) \cos (\log x) + (c_3 + c_4 \log x) \sin (\log x) + (\log x)^2 + 2 \log x - 3$

17. $y = x [c_1 \cos (\sqrt{3} \log x) + c_2 \sin (\sqrt{3} \log x)] + \frac{1}{13} [3 \cos (\log x) - 2 \sin (\log x)] + \frac{1}{2} x \sin (\log x).$

7.4. EQUATIONS REDUCIBLE TO HOMOGENEOUS LINEAR FORM

Consider the differential equation

$$P_0 (a + bx)^n \frac{d^n y}{dx^n} + P_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 (a + bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n\, y = Q^* \quad \ldots(1)$$

where $P_0 \neq 0$, P_1, P_2, ..., P_n are all constants and Q is a function of x. This equation can be reduced to a homogeneous linear equation by putting $a + bx$ equal to some other variable.

We solve the equation (1).

*A differential equation of this type is called a **Legendre linear equation.**

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Let $\quad z = \log(a + bx) \quad$ *i.e.*, $\quad a + bx = e^z$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{b}{a+bx} = \frac{b}{a+bx}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = \frac{b}{a+bx}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{b^2}{(a+bx)^2}\frac{dy}{dz}$$

$$= \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right]$$

$$= \frac{b^2}{(a+bx)^2}\left[\frac{d^3y}{dz^3}\frac{dz}{dx} - \frac{d^2y}{dz^2}\frac{dz}{dx}\right] - \frac{2b^3}{(a+bx)^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$= \frac{b^3}{(a+bx)^3}\left[\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2} - 2\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right]$$

$$= \frac{b^3}{(a+bx)^3}\left[\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right]$$

....

....

On putting D for $\frac{d}{dz}$ and clearing the fractions, we get

$$(a+bx)\frac{dy}{dx} = b\text{D}y$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2(\text{D}^2 - \text{D})\,y = b^2\,\text{D}(\text{D}-1)y$$

$$(a+bx)^3\frac{d^3y}{dx^3} = b^3(\text{D}^3 - 3\text{D}^2 + 2\text{D})\,y = b^3\,\text{D}(\text{D}-1)(\text{D}-2)y$$

....

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In general,

$$(a+bx)^n\frac{d^ny}{dx^n} = b^n\,\text{D}(\text{D}-1)(\text{D}-2)\,......\,(\text{D}-(n-1))y$$

$\therefore$ (1) becomes

$$[\text{P}_0b^n\,\text{D}(\text{D}-1)(\text{D}-2)\,......\,(\text{D}-(n-1)) + \text{P}_1b^{n-1}\,\text{D}(\text{D}-1)(\text{D}-2)\,......\,(\text{D}-(n-2))$$
$$+ \,......\, + \text{P}_{n-1}\,b\text{D} + \text{P}_n]\,y = \text{Z} \qquad ...(2)$$

where Z is the function of z into which Q is changed.

(2) is a linear differential equation with constant coefficients. This equation can be solved. Let the general solution of equation (2) be $y = f(z)$.

$\therefore$ The general solution of the given equation (1) is $y = f(\log(a + bx))$.

NOTES

Example 1. *Solve* $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x.$

Sol. We have $\qquad (x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x.$ ...(1)

Let $\qquad z = \log (x+a). \quad \therefore \quad x + a = e^z$

$\therefore \quad (x+a)\dfrac{d}{dx} = 1.D = D, \quad (x+a)^2 \dfrac{d^2}{dx^2} = 1^2 D(D-1) = D(D-1),$ where $D = \dfrac{d}{dz}.$

$\therefore \quad (1) \quad \Rightarrow \qquad [D(D-1) - 4D + 6]y = e^z - a$

$\Rightarrow \qquad (D^2 - 5D + 6)y = e^z - a.$ ...(2)

$\therefore$ The A.E. of (2) is $\quad D^2 - 5D + 6 = 0, \quad \therefore \quad D = 2, 3$

$\therefore \quad$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1(x+a)^2 + c_2(x+a)^3$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}(e^z - a) = \frac{1}{D^2 - 5D + 6} e^z - a\frac{1}{D^2 - 5D + 6} e^{0z}$$

$$= \frac{1}{1^2 - 5(1) + 6} e^z - a\frac{1}{0 - 0 + 6} e^{0z} = \frac{e^z}{2} - \frac{a}{6} = \frac{x+a}{2} - \frac{a}{6} = \frac{1}{6}(3x + 2a)$$

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$\therefore \qquad \mathbf{y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{6}(3x + 2a).}$

Example 2. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x).$

Sol. We have

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x). \quad ...(1)$$

Let $\qquad z = \log(1+x). \quad \therefore \quad 1 + x = e^z$

$\therefore \qquad (1+x)\dfrac{d}{dx} = 1.D = D, \quad (1+x)^2 \dfrac{d^2}{dx^2} = 1^2 \cdot D(D-1) = D(D-1).$

where $\qquad D = \dfrac{d}{dz}$

$\therefore \quad (1) \quad \Rightarrow \quad (D(D-1) + D + 1)\,y = 4\cos z \quad \Rightarrow \quad (D^2 + 1)\,y = 4\cos z$ ...(2)

$\therefore$ The A.E. of (2) is $\quad D^2 + 1 = 0, \quad \therefore \quad D = \pm i$

$\therefore \quad$ C.F. $= e^{0z}(c_1 \cos 1 \cdot z + c_2 \sin 1 \cdot z) = c_1 \cos z + c_2 \sin z$

$= c_1 \cos(\log(1+x)) + c_2 \sin(\log(1+x))$

$$\text{P.I.} = \frac{1}{D^2+1} 4\cos z = 4\frac{1}{D^2+1}\cos z = 4z \cdot \frac{1}{2D}\cos z$$

$= 2z \sin z = 2\log(1+x)\sin\log(1+x)$

$$\left(\because \frac{1}{D}\cos z = \int \cos z\, dz = \sin z\right)$$

The general solution of (1) is $y =$ C.F. + P.I.

$\therefore \qquad \mathbf{y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x)\sin\log(1+x).}$

NOTES

EXERCISE 2

Solve the following differential equations :

1. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$
2. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$
3. $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$
4. $(2x-1)^3 \dfrac{d^3y}{dx^3} + (2x-1)\dfrac{dy}{dx} - 2y = 0$
5. $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = 4x^2 + 14x + 12.$

Answers

1. $y = (1+2x)^2\,[(\log(1+2x))^2 + c_1 + c_2 \log(1+2x)]$
2. $y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2} + \dfrac{1}{108}\,[(3x+2)^2 \log(3x+2) + 1]$
3. $y = c_1(5+2x)^{2+\sqrt{2}} + c_3(2x-1)^{-\sqrt{3}/2}$
4. $y = (2x-1)[c_1 + c_2 (2x-1)^{\sqrt{3}/2} + c_3 (2x-1)^{-\sqrt{3}/2}]$
5. $y = c_1 + c_2 \log(1+x) + x^2 + 8x + \{\log(x+1)\}^2.$

UNIT 8 TOTAL DIFFERENTIAL EQUATIONS

NOTES

STRUCTURE

8.0. LEARNING OBJECTIVES

After going through this unit you will be able to:

- Define total differential equation
- Find integrability of a total differential equation
- Find solution of total differential equation and of exact total differential equation

8.1. INTRODUCTION

Till now we have been discussing the methods of solving differential equations involving only two variables. In the present chapter, we shall study differential equations involving more than two variables. A differential equation involving more than two variables is either *ordinary* or *partial*. If only one independent variable is involved then the differential equation is *ordinary* and the differential equation is *partial* if more than one independent variable are involved in the differential equation. We shall be discussing ordinary differential equations containing more than two variables.

8.2. TOTAL DIFFERENTIAL EQUATION

Let $P_1, P_2, \ldots\ldots, P_n$ be n functions of n variables $x_1, x_2, \ldots\ldots, x_n$. The ordinary differential equation of the form

$$P_1\, dx_1 + P_2\, dx_2 + \ldots\ldots + P_n\, dx_n = 0$$

is called a **total differential equation** in n variables.

In particular, if P, Q and R be functions of three variables x, y and z, then

$$Pdx + Qdy + Rdz = 0$$

is a total differential equation in three variables.

We shall be studying the methods of solving total differential equations involving only three variables.

8.3. HOMOGENEOUS TOTAL DIFFERENTIAL EQUATION

Let $Pdx + Qdy + Rdz = 0$ be a total differential equation in three variables x, y and z. Here each of P, Q and R is a function of x, y and z. This differential equation is called a **homogeneous total differential equation** or simply a **homogeneous equation** if the functions P, Q and R are homogeneous and of the same degree.

8.4. INTEGRABLE TOTAL DIFFERENTIAL EQUATION

Let $$Pdx + Qdy + Rdz = 0 \qquad \ldots(1)$$

be a total differential equation in three variables x, y and z. Here each of P, Q, R is a function of x, y and z.

The total differential equation (1) is said to be an **integrable total differential equation** if there exists a function $u(x, y, z)$ of x, y and z such that the equation (1) can be obtained by differentiating the relation $u(x, y, z) = c$, where c is an arbitrary constant.

In such a case, the relation $u(x, y, z) = c$ is called the **solution** of the total differential equation (1). The solution of the differential equation (1) is also called the **complete integral** of the total differential equation (1).

8.5. INTEGRABILITY OF A TOTAL DIFFERENTIAL EQUATION

Theorem. *Let $Pdx + Qdy + Rdz = 0$ be a total differential equation, where P, Q and R are functions of x, y and z. This equation is integrable if and only if*

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

NOTES

Proof. The given total differential equation is

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \ldots(1)$$

Necessity. Let the equation (1) be integrable and its solution be

$$u(x, y, z) = c \qquad \ldots(2)$$

Differentiating (2), we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \qquad \ldots(3)$$

Since (2) is a solution of (1), we have

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=\lambda(P\,dx+Q\,dy+R\,dz).$$

where λ is some function of x, y and z. In particular if no multiplication on the right side is required then we have $\lambda(x, y, z) = 1$

$$\therefore \qquad \frac{\partial u}{\partial x}=\lambda P,\quad \frac{\partial u}{\partial y}=\lambda Q,\quad \frac{\partial u}{\partial z}=\lambda R$$

Assuming $\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}$, we have $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$.

$$\Rightarrow \qquad \frac{\partial}{\partial y}(\lambda P)=\frac{\partial}{\partial x}(\lambda Q)$$

$$\Rightarrow \qquad \lambda\frac{\partial P}{\partial y}+P\frac{\partial \lambda}{\partial y}=\lambda\frac{\partial Q}{\partial x}+Q\frac{\partial \lambda}{\partial x}$$

$$\Rightarrow \qquad \lambda\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \lambda}{\partial x}-P\frac{\partial \lambda}{\partial y} \qquad \ldots(4)$$

Similarly, $\dfrac{\partial^2 u}{\partial z\partial y}=\dfrac{\partial^2 u}{\partial y\partial z}$ implies

$$\lambda\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \lambda}{\partial y}-Q\frac{\partial \lambda}{\partial z} \qquad \ldots(5)$$

and $\dfrac{\partial^2 u}{\partial x\partial z}=\dfrac{\partial^2 u}{\partial z\partial x}$ implies

$$\lambda\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \lambda}{\partial z}-R\frac{\partial \lambda}{\partial x} \qquad \ldots(6)$$

Multiplying (4), (5) and (6) by R, P and Q respectively and adding, we get

$$\lambda R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+\lambda P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+\lambda Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=0$$

$$\Rightarrow \qquad P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (\because\ \lambda\neq 0)$$

Sufficiency. Let

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad \ldots(7)$$

Let μ be any function of x, y and z.

Let $\qquad P_1=\mu P,\ Q_1=\mu Q,\ R_1=\mu R$

We shall show that (7) also hold if P, Q and R are replaced by P_1, Q_1 and R_1 respectively.

$$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \frac{\partial}{\partial z}(\mu Q) - \frac{\partial}{\partial y}(\mu R)$$

$$= \left(\mu \frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z}\right) - \left(\mu \frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}\right)$$

$$= \mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + \left(Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y}\right)$$

Similarly, $$\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z} = \mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \left(R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z}\right)$$

and $$\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x} = \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + \left(P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x}\right)$$

Multiplying these relations by P_1, Q_1 and R_1 and adding, we get

$$P_1\left(\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y}\right) + Q_1\left(\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z}\right) + R_1\left(\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x}\right)$$

$$= \mu^2\left[P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right]$$

$$+ \mu\left[P\left(Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y}\right) + Q\left(R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z}\right) + R\left(P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x}\right)\right]$$

$$= \mu^2(0) + \mu(0) = 0$$

$$\therefore \quad P_1\left(\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y}\right) + Q_1\left(\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z}\right) + R_1\left(\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x}\right) = 0$$

Thus if relation (7) holds for the equation $P dx + Q dy + R dz = 0$, then a similar relation holds for the equation

$$\mu P\, dx + \mu Q\, dy + \mu R\, dz = 0 \qquad \ldots(8)$$

where μ is any function of x, y and z.

Now $P\,dx + Q\,dy$ may or may not be an exact differential with respect to x and y. If $P\,dx + Q\,dy$ is not an exact differential with respect to x and y, an integrating factor μ can be found for it and equation (8) can then be taken as the equation to be considered. Hence there is no loss of generality in assuming $Pdx + Qdy$ as an axact differential with respect to x and y.

$$\therefore \quad \text{We have} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{and} \quad \int (Pdx + Qdy) = V, \text{ say}$$

$$\therefore \quad Pdx + Qdy = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$\Rightarrow \quad P = \frac{\partial V}{\partial x} \quad \text{and} \quad Q = \frac{\partial V}{\partial y}$$

$$\therefore \quad (7) \Rightarrow \frac{\partial V}{\partial x}\left(\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right) - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial x}\right)\right)$$

$$+ R\left(\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)\right) = 0$$

NOTES

$$\Rightarrow \quad \frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right) + R(0) = 0$$

$$\Rightarrow \quad \frac{\partial V}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right)\right) - \frac{\partial V}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right)\right) = 0$$

$$\Rightarrow \quad \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

∴ *There exists a relation between V and $\frac{\partial V}{\partial z} - R$ which is independent of x and y.

∴ $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone.

Let $$\frac{\partial V}{\partial z} - R = \phi(z, V)$$

∴ The given equation P dx + Q dy + R dz = 0 reduces to

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + R dz + \left(\frac{\partial V}{\partial z} dz - \frac{\partial V}{\partial z} dz\right) = 0$$

or $$\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right) + \left(R - \frac{\partial V}{\partial z}\right) dz = 0$$

or $$dV - \phi(z, V)\, dz = 0 \qquad \ldots(9)$$

This is an equation in two variables and hence solvable. Let the solution of (9) be $f(z, V) = 0$. Putting the value of V in $f(z, V) = 0$, we get the solution of the given total differential equation $Pdx + Qdy + Rdz = 0$.

∴ **The total differential equation Pdx + Qdy + Rdz = 0 is integrable i.e., solvable if and only if**

$$\mathbf{P}\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + \mathbf{Q}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \mathbf{R}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Remark. The necessary and sufficient condition for the equation P dx + Q dy + R dz = 0 to be integrable can be easily remembered because P, Q, R, x, y, z appear in it in a regular cyclical order.

8.6. EXACT TOTAL DIFFERENTIAL EQUATION

Let $$Pdx + Qdy + Rdz = 0 \qquad \ldots(1)$$

be an exact total differential equation in three variables x, y and z.

Since (1) is exact, the expression $Pdx + Qdy + Rdz$ is equal to the differential of a function say, $u(x, y, z)$ of x, y and z.

$$\therefore \quad Pdx + Qdy + Rdz = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\Rightarrow \quad P = \frac{\partial u}{\partial x}, Q = \frac{\partial u}{\partial y}, R = \frac{\partial u}{\partial z}$$

* This is a standard result.

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial R}{\partial y}$$

and
$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial P}{\partial z}$$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

∴ **If the equation P dx + Q dy + R dz = 0 is exact then we have**

$$\frac{\partial \mathbf{P}}{\partial \mathbf{y}} = \frac{\partial \mathbf{Q}}{\partial \mathbf{x}}, \frac{\partial \mathbf{Q}}{\partial \mathbf{z}} = \frac{\partial \mathbf{R}}{\partial \mathbf{y}} \textbf{ and } \frac{\partial \mathbf{R}}{\partial \mathbf{x}} = \frac{\partial \mathbf{P}}{\partial \mathbf{z}}$$

Remark. If the equation $Pdx + Qdy + Rdz = 0$ is exact then

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$= P(0) + Q(0) + R(0) = 0$ *i.e.*, the condition of integrability is trivially satisfied.

8.7. SOLUTION OF TOTAL DIFFERENTIAL EQUATION

Let $\quad P\,dx + Q\,dy + R\,dz = 0 \quad$ ...(1)

be a total differential equation in three variables x, y and z. The equation (1) is solvable only when the functions P, Q and R satisfy the relation

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

An equation of the form (1) and not satisfying above relation cannot be solved. There are number of methods of solving integrable total differential equations in there variables. Some of the methods depend upon the exactness of the given equation and some on the homogeneity of the functions P, Q and R.

8.8. (METHOD I) SOLUTION OF EXACT TOTAL DIFFERENTIAL EQUATION

Let $\quad P\,dx + Q\,dy + R\,dz = 0$ be an exact total differential equation.

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Since the given equation is exact, we would be able to solve this equation after some desired regrouping of terms.

The following is the list of some important exact differentials. These formulae are used to regroup the terms of the given equation. In the given formulae, we can replace x by y (or z) and so on as desired.

(*i*) $xdy + ydx = d(xy)$ (*ii*) $xydz + xzdy + yzdx = d(xyz)$

(*iii*) $y^2dx + 2xy\,dy = d(xy^2)$ (*iv*) $3x^2y\,dx + x^3\,dy = d(x^3y)$

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(v) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$ (vi) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(vii) $-\frac{xdy + ydx}{x^2y^2} = d\left(\frac{1}{xy}\right)$ (viii) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(ix) $\frac{xdx + ydy}{x^2 + y^2} = d\left(\frac{1}{2}\log(x^2 + y^2)\right)$ (x) $\frac{d(f(x, y, z))}{f(x, y, z)} = d\,(\log(f(x, y, z)))$

(xi) $\frac{d(f(x, y, z))}{(f(x, y, z))^n} = d\left(\frac{(f(x, y, z))^{n+1}}{n+1}\right), \quad n \neq -1$

(xii) $\frac{ye^x\,dx - e^x\,dy}{y^2} = d\left(\frac{e^x}{y}\right)$

Example 1. *Solve the following total differential equations*

(i) $(x - y)dx - x\,dy + z\,dz = 0$

(ii) $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$

(iii) $(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy + (x - y)(x + y - 2z)dz = 0$

(iv) $(3x^2y^2 - e^xz)dx + (2x^3y + \sin z)dy + (y\cos z - e^x)dz = 0.$

Sol. (i) We have $(x - y)dx - x\,dy + z\,dz = 0$ ...(1)

Here $P = x - y, Q = -x \quad R = z.$

$$\therefore \quad \frac{\partial P}{\partial y} = -1, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = -1, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = 0$$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$\therefore$ The equation (1) is exact.

On regrouping the terms,

$$(1) \Rightarrow \quad xdx - (ydx + xdy) + zdz = 0$$

$$\Rightarrow \quad d\left(\frac{x^2}{2}\right) - d(xy) + d\left(\frac{z^2}{2}\right) = 0 \Rightarrow d\left(\frac{x^2}{2} - xy + \frac{z^2}{2}\right) = 0$$

Integrating, we get

$$\frac{x^2}{2} - xy + \frac{z^2}{2} = k \Rightarrow x^2 - 2xy + z^2 = c. \qquad (\because \text{ Putting } c = 2k)$$

This is the required solution.

(ii) We have $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$ ...(1)

Here $P = yz + 2x, \quad Q = zx - 2z, \quad R = xy - 2y$

$$\therefore \quad \frac{\partial P}{\partial y} = z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = z, \quad \frac{\partial Q}{\partial z} = x - 2, \quad \frac{\partial R}{\partial x} = y, \quad \frac{\partial R}{\partial y} = x - 2$$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$\therefore$ The equation (1) is exact

On regrouping the terms,

$$(1) \Rightarrow (yz\,dx + zx\,dy + xy\,dz) + 2x\,dx - 2(z\,dy + y\,dz) = 0$$

$$\Rightarrow d(xyz) + d(x^2) - 2d(yz) = 0$$

$$\Rightarrow d(xyz + x^2 - 2yz) = 0$$

Integrating, we get. $\mathbf{xyz + x^2 - 2yz = c.}$

This is the required solution.

(*iii*) We have

$$(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy + (x - y)(x + y - 2z)dz = 0 \quad ...(1)$$

Here
$$P = (y - z)(y + z - 2x) = y^2 - z^2 - 2x(y - z)$$
$$Q = (z - x)(z + x - 2y) = z^2 - x^2 - 2y(z - x)$$
$$R = (x - y)(x + y - 2z) = x^2 - y^2 - 2z(x - y)$$

$\therefore$
$$\frac{\partial P}{\partial y} = 2y - 2x, \quad \frac{\partial P}{\partial z} = -2z + 2x,$$
$$\frac{\partial Q}{\partial x} = -2x + 2y, \quad \frac{\partial Q}{\partial z} = 2z - 2y,$$
$$\frac{\partial R}{\partial x} = 2x - 2z, \quad \frac{\partial R}{\partial y} = -2y + 2z$$

$\therefore$
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$\therefore$ The equation (1) is exact.

On regrouping the terms,

$$(1) \Rightarrow (y^2 - z^2 - 2xy + 2xz)\,dx + (z^2 - x^2 - 2yz + 2xy)\,dy$$
$$+ (x^2 - y^2 - 2zx + 2yz)\,dz = 0$$

$$\Rightarrow (y^2\,dx + 2xy\,dy) - (z^2\,dx + 2zx\,dz) - (2xy\,dx + x^2\,dy) + (2xz\,dx + x^2\,dz)$$
$$+ (z^2\,dy + 2yz\,dz) - (2yz\,dy + y^2\,dz) = 0$$

$$\Rightarrow d(y^2x) - d(z^2x) - d(x^2y) + d(x^2z) + d(z^2y) - d(y^2z) = 0$$

$$\Rightarrow d(y^2x - z^2x - x^2y + x^2z + z^2y - y^2z) = 0$$

Integrating, we get $y^2x - z^2x - x^2y + x^2z + z^2y - y^2z = c$

$$\Rightarrow \mathbf{x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2) = c.}$$

This is the required solution.

(*iv*) We have

$$(3x^2y^2 - e^x z)dx + (2x^3y + \sin z)dy + (y \cos z - e^x)dz = 0 \quad ...(1)$$

Here $P = 3x^2y^2 - e^xz$, $Q = 2x^3y + \sin z$ and $R = y \cos z - e^x$

$\therefore$
$$\frac{\partial P}{\partial y} = 6x^2y, \quad \frac{\partial P}{\partial z} = -e^x, \quad \frac{\partial Q}{\partial x} = 6x^2y, \quad \frac{\partial Q}{\partial z} = \cos z,$$
$$\frac{\partial R}{\partial x} = -e^x, \quad \frac{\partial R}{\partial y} = \cos z$$

$\therefore$
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

NOTES

$\therefore$ The equation (1) is exact. On regrouping the terms

$$(1) \Rightarrow (3x^2y^2\,dx + 2x^3y\,dy) - (e^xz\,dx + e^x\,dz) + (\sin z\,dy + y\cos z\,dz) = 0$$

$$\Rightarrow d(x^3y^2) - d(e^xz) + d(y\sin z) = 0$$

$$\Rightarrow d(x^3y^2 - e^x z + y\sin z) = 0$$

Integrating, we get $\mathbf{x^3y^2 - e^x z + y \sin z = c.}$

This is the required solution.

EXERCISE 1

Solve the following total differential equations:

1. $x\,dx + 6y^2\,dy + z^3\,dz = 0$
2. $\frac{1}{x}\,dx + y\,dz + z\,dy = 0$
3. $(y + 3z)dx + (x + 2z)dy + (3x + 2y)dz = 0$
4. $(2x + y^2 + 2xz)dx + 2xy\,dy + x^2\,dz = 0$
5. $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$
6. $(2x + y^2 + 2xz)dx + 2xy\,dy + x^2\,dz = 0$
7. $\frac{y}{x^2}\,dx - \frac{1}{x}\,dy + 2z\,dz = 0$
8. $(\cos x + e^xy)dx + (e^x + e^yz)dy + e^y\,dz = 0.$

Answers

1. $2x^2 + 8y^3 + z^4 = c$
2. $\log x + yz = c$
3. $xy + 2yz + 3xz = c$
4. $x^2 + xy^2 + x^2z = c$
5. $xyz + x^2 - 2yz = c$
6. $x^2 + xy^2 + zx^2 = c$
7. $\frac{y}{x} - z^2 = c$
8. $e^xy + e^yz + \sin x = c.$

8.9. (METHOD II) SOLUTION OF EXACT AND HOMOGENEOUS TOTAL DIFFERENTIAL EQUATION OF DEGREE ($n \neq -1$)

Let P dx + Q dy + R dz = 0 be an exact and homogeneous total differential equation of degree n ($\neq -1$).

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad \text{...(1)}$$

Since the given equation is homogeneous, so by Euler's theorem, we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = nP \qquad \text{...(2)}$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} = nQ \qquad \text{...(3)}$$

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = nR \qquad \text{...(4)}$$

Let $\quad xP + yQ + zR = u \qquad \text{...(5)}$

Differentiating (5), we get

$$\left(1\cdot P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x} + z\frac{\partial R}{\partial x}\right)dx + \left(x\frac{\partial P}{\partial y} + 1\cdot Q + y\frac{\partial Q}{\partial y} + z\frac{\partial R}{\partial y}\right)dy$$
$$+ \left(x\frac{\partial P}{\partial z} + y\frac{\partial Q}{\partial z} + 1\cdot R + z\frac{\partial R}{\partial z}\right)dz = 0$$

$$\Rightarrow \left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy$$
$$+ \left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0$$

[By using (1)]

$$\Rightarrow \quad (P + nP)\,dx + (Q + nQ)\,dy + (R + nR)\,dz = 0$$

[By using (2), (3) and (4)]

$$\Rightarrow \quad (n + 1)(P\,dx + Q\,dy + R\,dz) = 0$$

$$\Rightarrow \quad P\,dx + Q\,dy + R\,dz = 0 \qquad (\because \; n \neq -1)$$

$\therefore$ $xP + yQ + zR = c$ is a solution of the given equation.

Example 1. *Solve the following total differential equations :*

(i) $(4y + 2z)dx + (4x - z)dy + (2x - y)dz = 0$

(ii) $(y^2 + z^2 + 2xy + 2xz)dx + (x^2 + z^2 + 2xy + 2yz)dy + (x^2 + y^2 + 2xz + 2yz)dz = 0.$

Sol. *(i)* We have $\quad (4y + 2z)dx + (4x - z)dy + (2x - y)dz = 0 \qquad ...(1)$

Here $\quad P = 4y + 2z, \quad Q = 4x - z, \quad R = 2x - y$

$\therefore$ Equation (1) is a homogeneous equation of degree $1(\neq -1)$.

Also $$\frac{\partial P}{\partial y} = 4, \; \frac{\partial P}{\partial z} = 2, \; \frac{\partial Q}{\partial x} = 4, \; \frac{\partial Q}{\partial z} = -1, \; \frac{\partial R}{\partial x} = 2, \; \frac{\partial R}{\partial y} = -1$$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \; \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \; \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$\therefore$ The equation (1) is also exact.

$\therefore$ The solution of (1) is

$$xP + yQ + zR = c$$

i.e., $$x(4y + 2z) + y(4x - z) + z(2x - y) = c$$

or $$8xy - 2yz + 4zx = c$$

or $$\mathbf{4xy - yz + 2zx = k.} \qquad (\text{Putting } k = c/2)$$

This is the required solution.

(ii) We have $\quad (y^2 + z^2 + 2xy + 2xz)dx$
$$+ (x^2 + z^2 + 2xy + 2yz)dy + (x^2 + y^2 + 2xz + 2yz)dz = 0 \qquad ...(1)$$

Here $\quad P = y^2 + z^2 + 2xy + 2xz$

$Q = x^2 + z^2 + 2xy + 2yz$

$R = x^2 + y^2 + 2xz + 2yz$

$\therefore$ Equation (1) is a homogeneous equation of degree $2(\neq -1)$.

Also $$\frac{\partial P}{\partial y} = 2y + 2x, \; \frac{\partial P}{\partial z} = 2z + 2x, \; \frac{\partial Q}{\partial x} = 2x + 2y, \; \frac{\partial Q}{\partial z} = 2z + 2y,$$

$$\frac{\partial R}{\partial x} = 2x + 2z, \; \frac{\partial R}{\partial y} = 2y + 2z$$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \; \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \; \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$\therefore$ The equation (1) is also exact.

$\therefore$ The solution of (1) is $xP + yQ + zR = c$

i.e., $$\mathbf{x(y^2 + z^2 + 2xy + 2xz) + y(x^2 + z^2 + 2xy + 2yz) + z(x^2 + y^2 + 2xz + 2yz) = c.}$$

This is the required solution.

NOTES

EXERCISE 2

Solve the following total differential equations :

1. $(x-y)dx - x\,dy + z\,dz = 0$
2. $(y+3z)dx + (x+2z)dy + (3x+2y)dz = 0$
3. $(2y-z)dx + 2(x-z)dy - (x+2y)dz = 0$
4. $(y-7z)dx + (x-3z)dy - (7x+3y)dz = 0$
5. $(x-3y-z)dx + (2y-3x)dy + (z-x)dz = 0$
6. $(y-z)(y+z-2x)dx + (z-x)(z+x-2y)dy + (x-y)(x+y-2z)dz = 0.$

Answers

1. $x^2 - 2xy + z^2 = c$
2. [illegible]
3. $2xy - xz - 2yz = c$
4. [illegible]
5. $x^2 + 2y^2 + z^2 - 6xy - 2xz = c$
6. $x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2) = c$

8.10. (METHOD III) SOLUTION OF INTEGRABLE TOTAL DIFFERENTIABLE EQUATION BY FINDING INTEGRATING FACTOR BY INSPECTION

Let $P\,dx + Q\,dy + R\,dz = 0$ be an integrable total differential equation. If the given equation is not exact, we find an integrating factor $\lambda(x, y, z)$ by inspection. By multiplying the given equation by $\lambda(x, y, z)$, we get an exact differential equation and this equation is solved by using the method discussed earlier.

Example 1. *Solve the following total differential equations :*

(i) $y^2dx - z dy + y dz = 0$

(ii) $yz \log z dx - zx \log z dy + xy dz = 0$

(iii) $(2x^3y + 1)dx + x^4dy + x^2 \tan z dz = 0$

(iv) $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\, dz = 0.$

Sol. (*i*) We have $\quad y^2dx - z\,dy + y\,dz = 0 \qquad$ ...(1)

Here $P = y^2$, $Q = -z$, $R = y$

$$\therefore \quad \frac{\partial P}{\partial y} = 2y, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = -1, \quad \frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = 1$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{Equation (1) is not exact.}$$

$$\text{Now} \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= y^2(-1-1) + (-z)(0-0) + y(2y-0) = -2y^2 + 2y^2 = 0$$

$\therefore$ The equation (1) is integrable. We take $\frac{1}{y^2}$ as an integrating factor.

Multiplying (1) by $\frac{1}{y^2}$, we get $dx - \frac{z}{y^2}dy + \frac{1}{y}\,dz = 0$

$$\Rightarrow \quad dx + \frac{ydz - zdy}{y^2} = 0 \;\Rightarrow\; dx + d\left(\frac{z}{y}\right) = 0 \;\Rightarrow\; d\left(x + \frac{z}{y}\right) = 0$$

NOTES

Integrating, we get $\mathbf{x + \frac{z}{y} = c}$.

This is the required solution.

(*ii*) We have $\quad yz \log z\, dx - zx \log z\, dy + xy\, dz = 0 \qquad \ldots(1)$

Here $\quad P = yz \log z, \quad Q = -zx \log z, R = xy$

$$\therefore \quad \frac{\partial P}{\partial y} = z \log z\,, \quad \frac{\partial P}{\partial z} = y\left(z \cdot \frac{1}{z} + 1 \cdot \log z\right) = y + y \log z$$

$$\frac{\partial Q}{\partial x} = -z \log z\,, \quad \frac{\partial Q}{\partial z} = -x\left(z \cdot \frac{1}{z} + 1 \cdot \log z\right) = -x - x \log z$$

$$\frac{\partial R}{\partial x} = y\,, \quad \frac{\partial R}{\partial y} = x$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{Equation (1) is not exact.}$$

Now $\quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= yz \log z\,(-x - x \log z - x) - zx \log z(y - y - y \log z) + xy\,(z \log z + z \log z)$

$= -2xyz \log z - xyz\,(\log z)^2 + xyz\,(\log z)^2 + 2xyz \log z = 0$

$\therefore \quad$ The equation (1) is integrable. We take $\frac{1}{xyz \log z}$ as an integrating factor.

Multiplying (1) by $\frac{1}{xyz \log z}$, we get

$$\frac{1}{x}\,dx - \frac{1}{y}\,dy + \frac{1}{z \log z}\,dz = 0$$

$$\Rightarrow \quad d(\log x) - d(\log y) + d(\log \log z) = 0$$

$$\Rightarrow \quad d\,(\log x - \log y + \log \log z) = 0$$

$$\Rightarrow \quad d\left\{\log\left(\frac{x}{y} \log z\right)\right\} = 0$$

Integrating, we get $\qquad \log\left(\frac{x}{y} \log z\right) = c.$

$$\Rightarrow \quad \frac{x}{y} \log z = e^c \;\Rightarrow\; \frac{x}{y} \log z = k \qquad (\text{Putting } k = e^c)$$

$\Rightarrow \qquad \mathbf{x \log z = ky.}$

This is the required solution.

(*iii*) We have $\quad (2x^3y + 1)dx + x^4\,dy + x^2 \tan z\, dz = 0 \qquad \ldots(1)$

Here $P = 2x^3y + 1,\ Q = x^4,\ R = x^2 \tan z$

$$\therefore \quad \frac{\partial P}{\partial y} = 2x^3, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = 4x^3, \quad \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 2x \tan z, \quad \frac{\partial R}{\partial y} = 0$$

NOTES

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{Equation (1) is not exact.}$$

Now $\quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (2x^3y + 1)(0 - 0) + x^4 (2x \tan z - 0) + x^2 \tan z\,(2x^3 - 4x^3)$$

$$= 0 + 2x^5 \tan z - 2x^5 \tan z = 0$$

$\therefore$ The equation (1) is integrable. Taking integrating factor $\frac{1}{x^2}$, equation (1) becomes

$$\frac{2x^3y + 1}{x^2}\,dx + x^2 dy - \tan z\,dz = 0$$

$$\Rightarrow \quad 2xy\,dx + \frac{1}{x^2}\,dx + x^2\,dy + \tan z\,dz = 0$$

$$\Rightarrow \quad (2xy\,dx + x^2\,dy) + \frac{1}{x^2}\,dx + \tan z\,dz = 0$$

$$\Rightarrow \quad d(x^2 y) + d\left(-\frac{1}{x}\right) + d(\log \sec z) = 0$$

$$\Rightarrow \quad d\left(x^2y - \frac{1}{x} + \log \sec z\right) = 0$$

Integrating, we get $\quad \mathbf{x^2y - \frac{1}{x} + \log \sec z = c.}$

This is the required solution.

(*ii*) We have $\quad (2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z dz = 0 \quad$ ...(1)

Here $\quad P = 2x^2 + 2xy + 2xz^2 + 1 \quad Q = 1,\ R = 2z$

$$\therefore \quad \frac{\partial P}{\partial y} = 2x,\ \ \frac{\partial P}{\partial z} = 4xz,\ \ \frac{\partial Q}{\partial x} = 0,\ \ \frac{\partial Q}{\partial z} = 0,\ \ \frac{\partial R}{\partial x} = 0,\ \ \frac{\partial R}{\partial y} = 0$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{Equation (1) is not exact.}$$

Now $\quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1 \cdot (0 - 4xz) + 2z(2x - 0) = -4xz + 4xz = 0$$

$\therefore$ **The equation (1) is integrable**

(1) $\Rightarrow \quad (2x^2 + 2xy + 2xz^2)dx + (dx + dy + 2zdz) = 0 \quad$ **(Note this step)**

$\Rightarrow \quad 2x(x + y + z^2)\,dx + d(x + y + z^2) = 0 \quad$ ...(2)

Taking integrating factor $\frac{1}{x + y + z^2}$ equation (2) becomes

$$2xdx + \frac{d(x + y + z^2)}{x + y + z^2} = 0$$

$$\Rightarrow \quad d(x^2) + d(\log (x + y + z^2)) = 0$$

$$\Rightarrow \quad d(x^2 + \log (x + y + z^2)) = 0$$

Integrating, we get $\quad \mathbf{x^2 + \log (x + y + z^2) = c.}$

This is the required solution.

EXERCISE 3

Solve the following total differential equations :

1. $xdy - ydx - 2x^2z\,dz = 0$
2. $2yzdx + zxdy - xy(1 + z)dz = 0$
3. $(a - z)(ydx + xdy) + xydz = 0$
4. $2xzdx + zdy - dz = 0$
5. $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$
6. $(z + z^2)\cos x\,dx - (z + z^2)dy + (1 - z^2)(y - \sin x)dz = 0$
7. $2yzdx - 3zxdy - 4xydz = 0$
8. $(2x^3 - z)zdx + 2x^2yzdy + x(z + x)dz = 0.$

Answers

1. $\frac{y}{x} - z^2 = c$
2. $x^2y = cze^z$
3. $xy = c(a - z)$
4. $x^2 + y - \log z = c$
5. $(x^2 - z^2)(y^2 - a^2) = c$
6. $y - \sin x = cze^{-z}$
7. $x^2 = cy^3z^4$
8. $x^2 + y^2 + \log z + \frac{z}{x} = c.$

Hints

1. Use I.F. $= \frac{1}{x^2}$
2. Use I.F. $= \frac{1}{xyz}$
3. Use I.F. $= \frac{1}{xy(a - z)}$.
4. Use I.F. $= \frac{1}{z}$
5. Use I.F. $= \frac{1}{(y^2 - a^2)(x^2 - z^2)}$
6. Use I.F. $= \frac{1}{(y - \sin x)(z + z^2)}$
8. Use I.F. $= \frac{1}{x^2z}$.

8.11. (METHOD IV) SOLUTION OF INTEGRABLE HOMOGENEOUS TOTAL DIFFERENTIAL EQUATION

Let $$P\,dx + Q\,dy + R\,dz = 0 \quad \ldots(1)$$

be an integrable homogeneous total differential equation of degree n.

$\therefore$ The functions P, Q and R are homogeneous functions of degree n in the variables x, y and z.

Let $$x = uz \quad \text{and} \quad y = vz.$$

$$\therefore \quad dx = u\,dz + z\,du \quad \text{and} \quad dy = v\,dz + z\,dv$$

$$\therefore \quad (1) \Rightarrow P(uz, vz, z)\,[u\,dz + z\,du] + Q(uz, vz, z)\,[v\,dz + z\,dv] + R(uz, vz, z)dz = 0 \quad \ldots(2)$$

Since P, Q, R are homogeneous functions of degree n, we have

$P(uz, vz, z) = z^n P(u, v, 1)$, $Q(uz, vz, z) = z^n Q(u, v, 1)$

and $R(uz, vz, z) = z^n R(u, v, 1)$.

$$\therefore \quad (2) \Rightarrow z^n P(u, v, 1)\,[u\,dz + z\,du] + z^n Q(u, v, 1)\,[v\,dz + z\,dv] + z^n R(u, v, 1)dz = 0$$

Dividing by z^n, we get $P(u, v, 1)\,[u\,dz + z\,du] + Q(u, v, 1)\,[vdz + zdv] + R(u, v, 1)dz = 0$

$$\Rightarrow \quad P_1\,(u\,dz + z\,du) + Q_1\,(v\,dz + z\,dv) + R_1\,dz = 0$$

(Putting $P_1 = P(u, v, 1)$ etc.)

NOTES

$$\Rightarrow \quad z(P_1\,du + Q_1\,dv) + (uP_1 + vQ_1 + R_1)dz = 0$$

$$\Rightarrow \quad \frac{P_1}{uP_1 + vQ_1 + R_1}\,du + \frac{Q_1}{uP_1 + vQ_1 + R_1}\,dv + \frac{1}{z}\,dz = 0 \qquad ...(3)$$

In equation (3), the variable z has been separated from the variables u and v and it is occurring only in the last term

Since equation (1) is integrable, the equation (3) is also integrable.

$$\therefore \quad \frac{P_1}{uP_1 + vQ_1 + R_1}(0-0) + \frac{Q_1}{uP_1 + vQ_1 + R_1}(0-0) + \frac{1}{z}\left(\frac{\partial}{\partial v}\left(\frac{P_1}{uP_1 + vQ_1 + R_1}\right) - \frac{\partial}{\partial u}\left(\frac{Q_1}{uP_1 + vQ_1 + R_1}\right)\right) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial v}\left(\frac{P_1}{uP_1 + vQ_1 + R_1}\right) = \frac{\partial}{\partial u}\left(\frac{Q_1}{uP_1 + vQ_1 + R_1}\right)$$

$\Rightarrow \quad \dfrac{P_1}{uP_1 + vQ_1 + R_1}\,du + \dfrac{Q_1}{uP_1 + vQ_1 + R_1}\,dv$ is an exact differential. Also the third term of (3) is exact differential of $\log z$.

$\therefore$ The equation (3) is an exact equation and is thus solvable. In the solution of (3), we put $u = z/x$ and $v = z/y$ and get the required solution of the given equation (1).

Example 1. *Solve the following total differential equations :*

(*i*) $yzdx - z^2dy - xydz = 0$

(*ii*) $(yz + z^2)dx - xzdy + xydz = 0$

(*iii*) $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$

(*iv*) $yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0.$

Sol. (*i*) We have $\quad yzdx - z^2\,dy - xydz = 0 \qquad ...(1)$

Here $\quad P = yz, \quad Q = -z^2, \quad R = -xy$

$\therefore$ (1) is a homogeneous equation of degree 2.

$$\frac{\partial P}{\partial y} = z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = -2z, \quad \frac{\partial R}{\partial x} = -y, \quad \frac{\partial R}{\partial y} = -x$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= yz(-2z + x) - z^2(-y - y) - xy(z - 0)$$

$$= -2yz^2 + xyz + 2yz^2 - xyz = 0$$

$\therefore$ The equation (1) is integrable.

Let $\quad x = uz \quad$ and $\quad y = vz.$

$\therefore \quad dx = udz + zdu \quad$ and $\quad dy = vdz + zdv$

$\therefore$ (1) reduces to $\quad (vz)z[udz + zdu] - z^2[vdz + zdv] - (uz)(vz)dz = 0$

$\Rightarrow$ Dividing by z^2, we get $\quad v[udz + zdu] - [vdz + zdv] - uvdz = 0$

$$\Rightarrow \quad vzdu - zdv - vdz = 0$$

$$\Rightarrow \quad z(vdu - dv) - vdz = 0$$

$$\Rightarrow \quad \frac{v\,du - dv}{v} - \frac{1}{z}\,dz = 0$$

$$\Rightarrow \quad du - \frac{1}{v}\,dv - \frac{1}{z}\,dz = 0$$

$\Rightarrow \quad du - d(\log v) - d(\log z) = 0$

$\Rightarrow \quad d(u - \log v - \log z) = 0$

Integrating, we get $\quad u - \log v - \log z = \log k$

$\Rightarrow \quad u = \log (kvz) \quad \Rightarrow \quad kvz = e^u$

$\Rightarrow \quad ky = e^{x/z} \quad \Rightarrow \quad y = ce^{x/z}$. $\quad \left(\text{Putting } c = \frac{1}{k}\right)$

This is the required solution.

(*ii*) We have $\quad (yz + z^2)dx - xz\,dy + xy\,dz = 0$ ...(1)

Here $\quad P = yz + z^2, \quad Q = -xz, \quad R = xy$

$\therefore$ (1) is a homogeneous equation of degree 2.

$$\frac{\partial P}{\partial y} = z,\ \frac{\partial P}{\partial z} = y + 2z,\ \frac{\partial Q}{\partial x} = -z,\ \frac{\partial Q}{\partial z} = -x,\ \frac{\partial R}{\partial x} = y,\ \frac{\partial R}{\partial y} = x$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (yz + z^2)(-x - x) - xz(y - y - 2z) + xy(z + z)$$

$$= -2xyz - 2xz^2 + 2xz^2 + 2xyz = 0$$

$\therefore$ The equation (1) is integrable.

Let $\quad x = uz \quad$ and $\quad y = vz$.

$\therefore \quad dx = udz + zdu \quad$ and $\quad dy = vdz + zdv$

$\therefore$ (1) reduces to $\quad (vz^2 + z^2)\,[udz + zdu] - uz^2\,[vdz + zdv] + uvz^2dz = 0$

Dividing by z^2, we get $\quad (v + 1)\,[udz + zdu] - u[vdz + zdv] + uvdz = 0$

$\Rightarrow \quad z(v + 1)\,du - uzdv + (uv + u - uv + uv)\,dz = 0$

$\Rightarrow \quad z[(v + 1)du - udv] + u(v + 1)dz = 0$

$$\Rightarrow \quad \frac{(v + 1)du - u\,dv}{u(v + 1)} + \frac{1}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{1}{u}\,du - \frac{1}{v + 1}\,dv + \frac{1}{z}\,dz = 0$$

$\Rightarrow \quad d\,(\log u - \log (v + 1) + \log z) = 0$

Integrating, we get $\quad \log u - \log (v + 1) + \log z = \log k$

$$\Rightarrow \quad \frac{uz}{v + 1} = k \quad \Rightarrow \quad \frac{x}{\frac{y}{z} + 1} = k$$

$\Rightarrow \quad \mathbf{xz = k(y + z).}$

This is the required solution.

(*iii*) We have $\quad (2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$ ...(1)

Here $\quad P = 2xz - yz, \quad Q = 2yz - xz, \quad R = -(x^2 - xy + y^2)$

$\therefore$ (1) is a homogeneous equation of degree 2.

$$\frac{\partial P}{\partial y} = -z,\ \frac{\partial P}{\partial z} = 2x - y,\ \frac{\partial Q}{\partial x} = -z,\ \frac{\partial Q}{\partial z} = 2y - x,\ \frac{\partial R}{\partial x} = -2x + y,\ \frac{\partial R}{\partial y} = x - 2y$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (2xz - yz)(2y - x - x + 2y) + (2yz - xz)(-2x + y - 2x + y)$$
$$- (x^2 - xy + y^2)(-z + z)$$

$$= 2z(2x - y)(2y - x) + 2z(2y - x)(y - 2x) = 0$$

$\therefore$ The equation (1) is integrable

Let $\quad x = uz \quad$ and $\quad y = vz$.

$\therefore \quad dx = udz + zdu \quad$ and $\quad dy = vdz + zdv$

$\therefore$ (1) reduces to

$$(2uz^2 - vz^2)\,[udz + zdu] + (2vz^2 - uz^2)[v\,dz + z\,dv] - (u^2z^2 - uvz^2 + v^2z^2)dz = 0$$

Dividing by z^2, we get

$$(2u - v)[udz + zdu] + (2v - u)[vdz + zdv] - (u^2 - uv + v^2)dz = 0$$

$$\Rightarrow \quad z(2u - v)du + z(2v - u)dv + (2u^2 - uv + 2v^2 - uv - u^2 + uv - v^2)dz = 0$$

$$\Rightarrow \quad z[(2u - v)du + (2v - u)dv] + (u^2 - uv + v^2)dz = 0$$

$$\Rightarrow \quad \frac{2u\,du - (v\,du + u\,dv) + 2v dv}{u^2 - uv + v^2} + \frac{1}{z}\,dz = 0$$

(Note this step)

$$\Rightarrow \quad \frac{d(u^2) - d(uv) + d(v^2)}{u^2 - uv + v^2} + \frac{1}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{d(u^2 - uv + v^2)}{u^2 - uv + v^2} + \frac{1}{z}\,dz = 0$$

Integrating, we get $\quad \log\,(u^2 - uv + v^2) + \log z = \log c$

$$\Rightarrow \quad (u^2 - uv + v^2)\,z = c$$

$$\Rightarrow \quad \left(\frac{x^2}{z^2} - \frac{xy}{z^2} + \frac{y^2}{z^2}\right) z = c$$

$$\Rightarrow \quad \mathbf{x^2 - xy + y^2 = cz.}$$

This is the required solution.

(*iv*) We have $\quad yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0 \qquad ...(1)$

Here $\;$ $P = yz(y + z)$, $Q = xz(x + z)$, $R = xy(x + y)$

$\therefore$ (1) is a homogeneous equation of degree 3

$$\frac{\partial P}{\partial y} = 2yz + z^2, \quad \frac{\partial P}{\partial z} = y^2 + 2yz, \quad \frac{\partial Q}{\partial x} = 2xz + z^2,$$

$$\frac{\partial Q}{\partial z} = x^2 + 2xz, \quad \frac{\partial R}{\partial x} = 2xy + y^2, \quad \frac{\partial R}{\partial y} = x^2 + 2xy$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= yz(y + z)(x^2 + 2xz - x^2 - 2xy) + xz(x + z)(2xy + y^2 - y^2 - 2yz)$$
$$+ \; xy(x + y)(2yz + z^2 - 2xz - z^2)$$

$$= 2xyz(y + z)(z - y) + 2xyz(x + z)(x - z) + 2xyz(x + y)(y - x)$$

$$= 2xyz[z^2 - y^2 + x^2 - z^2 + y^2 - x^2] = 0$$

$\therefore$ The equation (1) is integrable.

Let $\quad x = uz \quad$ and $\quad y = vz$.

$\therefore \quad dx = u\,dz + z\,du \quad$ and $\quad dy = v\,dz + z\,dv$

$\therefore$ (1) reduces to

$$vz^2(vz + z)\,[u\,dz + z\,du] + uz^2(uz + z)[v\,dz + z\,dv] + u\,vz^2(uz + vz)\,dz = 0$$

Dividing by z^3, we get

$$v(v+1)[u\,dz + z\,du] + u(u+1)\{v\,dz + z\,dv\} + uv(u+v)dz = 0$$

$$\Rightarrow \quad zv(v+1)du + zu(u+1)dv + [uv(v+1) + uv(u+1) + uv(u+v)]dz = 0$$

$$\Rightarrow \quad z[v(v+1)du + u(u+1)dv] + 2uv(u+v+1)dz = 0$$

$$\Rightarrow \quad \frac{v(v+1)}{uv(u+v+1)}\,du + \frac{u(u+1)}{uv\,(u+v+1)}\,dv + \frac{2}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{v+1}{u(u+v+1)}\,du + \frac{u+1}{v(u+v+1)}\,dv + \frac{2}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{(u+v+1)-v}{u(u+v+1)}\,du + \frac{(u+v+1)-v}{v(u+v+1)}\,dv + \frac{2}{z}\,dz = 0$$

(Note this step)

$$\Rightarrow \quad \left(\frac{1}{u} - \frac{1}{u+v+1}\right)du + \left(\frac{1}{v} - \frac{1}{u+v+1}\right)dv + \frac{2}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{1}{u}\,du + \frac{1}{v}\,dv - \frac{1}{u+v+1}(du + dv) + \frac{2}{z}\,dz = 0$$

$$\Rightarrow \quad \frac{1}{u}\,du + \frac{1}{v}\,dv - \frac{1}{(u+v)+1}\,d(u+v) + \frac{2}{z}\,dz = 0$$

Integrating, we get $\quad \log u + \log v - \log(u+v+1) + 2\log z = \log c$

$$\Rightarrow \quad \frac{uvz^2}{u+v+1} = c \quad \Rightarrow \quad uvz^2 = c(u+v+1)$$

$$\Rightarrow \quad \frac{x}{z}\cdot\frac{y}{z}\cdot z^2 = c\left(\frac{x}{z} + \frac{y}{z} + 1\right) \quad \Rightarrow \quad \mathbf{xyz = c(x+y+z).}$$

This is the required solution.

EXERCISE 4

Solve the following total differential equations :

1. $(2y + x - z)dx - (x + z)dy + 2(x + y)dz = 0$
2. $2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0$
3. $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$
4. $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$
5. $z^2\,dx + (z^2 - 2yz)dy + (2y^2 - yz - zx)dz = 0$
6. $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$
7. $(2xyz + y^2z + yz^2)dx + (x^2z + 2xyz + xz^2)dy + (x^2y + xy^2 + 2xyz)dz = 0$
8. $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$

Answers

1. $(x + z)^2 = c(x + y)$
2. $(x + z)^2 = c(y + z)$
3. $y(x + z) = c(y + z)$
4. $z(x + y) = c(x + z)$
5. $xz + yz - y^2 = cz^2$
6. $x^2 + y^2 + yz + xz = cxy$
7. $xyz(x + y + z) = c$
8. $xy + yz + zx = c(x + y + z).$

8.12. (METHOD V) SOLUTION OF INTEGRABLE TOTAL DIFFERENTIAL EQUATION

NOTES

Let $$P\,dx + Q\,dy + R\,dz = 0 \qquad ...(1)$$
be an integrable total differential equation.

In this method, we consider any one of the variables, say z, as a constant for the moment.

$\therefore$ (1) reduces to $P\,dx + Q\,dy + R.0 = 0$

i.e., $$P\,dx + Q\,dy = 0 \qquad ...(2)$$

Let the solution of (2) be $u(x, y, z) = \phi(z)$, where $\phi(z)$ is any arbitrary function of z. We have taken $\phi(z)$ in the solution of (2) in place of an arbitrary constant because $\phi(z)$ is a constant with respect to x and y.

Let $\mu(x, y, z)$ be an integrating factor of (2)

$$\therefore \qquad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \mu P\,dx + \mu Q\,dy$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \mu P \text{ and } \frac{\partial u}{\partial y} = \mu Q \qquad ...(3)$$

Differentiating $u(x, y, z) = \phi(z)$ w.r.t. x, y and z, we get

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = d\phi$$

$$\Rightarrow \qquad \mu P dx + \mu Q\,dy + \frac{\partial u}{\partial z}\,dz = d\phi \qquad ...(4)$$

$$(1) \Rightarrow \qquad \mu P dx + \mu Q\,dy + \mu R\,dz = 0 \qquad ...(5)$$

Subtracting (5) from (4), we get

$$d\phi = \frac{\partial u}{\partial z}\,dz - \mu R dz$$

i.e., $$d\phi = \left(\frac{\partial u}{\partial z} - \mu R\right)dz$$

This equation is solved to get the value of the function $\phi(z)$. The value of $\phi(z)$ is substituted in $u(x, y, z) = \phi(z)$ to get the solution of the given equation.

Example 1. *Solve the following total differential equations :*

(*i*) $2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0$

(*ii*) $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0$

(*iii*) $y^2z(x \cos x - \sin x)dx + x^2z(y \cos y - \sin y)dy + xy(y \sin x + x \sin y + xy \cos z)dz = 0$

(*iv*) $2yz\,dx + zx\,dy - xy(1 + z)dz = 0$.

Sol. (*i*) We have $$2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0 \qquad ...(1)$$

Here $P = 2(y + z)$, $Q = -(x + z)$, $R = 2y - x + z$

$$\therefore \qquad \frac{\partial P}{\partial y} = 2,\ \frac{\partial P}{\partial z} = 2,\ \frac{\partial Q}{\partial x} = -1,\ \frac{\partial Q}{\partial z} = -1,\ \frac{\partial R}{\partial x} = -1,\ \frac{\partial R}{\partial y} = 2$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
$$= 2(y + z)(-1 - 2) - (x + z)(-1 - 2) + (2y - x + z)(2 + 1)$$
$$= -6y - 6z + 3x + 3z + 6y - 3x + 3z = 0$$

NOTES

$\therefore$ The equation (1) is integrable.

Considering z as a constant, we have $dz = 0$.

$\therefore$ (1) $\Rightarrow$ $2(y+z)dx - (x+z)dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(y+z)}{x+z} \Rightarrow \frac{dy}{dx} - \frac{2}{x+z}y = \frac{2z}{x+z} \quad \ldots(2)$$

This is a linear differential equation of first order.

$$\text{I.F} = e^{\int -\frac{2}{x+z}dx} = e^{-2\log(x+z)} = \frac{1}{(x+z)^2}$$

$\therefore$ The solution of (2) is

$$y \cdot \frac{1}{(x+z)^2} = \int \frac{2z}{x+z} \cdot \frac{1}{(x+z)^2}dx + \phi(z),$$

where $\phi(z)$ is any arbitrary function of z.

$$\Rightarrow \quad \frac{y}{(x+z)^2} = 2z \cdot \frac{(x+z)^{-2}}{-2} + \phi(z)$$

$$\Rightarrow \quad \frac{y}{(x+z)^2} = -\frac{z}{(x+z)^2} + \phi(z) \quad \ldots(3)$$

Differentiating (3) w.r.t. x, y and z, we get

$$\frac{1}{(x+z)^2}dy - \frac{2y}{(x+z)^3}dx - \frac{2y}{(x+z)^3}dz = \frac{2z}{(x+z)^3}dx - \left(1 \cdot \frac{1}{(x+z)^2} - \frac{2z}{(x+z)^3}\right)dz + d\phi$$

$$\Rightarrow \quad \frac{dy}{(x+z)^2} - \frac{2y}{(x+z)^3}(dx+dz) = -\frac{dz}{(x+z)^2} + \frac{2z}{(x+z)^3}(dx+dz) + d\phi$$

$$\Rightarrow \quad \frac{2(y+z)}{(x+z)^3}(dx+dz) - \frac{dy}{(x+z)^2} - \frac{dz}{(x+z)^2} + d\phi = 0$$

$$\Rightarrow \quad 2(y+z)dx - (x+z)dy + (2y-x+z)dz + (x+z)^3 d\phi = 0$$

$$\Rightarrow \quad (x+z)^3\, d\phi = 0 \quad \text{[By using (1)]}$$

$$\Rightarrow \quad d\phi = 0 \Rightarrow \phi = c$$

$$\therefore \text{ (3)} \Rightarrow \quad \frac{y}{(x+z)^2} = -\frac{z}{(x+z)^2} + c \Rightarrow \mathbf{y + z = c(x+z)^2}.$$

This is the required solution.

(*ii*) We have $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0$ ...(1)

Here $P = 2x^2 + 2xy + 2xz^2 + 1$, $Q = 1$, $R = 2z$

$$\therefore \quad \frac{\partial P}{\partial y} = 2x, \quad \frac{\partial P}{\partial z} = 4xz, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = 0$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (2x^2 + 2xy + 2xz^2 + 1)(0-0) + 1 \cdot (0 - 4xz) + 2z(2x - 0)$$

$$= -4xz + 4xz = 0$$

$\therefore$ The equation (1) is integrable.

NOTES

Considering x* as a constant, we have $dx = 0$

$\therefore$ (1) $\Rightarrow$ $dy + 2z\,dz = 0$

Integrating, we get $y + z^2 = \phi(x)$. ...(2)

where $\phi(x)$ is any arbitrary function of x

Differentiating (2) w.r.t. x, y and z, we get

$$1 \cdot dy + 2z\,dz = d\phi \qquad ...(3)$$

(1) − (3) $\Rightarrow$ $(2x^2 + 2xy + 2xz^2 + 1)\,dx = -\,d\phi$

$\Rightarrow$ $(2x^2 + 2xy + 2xz^2 + 1)dx = -\,\phi'(x)dx$

$\Rightarrow$ $\phi'(x) = -\,(2x^2 + 2xy + 2xz^2 + 1)$

$\Rightarrow$ $\phi'(x) = -\,(2x^2 + 2x(y + z^2) + 1)$

$\Rightarrow$ $\phi'(x) = -\,2x^2 - 2x\,\phi(x) - 1$ [Using (2)]

$\Rightarrow$ $\dfrac{d\phi}{dx} + 2x\phi = -\,(2x^2 + 1)$ ...(4)

(4) is a linear differential equation of first order

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ Solution of (4) is

$$\phi(x)\;e^{x^2} = \int -\,(2x^2 + 1)\,e^{x^2}dx + c$$

$\Rightarrow$ $\phi(x)\,e^{x^2} = -\int x \cdot 2x\,e^{x^2}\,dx - \int e^{x^2}\,dx + c$

$$= -\left[x \cdot e^{x^2} - \int 1 \,.\, e^{x^2}\,dx\right] - \int e^{x^2}\,dx + c = -\,xe^{x^2} + c$$

$\therefore$ $\phi(x)e^{x^2} = -\,xe^{x^2} + c \;\Rightarrow\; (y + z^2)e^{x^2} = -\,xe^{x^2} + c$

$\Rightarrow$ $\mathbf{(x + y + z^2)e^{x^2} = c.}$

This is the required solution.

(*iii*) We have $y^2z(x\cos x - \sin x)dx + x^2z\,(y\cos y - \sin y)dy + xy(y\sin x + x\sin y + xy\cos z)dz = 0$ ...(1)

It can be verified that the equation (1) is integrable

Considering z as a constant, we have $dz = 0$.

$\therefore$ (1) $\Rightarrow$ $y^2z\,(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy = 0$ ...(2)

$\Rightarrow$ $\dfrac{x\cos x - \sin x}{x^2}\,dx + \dfrac{y\cos y - \sin y}{y^2}\,dy = 0$

$\Rightarrow$ $d\left(\dfrac{\sin x}{x}\right) + d\left(\dfrac{\sin y}{y}\right) = 0$

Integrating, we get $\dfrac{\sin x}{x} + \dfrac{\sin y}{y} = \phi(z)$, ...(3)

where $\phi(z)$ is any arbitrary function of z.

*We have chosen x to be constant because P is more complex then Q and R.

Differentiating (3) w.r.t. x, y and z, we get

$$\frac{x\cos x-\sin x}{x^2}dx+\frac{y\cos y-\sin y}{y^2}dy=\phi'(z)dz$$

$$\Rightarrow \quad y^2z(x\cos x-\sin x)dx+x^2z(y\cos y-\sin y)dy-x^2y^2z\,\phi'(z)dz=0 \quad ...(4)$$

$$(1)-(4) \Rightarrow \quad xy(y\sin x+x\sin y+xy\cos z)\,dz+x^2y^2z\,\phi'(z)dz=0$$

$$\Rightarrow \quad y\sin x+x\sin y+xy\cos z=-xyz\,\phi'(z)$$

$$\Rightarrow \quad \frac{\sin x}{x}+\frac{\sin y}{y}+\cos z=-z\,\phi'(z)$$

$$\Rightarrow \quad \phi(z)+\cos z=-z\,\phi'(z)$$

$$\Rightarrow \quad \frac{d\phi}{dz}+\frac{1}{z}\phi=-\frac{\cos z}{z} \quad ...(5)$$

This is a linear equation of first order.

$$\text{I.F.}=e^{\int\frac{1}{z}dz}=e^{\log z}=z$$

$\therefore$ The solution of (5) is $\quad \phi(z)\cdot z=\int\left(-\frac{\cos z}{z}\right)z\,dz+c$

$$\Rightarrow \quad z\,\phi(z)=-\sin z+c$$

$$\Rightarrow \quad z\left(\frac{\sin x}{x}+\frac{\sin y}{y}\right)=-\sin z+c$$

$$\Rightarrow \quad \frac{\sin x}{x}+\frac{\sin y}{y}+\frac{\sin z}{z}=\frac{c}{z}.$$

This is the required solution.

(*ii*) We have $\quad 2yz\,dx+zx\,dy-xy(1+z)dz=0 \quad ...(1)$

Here $\quad P=2yz,\ Q=zx,\ R=-xy(1+z)$

$$\therefore \quad \frac{\partial P}{\partial y}=2z,\ \frac{\partial P}{\partial z}=2y,\ \frac{\partial Q}{\partial x}=z,\ \frac{\partial Q}{\partial z}=x,\ \frac{\partial R}{\partial x}=-y-yz,\ \frac{\partial R}{\partial y}=-x-xz$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$=2yz(x+x+xz)+zx(-y-yz-2y)-xy(1+z)(2z-z)$$

$$=4xyz+2xyz^2-3xyz-xyz^2-xyz-xyz^2=0$$

$\therefore$ The equation (1) is integrable. Considering z as a constant, we have $dz=0$.

$$(1) \Rightarrow \quad 2yz\,dx+zx\,dy=0$$

$$\Rightarrow \quad \frac{2}{x}dx+\frac{1}{y}dy=0$$

Integrating, we get $\quad 2\log x+\log y=\log\phi(z)$,

where $\phi(z)$ is any arbitrary function of z.

$$\therefore \quad x^2y=\phi(z) \quad ...(2)$$

Differentiating (2) w.r.t. x, y and z, we get

$$2xy\,dx+x^2\,dy=\phi'(z)dz$$

$$\Rightarrow \quad 2xy\,dx+x^2\,dy-\phi'(z)\,dz=0$$

NOTES

Multiplying by $\frac{z}{x}$, we get

$$2yz\,dx + xz\,dy - \frac{z}{x}\,\phi'(z)\,dz = 0 \qquad \ldots(3)$$

$$(1) - (3) \quad \Rightarrow \quad -xy(1+z)dz + \frac{z}{x}\,\phi'(z)\,dz = 0$$

$$\Rightarrow \qquad x^2y(1+z) = z\,\phi'(z) \quad \Rightarrow \quad \phi(z)(1+z) = z\,\phi'(z)$$

$$\Rightarrow \qquad \frac{\phi'(z)}{\phi(z)} = \frac{1+z}{z} \quad \Rightarrow \quad \frac{\phi'(z)}{\phi(z)} = \frac{1}{z} + 1$$

Integrating, we get

$$\log \phi(z) = \log z + z + c$$

$$\Rightarrow \qquad \log x^2y = \log z + z + c$$

$$\Rightarrow \qquad \log \frac{x^2y}{z} = z + c.$$

This is the required solution.

EXERCISE 5

Solve the following total differential equations

1. $(y + z)dx + (z + x)dy + (x + y)dz = 0$
2. $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$
3. $xz^3\,dx - z\,dy + 2y\,dz = 0$
4. $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$
5. $(x^2 + y^2 + z^2)dx - 2xy\,dy - 2xz\,dz = 0$
6. $(e^xy + e^z)dx + (e^yz + e^x)dy + (e^y - e^xy - e^yz)dz = 0$
7. $(z + z^3)\cos x\,dx - (z + z^3)dy + (1 - z^2)(y - \sin x)dz = 0$
8. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$

Answers

1. $xy + yz + zx = c$
2. $(x + y)z = c(x + z)$
3. $2y - x^2z^2 = 2cz^2$
4. $x^3 + y^3 = e^{2z} + ce^z$
5. $y^2 + z^2 = x^2 + cx$
6. $e^xy + e^zx + e^yz = ce^z$
7. $(\sin x - y)(z^2 + 1) = cz$
8. $x^2 - xy + y^2 = cz.$

8.13. (METHOD VI) SOLUTION OF INTEGRABLE NON-EXACT TOTAL DIFFERENTIAL EQUATION BY THE METHOD OF AUXILIARY EQUATIONS

Let $\quad P\,dx + Q\,dy + R\,dz = 0 \qquad \ldots(1)$

be an integrable non-exact total differential equation

$$\therefore \qquad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad \ldots(2)$$

Comparing (1) and (2), we have

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \qquad \ldots(3)$$

These are called the **auxiliary equations** of the given equation (1).

Since (1) is non-exact, $\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}$ and $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ are not all zero.

Let $u(x, y, z) = a$ and $v(x, y, z) = b$ be two solutions* of (3).

In the next step, we form the equation $A du + B dv = 0$ and find the values of A and B by comparing this equation with the given equation (1). Using $u(x, y, z) = a$ and $v(x, y, z) = b$, we find the values of A and B in terms of u and v. These values of A and B are substituted in the equation $A du + B dv = 0$. The solution of this equation gives the solution of the given equation (1)

Example 1. *Solve the following total differential equations :*

(*i*) $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)\, dz = 0$

(*ii*) $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)\, dy + (xy^2 + x^2y)dz = 0.$

Sol. (*i*) We have $\quad (y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0 \qquad \ldots(1)$

Here $\quad P = y^2 + yz,\ Q = z^2 + zx,\ R = y^2 - xy$

$$\therefore \quad \frac{\partial P}{\partial y} = 2y + z, \frac{\partial P}{\partial z} = y, \frac{\partial Q}{\partial x} = z, \frac{\partial Q}{\partial z} = 2z + x, \frac{\partial R}{\partial x} = -y, \frac{\partial R}{\partial y} = 2y - x$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial P}{\partial x}$, the equation (1) is not exact.

$$\text{Now} \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)(2z + x - 2y + x) + (z^2 + zx)(-y - y) + (y^2 - xy)(2y + z - z)$$

$$= 2y(y + z)(z - y + x) - 2yz(z + x) + 2y^2(y - x)$$

$$= 2y[(y + z)(z - y + x) - z(z + x) + y(y - x)]$$

$$= 2y[z^2 - y^2 + xy + xz - z^2 - xz + y^2 - xy]$$

$$= 2y \times 0 = 0$$

$\therefore$ The equation (1) is integrable

The auxiliary equations of (1) are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

$$\Rightarrow \quad \frac{dx}{2z + x - 2y + x} = \frac{dy}{-y - y} = \frac{dz}{2y + z - z}$$

$$\Rightarrow \quad \frac{dx}{2(x - y + z)} = \frac{dy}{-2y} = \frac{dz}{2y}$$

$$\Rightarrow \quad \frac{dx}{x - y + z} = \frac{dy}{-y} = \frac{dz}{y} \qquad \ldots(2)$$

Taking last two fractions of (2), we get $-dy = dz$.

$$\Rightarrow \quad d(y + z) = 0$$

Integrating, we get $\quad y + z = a$

*Details regarding the methods of finding such solutions is given in chapter 10.

NOTES

Taking 1, 0, 1 as multipliers, each fraction of (2)

$$= \frac{1\cdot dx + 0\cdot dy + 1\cdot dz}{1.(x-y+z)+0.(-y)+1.(y)} = \frac{dx+dz}{x+z} \quad ...(3)$$

$\therefore$ We have $\frac{dx+dz}{x+z} = \frac{dy}{-y} \Rightarrow \frac{d(x+z)}{x+z} + \frac{dy}{y} = 0$

Integrating, we get

$\log(x+z) + \log y = \log b$ or $(x+z)y = b$

Let $u(x, y, z) = y + z$ and $v(x, y, z) = (x+z)y$

$\therefore$ $du = dy + dz$ and $dv = ydx + (x+z)dy + ydz$

Let $A du + B dv = 0$ ...(4)

$\Rightarrow$ $A(dy + dz) + B(ydx + (x+z)dy + ydz) = 0$

$\Rightarrow$ $Bydx + (A + B(x+z))\, dy + (A + By)dz = 0$ ...(5)

Comparing (1) and (5), we have

$By = y^2 + yz$ ...(6)

$A + B(x+z) = z^2 + zx$ ...(7)

and $A + By = y^2 - xy$ ...(8)

(6) $\Rightarrow$ $B = y + z = u$

(7) $\Rightarrow$ $A = z^2 + zx - (y+z)(x+z) = -xy - yz = -y(x+z) = -v$

Putting the values of A and B in (4), we get

$-vdu + udv = 0$

$\Rightarrow$ $\frac{du}{u} - \frac{dv}{v} = 0$

Integrating, we get

$\log u - \log v = \log c$

$\Rightarrow$ $\frac{u}{v} = c$ or $\frac{y+z}{(x+z)y} = c$ or $\mathbf{y + z = c(x+z)y}$.

This is the required solution.

(*ii*) We have $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$ ...(1)

Here $P = x^2y - y^3 - y^2z$, $Q = xy^2 - x^2z - x^3$, $R = xy^2 + x^2y$

$\frac{\partial P}{\partial y} = x^2 - 3y^2 - 2yz$, $\frac{\partial P}{\partial z} = -y^2$, $\frac{\partial Q}{\partial x} = y^2 - 2xz - 3x^2$, $\frac{\partial Q}{\partial z} = -x^2$, $\frac{\partial R}{\partial x} = y^2 + 2xy$,

$\frac{\partial R}{\partial y} = 2xy + x^2$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the equation (1) is not exact.

It can be verified that $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$

$\therefore$ The equation (1) is integrable.

The auxiliary equations of (1) are

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

$$\Rightarrow \quad \frac{dx}{-x^2-2xy-x^2}=\frac{dy}{y^2+2xy+y^2}=\frac{dz}{x^2-3y^2-2yz-y^2+2xz+3x^2}$$

$$\Rightarrow \quad \frac{dx}{-2x(x+y)}=\frac{dy}{2y(x+y)}=\frac{dz}{4x^2-4y^2-2yz+2xz}$$

$$\Rightarrow \quad \frac{dx}{-x(x+y)}=\frac{dy}{y(x+y)}=\frac{dz}{2x^2-2y^2-yz+xz} \qquad \ldots(2)$$

Taking first two fractions of (2), we get $\frac{dx}{x}+\frac{dy}{y}=0$.

Integrating, we get

$$\log x+\log y=\log a \quad \Rightarrow \quad xy=a$$

Taking 1, 1, 0 as multipliers, each fraction of (2)

$$=\frac{1\cdot dx+1\cdot dy+0\cdot dz}{-x(x+y)+y(x+y)+0}=\frac{dx+dy}{(y-x)(x+y)}$$

Taking 1, 1, 1 as multipliers, each fraction of (2)

$$=\frac{1\cdot dx+1\cdot dy+1\cdot dz}{-x(x+y)+y(x+y)+2x^2-2y^2-yz+xz}$$

$$=\frac{dx+dy+dz}{x^2-y^2+xz-yz}=\frac{dx+dy+dz}{(x-y)(x+y)+z(x-y)}$$

$$=\frac{dx+dy+dz}{(x-y)(x+y+z)}$$

$$\therefore \quad \frac{dx+dy}{(y-x)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y+z)} \quad \Rightarrow \quad \frac{d(x+y)}{x+y}+\frac{d(x+y+z)}{x+y+z}=0$$

Integrating, we get

$$\log(x+y)+\log(x+y+z)=\log b$$

$$\Rightarrow \quad (x+y)(x+y+z)=b$$

Let $u(x,y,z)=xy$ and $v(x,y,z)=(x+y)(x+y+z)$

$$\therefore \quad du=y\,dx+x\,dy$$

and $dv=(x+y)(dx+dy+dz)+(x+y+z)(dx+dy)$

$$=(2x+2y+z)dx+(2x+2y+z)dy+(x+y)dz$$

Let $\quad A\,du+B\,dv=0 \qquad \ldots(3)$

$$\Rightarrow \quad A(y\,dx+x\,dy)+B((2x+2y+z)dx+(2x+2y+z)dy+(x+y)dz)=0$$

$$\Rightarrow \quad (Ay+B(2x+2y+z))dx+(Ax+B(2x+2y+z))dy+B(x+y)dz=0 \qquad \ldots(4)$$

Comparing (1) and (4), we have

$$Ay+B(2x+2y+z)=x^2y-y^3-y^2z \qquad \ldots(5)$$

$$Ax+B(2x+2y+z)=xy^2-x^2z-x^3 \qquad \ldots(6)$$

$$B(x+y)=xy^2+x^2y \qquad \ldots(7)$$

$$(7) \quad \Rightarrow \quad B=xy=u$$

$$(5)-(6) \quad \Rightarrow \quad A(y-x)=x^2y-y^3-y^2z-xy^2+x^2z+x^3$$

NOTES

$$= (x^3 - y^3) + (x^2y - xy^2) + (x^2z - y^2z)$$
$$= (x - y)[x^2 + xy + y^2 + xy + z(x + y)]$$
$$= (x - y)[(x + y)^2 + z(x + y)] = (x - y)(x + y)(x + y + z)$$

$\Rightarrow \qquad A = -(x - y)(x + y + z) = -v$

Putting the values of A and B in (3), we get

$$-v\,du + u\,dv = 0$$

$$\Rightarrow \qquad \frac{du}{u} - \frac{dv}{v} = 0$$

Integrating, we get

$$\log u - \log v = \log c$$

$$\Rightarrow \qquad \frac{u}{v} = c \;\Rightarrow\; u = cv \;\Rightarrow\; \mathbf{xy = c(x + y)(x + y + z).}$$

This is the required solution.

EXERCISE 6

Solve the following total differential equations :

1. $(y + z)dx + dy + dz = 0$
2. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$
3. $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$
4. $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$

Answers

1. $e^x (y + z) = c$
2. $x^2 + y^2 - xy = cz$
3. $(x + y)z = c(x + z)$
4. $xy + yz + zx = c(x + y + z).$

UNIT 9 LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

9 LINEAR DIFFERENTIAL [illegible]

9.0. LEARNING OBJECTIVES

NOTES

After going through this unit you will be able to:

- Find methods of solving linear differential equation of the second order
- Discribe method of finding a particular integral of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$
- Explain working rules for solving $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

9.1. INTRODUCTION

In this chapter, we shall learn the methods of solving linear differential equation of the second order.

If P, Q and R are functions of x then

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

is a linear differential equation of the second order. There is no general method of solving a linear differential equation of the second order. We shall study certain procedures which at times will yield a solution.

9.2. METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

In the present text, we shall discuss the following methods of solving a linear differential equation of the second order:

(*i*) Method of changing the dependent variable

(*ii*) Method of changing the independent variable

(*iii*) Method of variation of parameters.

9.3. FIRST METHOD OF SOLVING A LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER BY CHANGING THE DEPENDENT VARIABLE (BY USING A PARTICULAR INTEGRAL OF A CORRESPONDING EQUATION OF THE GIVEN EQUATION)

Let $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(1)$$

be a linear differential equation of the second order where P, Q, and R are functions of x.

Let $y = uv$ where u and v are functions of x.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

and
$$\frac{d^2y}{dx^2} = \left(u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx}\right) + \left(v\frac{d^2u}{dx^2} + \frac{dv}{dx}\frac{du}{dx}\right) = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}.$$

$$\therefore \ (1) \Rightarrow \quad \left(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = \frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + P_1\frac{dv}{dx} + Q_1 v = R_1 \qquad \ldots(2)$$

where
$$P_1 = \frac{2}{u}\frac{du}{dx} + P \quad Q_1 = \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) \quad \text{and} \quad R_1 = \frac{R}{u}.$$

Let u be any particular integral of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

$$\therefore \quad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \text{Also} \quad Q_1 = \frac{1}{u}\times 0 = 0$$

$$\therefore \ (2) \text{ becomes} \quad \frac{d^2v}{dx^2} + P_1\frac{dv}{dx} = R_1$$

$$\Rightarrow \quad \frac{dp}{dx} + P_1 p = R_1, \quad \text{where } p = \frac{dv}{dx} \qquad \ldots(3)$$

(3) is a linear differential equation of first order.

$$\therefore \quad \text{I.F.} = e^{\int P_1 dx} = e^{\int\left(\frac{2}{u}\frac{du}{dx} + P\right)dx} = e^{2\log u + \int P dx} = u^2 e^{\int P dx}$$

$\therefore$ The solution of (3) is

$$p \cdot u^2 e^{\int P dx} = \int\left(\frac{R}{u}\cdot u^2 e^{\int P dx}\right)dx + c_1$$

$$\Rightarrow \quad \frac{dv}{dx} = u^{-2} e^{-\int P dx}\int\left(R u e^{\int P dx}\right)dx + c_1 u^{-2} e^{-\int P dx}.$$

Integrating this equation, we get the value of v.

Hence the solution $y = uv$ is known. This solution will involve two arbitrary constants.

9.4. METHOD OF FINDING A PARTICULAR INTEGRAL OF $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

NOTES

I. Let $y = e^{mx}$ be a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

$\therefore \quad m^2 e^{mx} + Pme^{mx} + Qe^{mx} = 0 \quad$ or $\quad m^2 + Pm + Q = 0$.

$\therefore$ **e^{mx} is a solution if** $m^2 + Pm + Q = 0$

Particular cases :

(*i*) e^x is a solution if $1 + P + Q = 0$

(*ii*) e^{-x} is a solution if $1 - P + Q = 0$

II. Let x^m be a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

$\therefore \quad m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$

or $\quad m(m-1) + Pmx + Qx^2)\, x^{m-2} = 0$

or $\quad m(m-1) + Pmx + Qx^2 = 0$

$\therefore$ **x^m is a solution if** $m(m-1) + Pmx + Qx^2 = 0$.

Particular cases :

(*i*) x is a solution if $P + Qx = 0$

(*ii*) x^2 is a solution if $2 + 2Px + Qx^2 = 0$

$\therefore$ 1. If $m^2 + Pm + Q = 0$ then e^{mx} is a solution.

2. If $1 + P + Q = 0$, then e^x is a solution

3. If $1 - P + Q = 0$, then e^{-x} is a solution.

4. If $m(m-1) + Pmx + Qx^2 = 0$, then x^m is a solution.

5. If $P + Qx = 0$, then x is a solution.

6. If $2 + 2Px + Qx^2 = 0$, then x^2 is a solution.

9.5. WORKING RULES FOR SOLVING $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Step I. Find a particular integral of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$. Call this function as u. Sometimes it is given with the problem itself.

Step II. Take $y = uv$ and change the dependent variable y to v. The resultant equation will be a linear equation of first order with dependent variable $\frac{dv}{dx}$.

Step III. Put $p = \frac{dv}{dx}$. Solve the equation and find p. Put $p = \frac{dv}{dx}$ and integrate to find the value of v.

Step IV. Find $y = uv$. This gives the general solution of the given equation.

Example 1. *Solve* $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0$.

Sol. We have $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0.$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-\frac{2x-1}{x}\frac{dy}{dx}+\frac{x-1}{x}y=0 \qquad \text{...(1)}$$

Comparing (1) with

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \quad \text{we get}$$

$$P=-\frac{2x-1}{x},\quad Q=\frac{x-1}{x},\quad R=0$$

Here $\quad 1+P+Q=1-\frac{2x-1}{x}+\frac{x-1}{x}=0$

$\therefore$ e^x is a particular integral of (1), whose right member is already zero.

Let $y=e^x v$.

$$\therefore \quad \frac{dy}{dx}=e^x\frac{dv}{dx}+ve^x=e^x\left(\frac{dv}{dx}+v\right)$$

and

$$\frac{d^2y}{dx^2}=e^x\left(\frac{d^2v}{dx^2}+\frac{dv}{dx}\right)+e^x\left(\frac{dv}{dx}+v\right)=e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right).$$

$\therefore$ (1) becomes

$$e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\frac{2x-1}{x}e^x\left(\frac{dv}{dx}+v\right)+\frac{x-1}{x}e^xv=0$$

$$\Rightarrow \quad e^x\frac{d^2v}{dx^2}+\left(2e^x-\frac{2x-1}{x}e^x\right)\frac{dv}{dx}+\left(e^x-\frac{2x-1}{x}e^x+\frac{x-1}{x}e^x\right)v=0$$

$$\Rightarrow \quad e^x\frac{d^2v}{dx^2}+\frac{e^x}{x}\frac{dv}{dx}=0 \quad \Rightarrow \quad \frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0$$

$$\Rightarrow \quad \frac{dp}{dx}+\frac{1}{x}p=0 \text{ where } p=\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p}+\frac{dx}{x}=0 \quad \Rightarrow \quad \log p+\log x=\log c_1$$

$$\Rightarrow \quad px=c_1 \quad \Rightarrow \quad \frac{dv}{dx}x=c_1$$

$$\Rightarrow \quad dv=c_1\frac{dx}{x} \quad \Rightarrow \quad v=c_1\log x+c_2.$$

$$\therefore \quad y=e^xv=e^x(c_1\log x+c_2)$$

$\therefore$ The general solution of (1) is $\mathbf{y=c_1e^x\log x+c_2e^x}$.

Example 2. *Solve* $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)\,e^x$.

Sol. Dividing by $x+2$, the given equation in standard form is

NOTES

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x \qquad \ldots(1)$$

Comparing (1) with $\quad \dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R\quad$ we get

$$P = -\frac{2x+5}{x+2} \qquad Q = \frac{2}{x+2} \qquad R = \frac{x+1}{x+2}e^x$$

Here $\quad 4 + 2P + Q = 4 + 2\left(-\dfrac{2x+5}{x+2}\right) + \dfrac{2}{x+2} = 0$

$\therefore\quad e^{2x}$ is a particular integral of (1) with its right member replaced by zero.

Let $\qquad y = e^{2x}\,v$

$$\therefore \qquad \frac{dy}{dx} = e^{2x}\left(\frac{dv}{dx} + 2v\right) \quad \text{and} \quad \frac{d^2y}{dx^2} = e^{2x}\left(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v\right).$$

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$e^{2x}\left(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v\right) - \frac{2x+5}{x+2}e^{2x}\left(\frac{dv}{dx} + 2v\right) + \frac{2}{x+2}e^{2x}\,v = \frac{x+1}{x+2}e^x$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v - \frac{2x+5}{x+2}\left(\frac{dv}{dx} + 2v\right) + \frac{2}{x+2}v = \frac{x+1}{x+2}e^{-x}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\Rightarrow \qquad \frac{dp}{dx} + \frac{2x+3}{x+2}p = \frac{x+1}{x+2}e^{-x} \qquad \ldots(2)$$

where $\quad p = \dfrac{dv}{dx}$.

(2) is a linear differential equation of the first order

$$\text{I.F.} = e^{\int \frac{2x+3}{x+2}dx} = e^{\int\left(2 - \frac{1}{x+2}\right)dx} = e^{2x - \log(x+2)} = e^{2x}\cdot e^{-\log(x+2)} = \frac{e^{2x}}{x+2}.$$

$\therefore\quad$ The solution of (2) is

$$p\,\frac{e^{2x}}{x+2} = \int \frac{x+1}{x+2}e^{-x}\cdot\frac{e^{2x}}{x+2}dx + c_1$$

$$\Rightarrow \qquad p\,\frac{e^{2x}}{x+2} = \int \frac{x+1}{(x+2)^2}e^x\,dx + c_1 = \int\left(\frac{1}{x+2} - \frac{1}{(x+2)^2}\right)e^x\,dx + c_1$$

$$= \frac{e^x}{x+2} + c_1$$

$\therefore \qquad p = e^{-x} + c_1 e^{-2x}(x+2) \quad \text{or} \quad \dfrac{dv}{dx} = e^{-x} + c_1 e^{-2x}(x+2)$

$\Rightarrow \qquad \int dv = \int (e^{-x} + c_1 e^{-2x}(x+2))\, dx + c_2$

$\therefore \qquad v = -e^{-x} + c_1 \int (x+2) e^{-2x}\, dx + c_2$

$$= -e^{-x} + c_1 \left[(x+2) \cdot \frac{e^{-2x}}{-2} - \int 1 \cdot \frac{e^{-2x}}{-2}\, dx \right] + c_2$$

$$= -e^{-x} - \frac{c_1}{2}(x+2) e^{-2x} - \frac{c_1}{4} e^{-2x} + c_2$$

$\therefore \qquad v = -e^{-x} - \dfrac{c_1}{4}(2x+5) e^{-2x} + c_2$

$\therefore \qquad y = e^{2x} v = -e^{x} - \dfrac{c_1}{4}(2x+5) + c_2 e^{2x}$

$\therefore$ The general solution of (1) is

$$\mathbf{y = -e^{x} - \frac{c_1}{4}(2x+5) + c_2 e^{2x}.}$$

Example 3. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3$.

Sol. Dividing by x^2, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2} y = x \qquad \ldots(1)$$

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Q\,y = R$, we get

$$P = -\frac{2(1+x)}{x},\ Q = \frac{2(1+x)}{x^2},\ R = x.$$

Here $\qquad P + xQ = -\dfrac{2(1+x)}{x} + \dfrac{2(1+x)}{x} = 0.$

$\therefore$ x is a particular integral of (1) with its right member replaced by zero.

Let $y = xv$.

$\therefore \qquad \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v \quad \text{and} \quad \dfrac{d^2y}{dx^2} = \left(x\dfrac{d^2v}{dx^2} + 1 \cdot \dfrac{dv}{dx} \right) + \dfrac{dv}{dx} = x\dfrac{d^2v}{dx^2} + 2\dfrac{dv}{dx}$

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\left(x\frac{dv}{dx} + v \right) + \frac{2(1+x)}{x^2} xv = x$$

$\Rightarrow \qquad x\dfrac{d^2v}{dx^2} - 2x\dfrac{dv}{dx} = x \quad \Rightarrow \quad \dfrac{d^2v}{dx^2} - 2\dfrac{dv}{dx} = 1 \quad \Rightarrow \quad \dfrac{dp}{dx} - 2p = 1 \qquad \ldots(2)$

where $p = \dfrac{dv}{dx}$

NOTES

(2) is a linear differential equation of the first order

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

∴ The solution of (2) is

$$p\,.\,e^{-2x} = \int e^{-2x}\,dx + c_1 \quad \Rightarrow \quad pe^{-2x} = \frac{e^{-2x}}{-2} + c_1$$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x} \quad \Rightarrow \quad \int dv = \int\left(-\frac{1}{2} + c_1 e^{2x}\right)dx + c_2$$

$$\therefore \quad v = -\frac{x}{2} + c_1 \frac{e^{2x}}{2} + c_2$$

$$\therefore \quad y = xv = -\frac{x^2}{2} + c_1 \frac{xe^{2x}}{2} + xc_2$$

∴ The general solution of (1) is

$$y = -\frac{x^2}{2} + \frac{c_1 x e^{2x}}{2} + xc_2.$$

Example 4. *Solve* $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - (1 - \cot x)\, y = e^x \sin x$.

Sol. We have $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - (1 - \cot x)\, y = e^x \sin x$. ...(1)

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = -\cot x,\ Q = -1 + \cot x,\ R = e^x \sin x$$

Here $1 + P + Q = 1 - \cot x - 1 + \cot x = 0.$

∴ e^x is a particular integral of (1) with its right member replaced by zero.

Let $y = e^x v$.

$$\therefore \quad \frac{dy}{dx} = e^x\left(\frac{dv}{dx} + v\right) \quad \text{and} \quad \frac{d^2y}{dx^2} = e^x\left(\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right)$$

∴ (1) becomes

$$e^x\left(\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right) - (\cot x)\, e^x\left(\frac{dv}{dx} + v\right) - (1 - \cot x)\, e^x v = e^x \sin x$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x \quad \Rightarrow \quad \frac{dp}{dx} + (2 - \cot x)p = \sin x \quad \text{...(2)}$$

where $p = \dfrac{dv}{dx}$.

(2) is a linear differential equation of the first order.

$$\text{I.F.} = e^{\int (2 - \cot x)dx} = e^{2x - \log \sin x} = e^{2x} \div e^{\log \sin x} = \frac{e^{2x}}{\sin x}.$$

∴ The solution of (2) is

$$p\,\frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\,dx + c_1 \quad \Rightarrow \quad p\,\frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\sin x}{2} + \frac{c_1 \sin x}{e^{2x}} \Rightarrow \int dv = \int \left(\frac{\sin x}{2} + \frac{c_1 \sin x}{e^{2x}} \right) dx + c_2$$

$$\Rightarrow \quad v = -\frac{\cos x}{2} + c_1 \int e^{-2x} \sin x dx + c_2 .$$

Let $\quad I = \int e^{-2x} \sin x \, dx.$

$$\therefore \quad I = e^{-2x} \cdot -\cos x - \int -2 e^{-2x} \cdot -\cos x dx$$

$$= -e^{-2x} \cdot \cos x - 2 \int e^{-2x} \cos x dx$$

$$= -e^{-2x} \cdot \cos x - 2 \left[e^{-2x} \cdot \sin x - \int -2 e^{-2x} \sin x dx \right]$$

$$= -e^{-2x} \cos x - 2 e^{-2x} \sin x - 4 \int e^{-2x} \sin x dx$$

$$\therefore \quad I = -e^{-2x} (\cos x - 2 \sin x) - 4I.$$

$$\therefore \quad I = -\frac{e^{-2x}}{5} (\cos x - 2 \sin x).$$

$$\therefore \quad v = -\frac{\cos x}{2} - \frac{c_1 e^{-2x}}{5} (\cos x - 2 \sin x) + c_2$$

$$\therefore \quad \mathbf{y = e^x v = -\frac{e^x \cos x}{2} - \frac{c_1 e^{-x}}{5} (\cos x - 2 \sin x) + c_2 e^x.}$$

This gives the general solution of the given equation.

Example 5. *Solve* $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0$ *given that* $x + \frac{1}{x}$ *is one integral.*

Sol. We have $\quad x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0. \qquad \ldots(1)$

Let $\quad y = \left(x + \frac{1}{x} \right) v.$

$$\therefore \quad \frac{dy}{dx} = \left(x + \frac{1}{x} \right) \frac{dv}{dx} + \left(1 - \frac{1}{x^2} \right) v$$

and

$$\frac{d^2y}{dx^2} = \left(x + \frac{1}{x} \right) \frac{d^2v}{dx^2} + \left(1 - \frac{1}{x^2} \right) \frac{dv}{dx} + \left(1 - \frac{1}{x^2} \right) \frac{dv}{dx} + \frac{2}{x^3} v$$

$$= \left(x + \frac{1}{x} \right) \frac{d^2v}{dx^2} + 2 \left(1 - \frac{1}{x^2} \right) \frac{dv}{dx} + \frac{2}{x^3} v.$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(x^3 + x) \frac{d^2v}{dx^2} + (2x^2 - 2) \frac{dv}{dx} + \frac{2}{x} v + (x^2 + 1) \frac{dv}{dx} + \left(x - \frac{1}{x} \right) v - \left(x + \frac{1}{x} \right) v = 0$$

$$\therefore \quad (x^3 + x) \frac{d^2v}{dx^2} + (3x^2 - 1) \frac{dv}{dx} = 0 .$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \frac{3x^2-1}{x(x^2+1)}\frac{dv}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} + \frac{3x^2-1}{x(x^2+1)}p = 0 \qquad ...(2)$$

NOTES

where $p = \frac{dv}{dx}$.

$$\Rightarrow \quad \frac{dp}{p} + \left(\frac{4x}{x^2+1} - \frac{1}{x}\right)dx = 0$$

(On resolving into partial fractions)

Integrating, we get

$$\log p + 2\log(x^2+1) - \log x = \log c_1$$

$$\Rightarrow \quad \frac{p(x^2+1)^2}{x} = c_1 \quad \Rightarrow \quad p = \frac{c_1 x}{(x^2+1)^2}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{c_1 x}{(x^2+1)^2} \quad \Rightarrow \quad dv = \frac{c_1}{2}\cdot\frac{2x}{(x^2+1)^2}dx$$

Integrating, we get

$$v = \frac{c_1}{2}\cdot\frac{(x^2+1)^{-1}}{-1} + c_2 \quad \Rightarrow \quad v = -\frac{c_1}{2(x^2+1)} + c_2$$

$$\therefore \quad y = \left(x + \frac{1}{x}\right)v = \frac{x^2+1}{x}v = -\frac{c_1}{2x} + \frac{c_2(x^2+1)}{x}$$

$\therefore$ The general solution of (1) is $\mathbf{y = \frac{c_1'}{x} + c_2\left(x + \frac{1}{x}\right)}$.

Example 6. *Solve* $\sin^2 x \frac{d^2y}{dx^2} = 2y$ *given that* $y = \cot x$ *is a solution.*

Sol. We have $\sin^2 x \frac{d^2y}{dx^2} - 2y = 0$ or $\frac{d^2y}{dx^2} - \frac{2}{\sin^2 x}y = 0$ ...(1)

Let $y = v\cot x$.

$$\therefore \quad \frac{dy}{dx} = -v\,\text{cosec}^2 x + \cot x\frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v\cdot 2\,\text{cosec}^2 x\cot x - \text{cosec}^2 x\frac{dv}{dx} + \cot x\frac{d^2v}{dx^2} - \text{cosec}^2 x\frac{dv}{dx}$$

$$= 2v\,\text{cosec}^2 x\cot x - 2\,\text{cosec}^2 x\frac{dv}{dx} + \cot x\frac{d^2v}{dx^2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$2v\,\text{cosec}^2 x\cot x - 2\,\text{cosec}^2 x\frac{dv}{dx} + \cot x\frac{d^2v}{dx^2} - \frac{2}{\sin^2 x}v\cot x = 0$$

$$\Rightarrow \quad \cot x\frac{d^2v}{dx^2} - 2\,\text{cosec}^2 x\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \cot x\frac{dp}{dx} - 2p\,\text{cosec}^2 x = 0 \qquad ...(2)$$

where $p = \dfrac{dv}{dx}$

$$\Rightarrow \quad \frac{dp}{p} - \frac{2\,\text{cosec}^2 x}{\cot x}\,dx = 0.$$

Integrating, we get

$$\log p + 2\log\cot x = \log c_1$$

$$\Rightarrow \quad p\cot^2 x = c_1 \quad \Rightarrow \quad \frac{dv}{dx} = c_1 \tan^2 x \quad \Rightarrow \quad dv = c_1(\sec^2 x - 1)\,dx$$

Integrating, we get

$$v = c_1(\tan x - x) + c_2$$

$$\therefore \quad y = v\cot x = c_1(1 - x\cot x) + c_2 \cot x$$

$\therefore$ The general solution of (1) is $\mathbf{y = c_1(1 - x\cot x) + c_2\cot x}$.

EXERCISE 1

Solve the following differential equations by changing the dependent variable :

1. $\dfrac{d^2y}{dx^2} - \dfrac{3}{x}\dfrac{dy}{dx} + \dfrac{3y}{x^2} = 2x - 1$
2. $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$
3. $x^2\dfrac{d^2y}{dx^2} - (x^2 + 2x)\dfrac{dy}{dx} + (x+2)y = x^3 e^x$
4. $(x+1)\dfrac{d^2y}{dx^2} - 2(x+3)\dfrac{dy}{dx} + (x+5)y = e^x$
5. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + (1-x)y = x^2 e^{-x}$
6. $x\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - 2y = x^3$
7. $x\dfrac{dy}{dx} - y = (x-1)\left[\dfrac{d^2y}{dx^2} - x + 1\right]$
8. $\dfrac{d^2y}{dx^2} + (1 - \cot x)\dfrac{dy}{dx} - y\cot x = \sin^2 x$
9. $x\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + (x+2)y = (x-2)e^x$
10. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$ given that $y = x^3$ is a solution of the given equation.

Answers

1. $y = c_1 x^3 + c_2 x + x^3\log x + x^2$
2. $y = c_1\cos x + c_2 x$
3. $y = c_1 x e^x + c_2 x + (x-1)\,x e^x$
4. $y = c_1 e^x (x+1)^5 - \dfrac{1}{4}\,x e^x + c_2 e^x$
5. $y = c_1(2x+1)e^{-x} + c_2 e^x - \dfrac{1}{4}e^{-x}(2x^2 + 2x + 1)$
6. $y = c_1(x^2 - 2x + 2) + c_2 e^{-x} + x^3 - 3x^2 + 6x - 6$
7. $y = c_1 e^x + c_2 x - 1 - x^2$
8. $y = c_1(\sin x - \cos x) + c_2 e^{-x} - \dfrac{1}{10}(\sin 2x - 2\cos 2x)$
9. $y = c_1 x^3 e^x + c_2 e^x - \dfrac{1}{2}x^2 e^x + x e^x$
10. $y = c_1 x^3 + c_2 x^{-3}$.

NOTES

9.6. SECOND METHOD OF SOLVING A LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER BY CHANGING THE DEPENDENT VARIABLE (BY THE REMOVAL OF THE FIRST DERIVATIVE)

Let $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(1)$$

be a linear differential equation of the second order where P, Q and R are functions of x.

This method is used when a particular integral of the given equation when R is replaced by zero is neither given nor is easily found.

Let $y = uv$ where u and v are functions of x.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$\therefore \quad (1) \Rightarrow \left(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = \frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + P_1\frac{dv}{dx} + Q_1 v = R_1 \quad ...(2)$$

where $$P_1 = \frac{2}{u}\frac{du}{dx} + P,\ Q_1 = \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) \quad \text{and} \quad R_1 = \frac{R}{u}$$

Let u be any function such that $P_1 = \frac{2}{u}\frac{du}{dx} + P = 0$ ...(3)

$$\therefore \quad \frac{du}{u} = -\frac{1}{2}P\,dx$$

Integrating, we get

$$\log u = -\frac{1}{2}\int P\,dx \quad \therefore \quad u = e^{-\frac{1}{2}\int P\,dx}$$

$$(3) \quad \Rightarrow \frac{du}{dx} = -\frac{1}{2}uP \quad \text{and} \quad \frac{d^2u}{dx^2} = -\frac{1}{2}\left(u\frac{dP}{dx} + P\frac{du}{dx}\right) = -\frac{1}{2}u\frac{dP}{dx} - \frac{1}{2}P\frac{du}{dx}$$

$$\therefore \quad Q_1 = \frac{1}{u}\left(-\frac{1}{2}u\frac{dP}{dx} - \frac{1}{2}P\frac{du}{dx}\right) + \frac{P}{u}\frac{du}{dx} + Q$$

$$= -\frac{1}{2}\frac{dP}{dx} - \frac{P}{2u}\frac{du}{dx} + \frac{P}{u}\frac{du}{dx} + Q = -\frac{1}{2}\frac{dP}{dx} + \frac{P}{2u}\frac{du}{dx} + Q$$

$$= -\frac{1}{2}\frac{dP}{dx} + \frac{P}{2u}\left(-\frac{1}{2}uP\right) + Q \quad = -\frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 + Q$$

$\therefore \qquad Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

Also, $\qquad R_1 = \frac{R}{u} = \frac{R}{e^{-\frac{1}{2}\int P\,dx}} = R\,e^{\frac{1}{2}\int P\,dx}$

$\therefore$ (2) becomes $\quad \frac{d^2v}{dx^2} + Q_1 v = R_1 \qquad \ldots(4)$

where $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $R_1 = R\,e^{\frac{1}{2}\int P\,dx}$

Equation (4) is called the **normal form** of the equation (1). In the normal form (4), the first derivative term has been removed. This equation is solved to find the value of v.

Hence the solution $y = uv$ is known. This solution will involve two arbitrary constants.

Particular cases :

(*i*) If $Q_1 = \lambda$, a constant, then (4) becomes $\frac{d^2v}{dx^2} + \lambda v = R_1$.

This is a linear equation with constant coefficients.

(*ii*) If $Q_1 = \frac{\lambda}{x^2}$, then (4) becomes $x^2\frac{d^2v}{dx^2} + \lambda v = x^2 R_1$.

This is a homogeneous linear equation. The substitution $z = \log x$ will reduce it to one with constant coefficients.

Remark. In order to apply this method, it is advisable to calculate $Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$. If this quantity is either a constant λ or $\frac{\lambda}{x^2}$, then this method should be used. In such a case the transformation $y = vu = ve^{-\frac{1}{2}\int P\,dx}$ will reduce the given equation to a linear equation with constant coefficients or to a homogenous linear equation.

9.7. WORKING RULES FOR SOLVING $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Step I. Find $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ If this quantity is of the form λ or $\frac{\lambda}{x^2}$, then use this method.

Step II. Find $e^{-\frac{1}{2}\int P\,dx}$ and call this function as u.

Step III. Take $y = uv$. The equation with v as the dependent variable is $\frac{d^2v}{dx^2} + Q_1 v = R_1$, where $R_1 = \frac{R}{u}$. Solve this equation to find v.

Step IV. Find $y = uv$. This gives the general solution of the given equation.

NOTES

Example 1. *Solve* $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0.$

Sol. We have $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0.$ ...(1)

Here $P = -2\tan x,\ Q = 5,\ R = 0.$

Now $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 5 - \frac{1}{4}(4\tan^2 x) - \frac{1}{2}(-2\sec^2 x)$

$= 5 - \tan^2 x + \sec^2 x = 6$, which is a constant.

Now $\int P\,dx = \int -2\tan x\,dx = -2\log\sec x$

Let $u = e^{-\frac{1}{2}\int P\,dx}$ $\therefore$ $u = e^{-\frac{1}{2}(-2\log\sec x)} = \sec x$

Let $y = uv = (\sec x)v$

$\therefore$ (1) reduces to $\frac{d^2v}{dx^2} + Q_1 v = R_1$ ...(2)

where $R_1 = \frac{R}{u} = \frac{0}{\sec x} = 0.$

$\therefore$ (2) $\Rightarrow$ $\frac{d^2v}{dx^2} + 6v = 0 \Rightarrow (D^2 + 6)v = 0$ ...(3)

$\therefore$ The A.E. of (3) is $D^2 + 6 = 0.$ $\therefore$ $D = \pm\sqrt{6}\,i$

$\therefore$ $v = c_1\cos\sqrt{6}\,x + c_2\sin\sqrt{6}\,x$

$\therefore$ $y = uv$ implies $\mathbf{y = \sec x(c_1\cos\sqrt{6}\,x + c_2\sin\sqrt{6}\,x).}$

This is the general solution of the given equation.

Example 2. *Solve* $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x.$

Sol. Given equation is same as

$$\cot x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (\cot x + 2\tan x)\,y = \sec x.$$

Dividing by $\cot x$, we get

$$\frac{d^2y}{dx^2} + 2\tan x\frac{dy}{dx} + (1 + 2\tan^2 x)\,y = \sec x\tan x \quad ...(1)$$

Here $P = 2\tan x,\ Q = 1 + 2\tan^2 x,\ R = \sec x\tan x$

Now $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 1 + 2\tan^2 x - \frac{1}{4}\cdot 4\tan^2 x - \frac{1}{2}\cdot 2\sec^2 x$

$= 0$, which is a constant

Now $\int P\,dx = \int 2\tan x\,dx = 2\log\sec x$

Let $u = e^{-\frac{1}{2}\int P\,dx}$ $\therefore$ $u = e^{-\frac{1}{2}\cdot 2\log\sec x} = \cos x.$

NOTES

Let $y = uv = (\cos x)\, v.$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1 v = R_1$ ...(2)

where $R_1 = \dfrac{R}{u} = \dfrac{\sec x \tan x}{\cos x} = \sec^2 x \tan x.$

$\therefore$ (2) $\Rightarrow$ $\dfrac{d^2v}{dx^2} + 0\,.\,v = \sec^2 x \tan x \quad \Rightarrow \quad \dfrac{d^2v}{dx^2} = \sec x\,(\sec x \tan x).$

Integrating, we get $\dfrac{dv}{dx} = \dfrac{1}{2} \sec^2 x + c_1$

Integrating again, we get $v = \dfrac{1}{2} \tan x + c_1 x + c_2.$

$\therefore$ $y = uv$ implies $\mathbf{y = \cos x \left(\dfrac{1}{2} \tan x + c_1 x + c_2\right)}.$

This is the general solution of the given equation.

Example 3. *Solve* $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y \cos^2 x = 0$.

Sol. We have $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y \cos^2 x = 0$

$\Rightarrow$ $\cos^2 x \dfrac{d^2y}{dx^2} + (-2 \cos x \sin x)\dfrac{dy}{dx} + y \cos^2 x = 0$

Dividing by $\cos^2 x$, we get

$$\frac{d^2y}{dx^2} - 2 \tan x \frac{dy}{dx} + y = 0 \quad \text{...(1)}$$

Here $P = -2 \tan x,\ Q = 1,\ R = 0.$

Now $Q_1 = Q - \dfrac{1}{4} P^2 - \dfrac{1}{2}\dfrac{dP}{dx} = 1 - \dfrac{1}{4}(4 \tan^2 x) - \dfrac{1}{2}(-2 \sec^2 x)$

$= 2$, which is a constant.

Now $\int P\, dx = \int -2 \tan x\, dx = -2 \log \sec x$

Let $u = e^{-\frac{1}{2}\int P\,dx}$ $\therefore$ $u = e^{-\frac{1}{2}(-2 \log \sec x)} = \sec x.$

Let $y = uv = (\sec x)\, v$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1 v = R_1$ ...(2)

where $R_1 = \dfrac{R}{u} = \dfrac{0}{\sec x} = 0.$

$\therefore$ (2) $\Rightarrow$ $\dfrac{d^2v}{dx^2} + 2v = 0 \quad \Rightarrow \quad (D^2 + 2)v = 0$ ...(3)

$\therefore$ The A.E. of (3) is $D^2 + 2 = 0$. $\therefore$ $D = \pm \sqrt{2}\, i$

$\therefore$ $v = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x.$

$\therefore$ $y = uv$ implies

$\mathbf{y = \sec x\, [c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x\,]}$

This is the general solution of the given equation.

NOTES

Example 4. *Solve* $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + (x^2+1)\,y = x^3 + 3x$.

Sol. We have

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2+1)\,y = x^3 + 3x \qquad ...(1)$$

Here $\quad P = 2x,\ Q = x^2 + 1,\ R = x^3 + 3x.$

Now $\quad Q_1 = Q - \dfrac{1}{4}P^2 - \dfrac{1}{2}\dfrac{dP}{dx} = x^2 + 1 - \dfrac{1}{4}\cdot 4x^2 - \dfrac{1}{2}\cdot 2 = 0$, which is a constant.

Now $\int P\,dx = \int 2x\,dx = x^2$

Let $\quad u = e^{-\frac{1}{2}\int P\,dx}. \quad \therefore \quad u = e^{-\frac{1}{2}x^2}$

Let $\quad y = uv = e^{-\frac{1}{2}x^2}\,.\,v$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1 v = R_1 \qquad ...(2)$

where $\ R_1 = \dfrac{R}{u} = (x^3 + 3x)\,e^{x^2/2}$.

$\therefore$ (2) $\Rightarrow \quad \dfrac{d^2v}{dx^2} + 0\cdot v = (x^5 + 3x)\ e^{x^2/2} \quad \Rightarrow \quad \dfrac{d^2v}{dx^2} = (x^3+3x)\,e^{x^2/2}$.

Integrating, we get

$$\frac{dv}{dx} = \int (x^3 + 3x)\,e^{x^2/2}\,dx + c_1 = \int (x^2+3)\,x\,e^{x^2/2}\,dx + c_1$$

$$= \int (2t+3)\,e^t\,dt + c_1 \quad \text{where } t = x^2/2$$

$$= (2t+3)e^t - \int 2\cdot e^t\,dt + c_1 = (2t+3)e^t - 2e^t + c_1 = (2t+1)\,e^t + c_1$$

$$= (x^2+1)\ e^{x^2/2} + c_1$$

Integrating again, we get

$$v = \int (x^2+1)\,e^{x^2/2}\,dx + c_1x + c_2\,.$$

Now, $\quad \int x^2\,e^{x^2/2}\,dx = \int x\,(x\,e^{x^2/2})\,dx = x\cdot e^{x^2/2} - \int 1\cdot e^{x^2/2}\,dx$

$\therefore \qquad \int (x^2+1)\,e^{x^2/2}\,dx = xe^{x^2/2}$

$\therefore \qquad v = xe^{x^2/2} + c_1x + c_2$

$\therefore \quad y = uv$ implies $\qquad y = e^{-x^2/2}\,[xe^{x^2/2} + c_1x + c_2]$

or $\qquad y = x + (c_1x + c_2)\ e^{-x^2/2}$.

This is the general solution of the given equation.

Example 5. *Solve:*

$$x^2\,(\log x)^2\,\frac{d^2y}{dx^2} - 2x\,\log x\,\frac{dy}{dx} + (2 + \log x) - 2(\log x)^2\ y = x^2\,(\log x)^3.$$

Sol. Dividing the given equation by $x^2 (\log x)^2$, we get

$$\frac{d^2y}{dx^2} - \frac{2}{x \log x}\frac{dy}{dx} + \frac{2 + \log x - 2(\log x)^2}{x^2 (\log x)^2} y = \log x \qquad ...(1)$$

Here $\quad P = -\frac{2}{x \log x},\ Q = \frac{2 + \log x - 2(\log x)^2}{x^2 (\log x)^2},\ R = \log x$

Now, $\quad Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$$= \frac{2 + \log x - 2(\log x)^2}{x^2 (\log x)^2} - \frac{1}{4}\cdot\frac{4}{x^2 (\log x)^2}$$

$$- \frac{1}{2}\cdot(-2)\left[\frac{1}{x}\cdot(-1)(\log x)^{-2}\cdot\frac{1}{x} + \frac{1}{\log x}\left(\frac{-1}{x^2}\right)\right]$$

$$= \frac{2 + \log x - 2(\log x)^2}{x^2 (\log x)^2} - \frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 \log x}$$

$$= -\frac{2}{x^2}\text{, which is of the form } \frac{\lambda}{x^2}.$$

Now $\int P dx = \int -\frac{2}{x \log x} dx = -2 \log \log x$

Let $\quad u = e^{-\frac{1}{2}\int P dx} \quad \therefore \quad u = e^{-\frac{1}{2}\cdot -2 \log \log x} = e^{\log \log x} = \log x$

Let $\quad y = uv = (\log x)\, v.$

$\therefore$ (1) reduces to $\quad \frac{d^2v}{dx^2} + Q_1 v = R_1 \qquad ...(2)$

where $\quad R_1 = \frac{R}{u} = \frac{\log x}{\log x} = 1$

$\therefore$ (2) $\Rightarrow \quad \frac{d^2v}{dx^2} - \frac{2}{x^2}v = 1 \quad \Rightarrow \quad x^2 \frac{d^2v}{dx^2} - 2v = x^2 \qquad ...(3)$

Let $\quad z = \log x. \quad \therefore \quad x = e^z$

$\therefore \quad x^2 \frac{d^2}{dx^2} = D(D-1), \quad \text{where } D = \frac{d}{dz}.$

$\therefore$ (3) $\Rightarrow \quad (D(D-1) - 2)v = e^{2z} \quad \Rightarrow \quad (D^2 - D - 2)\, v = e^{2z} \qquad ...(4)$

$\therefore$ The A.E. of (4) is $\quad D^2 - D - 2 = 0. \quad \therefore \quad D = -1, 2$

$\therefore \quad C.F. = c_1 e^{-z} + c_2 e^{2z} = c_1 (e^z)^{-1} + c_2(e^z)^2 = c_1 x^{-1} + c_2 x^2$

$$P.I. = \frac{1}{D^2 - D - 2}e^{2z} = z\cdot\frac{1}{2D-1}e^{2z} = z\cdot\frac{1}{2(2)-1}e^{2z}$$

$$= \frac{z}{3}e^{2z} = \frac{\log x}{3}\, x^2$$

$\therefore \quad v = C.F. + P.I. = c_1 x^{-1} + c_2 x^2 + \frac{x^2 \log x}{3}$

$\therefore \quad y = uv$ implies $\mathbf{y = \log x\left(\frac{c_1}{x} + c_2 x^2 + \frac{x^2 \log x}{3}\right)}.$

This is the general solution of the given equation.

NOTES

Example 6. *Solve*

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2}\ (\sin 2x + 5e^{-2x} + 6).$$

Sol. We have

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2}\ (\sin 2x + 5e^{-2x} + 6). \qquad \ldots(1)$$

Here $\quad P = -4x,\ Q = 4x^2 - 1,\ R = -3\,e^{x^2}\ (\sin 2x + 5e^{-2x} + 6)$

Now $\quad Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 4x^2 - 1 - \frac{1}{4}\ 16x^2 - \frac{1}{2}(-4)$

$= 4x^2 - 1 - 4x^2 + 2 = 1$, which is a constant.

Now $\quad \int P\,dx = \int -4x\,dx = -2x^2$

Let $\quad u = e^{-\frac{1}{2}\int P\,dx} \quad \therefore \quad u = e^{-\frac{1}{2}\cdot -2x^2} = e^{x^2}$

Let $\quad y = uv = e^{x^2}\,v.$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1\, v = R_1 \qquad \ldots(2)$

where $\quad R_1 = \dfrac{R}{u} = \dfrac{-3e^{x^2}\ (\sin 2x + 5e^{-2x} + 6)}{e^{x^2}} = -3(\sin 2x + 5e^{-2x} + 6)$

$\therefore$ (2) $\Rightarrow \dfrac{d^2v}{dx^2} + 1\cdot v = -3(\sin 2x + 5e^{-2x} + 6)$

$\Rightarrow \qquad (D^2 + 1)v = -3\sin 2x - 15\,e^{-2x} - 18. \qquad \ldots(3)$

$\therefore$ The A.E. of (3) is $D^2 + 1 = 0. \quad \therefore \quad D = \pm i$

$\therefore \qquad C.F. = c_1 \cos x + c_2 \sin x$

$$P.I. = \frac{1}{D^2 + 1}\ (-3\sin 2x - 15\,e^{-2x} - 18)$$

$$= -3\frac{1}{D^2 + 1}\sin 2x - 15\frac{1}{D^2 + 1}e^{-2x} - 18\frac{1}{D^2 + 1}e^{0x}$$

$$= -3\frac{1}{(-4) + 1}\sin 2x - 15\frac{1}{(-2)^2 + 1}e^{-2x} - 18\frac{1}{0 + 1}e^{0x}$$

$= \sin 2x - 3e^{-2x} - 18.$

$\therefore \qquad v = C.F. + P.I. = c_1 \cos x + c_2 \sin x + \sin 2x - 3e^{-2x} - 18$

$\therefore \qquad y = uv$ implies

$$\mathbf{y = e^{x^2}\ (c_1 \cos x + c_2 \sin x + \sin 2x - 3e^{-2x} - 18)}$$

This is the general solution of the given equation.

NOTES

EXERCISE 2

Solve the following differential equations by removing the first derivative :

1. $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x$

2. $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(n^2 + \frac{2}{x^2}\right)y = 0$

3. $x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2y = 0$

4. $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$

5. $\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}}\frac{dy}{dx} + \frac{x + \sqrt{x} - 8}{4x^2}y = 0$

6. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = e^{\frac{1}{2}(x^2 + 2x)}$

7. $\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 2x - 1$

8. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2 y = x\,e^{x^2}$

9. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$

10. $x^2\frac{d^2y}{dx^2} - 2x(3x - 2)\frac{dy}{dx} + 3x(3x - 4)y = e^{3x}$

Answers

1. $y = \sec x\left[c_1\cos\sqrt{6}\,x + c_2\sin\sqrt{6}\,x + \frac{e^x}{7}\right]$

2. $y = x[c_1\cos nx + c_2\sin nx]$

3. $y = x[c_1\cos x + c_2\sin x]$

4. $y = x\left[c_1\cos x + c_2\sin x + \frac{1}{2}e^x\right]$

5. $y = e^{\sqrt{x}}(c_1x^2 + c_2x^{-1})$

6. $y = e^{x^2/2}\left(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + \frac{1}{4}e^x\right)$

7. $y = c_1x + c_2x^3 + x^3\log x + x^2$

8. $y = e^{x^2}\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{x}{2}\right)$

9. $y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)$

10. $y = \frac{e^{3x}}{x^2}\left(c_1x^2 + c_2x^{-1} + \frac{1}{3}x^2\log x\right)$.

9.8. METHOD OF SOLVING A LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER BY CHANGING THE INDEPENDENT VARIABLE

Let
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \ldots(1)$$

be a linear differential equation of the second order, where P, Q and R are functions of x.

NOTES

Let $\quad z = \phi(x)$ be a function of x.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

and
$$\frac{d^2y}{dx^2} = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{dy}{dz}\frac{d^2z}{dx^2} + \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2$$

$$\therefore \quad (1) \quad \Rightarrow \quad \frac{dy}{dz}\frac{d^2z}{dx^2} + \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \text{P}\frac{dy}{dz}\frac{dz}{dx} + \text{Q}y = \text{R}$$

$$\Rightarrow \quad \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}\right)\frac{dy}{dz} + \text{Q}y = \text{R}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{\text{Q}}{\left(\frac{dz}{dx}\right)^2}y = \frac{\text{R}}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \text{P}_1\frac{dy}{dz} + \text{Q}_1 y = \text{R}_1 \qquad \ldots(2)$$

where $\text{P}_1 = \dfrac{\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $\quad \text{Q}_1 = \dfrac{\text{Q}}{\left(\frac{dz}{dx}\right)^2}$ $\quad$ and $\quad \text{R}_1 = \dfrac{\text{R}}{\left(\frac{dz}{dx}\right)^2}$.

We choose $z = \phi(x)$ so that $\text{Q}_1 = \pm a^2$ *i.e.*, $\dfrac{\text{Q}}{\left(\frac{dz}{dx}\right)^2} = \pm a^2$ or $\dfrac{dz}{dx} = \sqrt{\pm\dfrac{\text{Q}}{a^2}}$, the sign being that which makes square root real and a^2 is any positive constant. This positive constant may be taken as per our convenience. We may consistently take $a^2 = 1$.

Integrating $\dfrac{dz}{dx} = \sqrt{\pm\dfrac{\text{Q}}{a^2}}$, we get z in terms of x *i.e.*, $z = \displaystyle\int\sqrt{\pm\frac{\text{Q}}{a^2}}\,dx$.

For this value of z, we calculate P_1. Let P_1 comes out to be a constant, say p_1.

$$\therefore \quad (2) \text{ becomes } \frac{d^2y}{dz^2} + p_1\frac{dy}{dz} \pm a^2y = \frac{\text{R}}{\left(\frac{dz}{dx}\right)^2}.$$

This is a linear differential equation with constant coefficients. This gives the value of y in terms of z. The value of z is put in terms of x so that we may get the value of y in terms of x.

9.9. WORKING RULES FOR SOLVING $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

Step I. Changing x to z the given equation reduces to $\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$

where $$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\quad Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}\quad\text{and}\quad R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}.$$

Step II. Put $\frac{dz}{dx}=\sqrt{\pm\frac{Q}{a^2}}$ and choose sign out of '±' and the value of a^2. For this choice of $\frac{dz}{dx}$, the value of P_1 must come out to be a constant.

Step III. The resultant equation is a linear equation with constant coefficients. Solve this equation and get the value of y in terms of z.

Step IV. Put the value of z and get y in terms of x.

Example 1. *Solve* $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0$.

Sol. We have $$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0. \qquad ...(1)$$

Here $P=\frac{2}{x}$, $Q=\frac{a^2}{x^4}$, $R=0$.

Let $z=\phi(x)$.

$$\therefore\ (1)\ \Rightarrow\ \frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \qquad ...(2)$$

where $$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\ Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2},\ R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let $\frac{dz}{dx}=\sqrt{\pm\frac{Q}{\lambda^2}}$ $\quad\therefore\quad \frac{dz}{dx}=\sqrt{\pm\frac{a^2}{x^4\lambda^2}}$ We choose '+' sign and $\lambda^2=a^2$

$$\therefore\quad \frac{dz}{dx}=\frac{1}{x^2}\ ,\quad \therefore\quad z=\frac{x^{-1}}{-1}=-\frac{1}{x}\quad\text{and}\quad \frac{d^2z}{dx^2}=-\frac{2}{x^3}$$

$$\therefore\quad P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{-\frac{2}{x^3}+\frac{2}{x}\cdot\frac{1}{x^2}}{\left(\frac{1}{x^2}\right)^2}=0,\quad Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{a^2}{x^4}}{\left(\frac{1}{x^2}\right)^2}=a^2,$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\left(\frac{1}{x^2}\right)^2} = 0.$$

NOTES

$\therefore$ (2) becomes $\frac{d^2y}{dz^2} + 0.\frac{dy}{dz} + a^2y = 0$ or $(D^2 + a^2)\, y = 0.$ ...(3)

$\therefore$ The A.E. of (3) is $D^2 + a^2 = 0$. $\therefore$ $D = \pm ai$

$\therefore$ $y = c_1 \cos az + c_2 \sin az$

$\therefore$ $y = c_1 \cos\left(-\frac{a}{x}\right) + c_2 \sin\left(-\frac{a}{x}\right)$ $\left(\because \; z = -\frac{1}{x}\right)$

$\therefore$ $y = c_1 \cos\left(\frac{a}{x}\right) - c_2 \sin\left(\frac{a}{x}\right)$.

This is the general solution of the given equation.

Example 2. *Solve* $(1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + 4y = 0$

Sol. Dividing by $(1 + x^2)^2$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}\, y = 0 \qquad ...(1)$$

Here $P = \frac{2x}{1+x^2}$, $Q = \frac{4}{(1+x^2)^2}$, $R = 0$.

Let $z = \varphi(x)$.

$\therefore$ (1) $\Rightarrow$ $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ ...(2)

where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Let $\frac{dz}{dx} = \sqrt{\pm\frac{Q}{a^2}}$. $\therefore$ $\frac{dz}{dx} = \sqrt{\pm\frac{4}{(1+x^2)^2\, a^2}}$. We choose '+' sign and $a^2 = 4$.

$\therefore$ $\frac{dz}{dx} = \frac{1}{1+x^2}$

$\therefore$ $z = \int \frac{dx}{1+x^2} = \tan^{-1} x$ and $\frac{d^2z}{dx^2} = -\frac{1}{(1+x^2)^2}\cdot 2x = -\frac{2x}{(1+x^2)^2}$

$$\therefore \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{2x}{(1+x^2)^2} + \frac{2x}{1+x^2}\cdot\frac{1}{1+x^2}}{\left(\frac{1}{1+x^2}\right)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{4}{(1+x^2)^2}}{\frac{1}{(1+x^2)^2}} = 4, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\frac{1}{(1+x^2)^2}} = 0.$$

$\therefore$ (2) becomes $\dfrac{d^2y}{dz^2} + 0 \cdot \dfrac{dy}{dz} + 4y = 0$ or $(D^2 + 4)\, y = 0$ ...(3)

$\therefore$ The A.E. of (3) is $D^2 + 4 = 0$. $\therefore$ $D = \pm 2i$

$\therefore$ $$y = c_1 \cos 2z + c_2 \sin 2z$$

Putting $z = \tan^{-1} x$, we get

$$\mathbf{y = c_1 \cos (2 \tan^{-1} x) + c_2 \sin (2 \tan^{-1} x).}$$

This is the general solution of the given equation.

Example 3. *Solve* $x \dfrac{d^2y}{dx^2} + (4x^2 - 1) \dfrac{dy}{dx} + 4x^3 y = 2x^3$.

Sol. Dividing by x, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{4x^2 - 1}{x} \frac{dy}{dx} + 4x^2 y = 2x^2 \qquad ...(1)$$

Here $P = \dfrac{4x^2 - 1}{x}$, $Q = 4x^2$, $R = 2x^2$.

Let $z = \phi(x)$.

$\therefore$ (1) $\Rightarrow$ $\dfrac{d^2y}{dz^2} + P_1 \dfrac{dy}{dz} + Q_1 y = R_1$ ...(2)

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Let $\dfrac{dz}{dx} = \sqrt{\pm \dfrac{Q}{a^2}}$ $\therefore$ $\dfrac{dz}{dx} = \sqrt{\pm \dfrac{4x^2}{a^2}}$ We choose '+' sign and $a^2 = 1$.

$\therefore$ $\dfrac{dz}{dx} = 2x$ $\therefore$ $z = x^2$ and $\dfrac{d^2z}{dx^2} = 2$

$\therefore$ $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \frac{4x^2 - 1}{x} \cdot 2x}{4x^2} = 2.$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1 \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{4x^2} = \frac{1}{2}.$$

$\therefore$ (2) becomes $\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz} + 1 \cdot y = \dfrac{1}{2}$ or $(D^2 + 2D + 1)y = \dfrac{1}{2}$ ...(3)

NOTES

$\therefore$ The A.E. of (3) is $D^2 + 2D + 1 = 0$ $\therefore$ $D = -1, -1$.

$\therefore$ $$\text{C.F.} = (c_1 + c_2 z)e^{-z}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{(D+1)^2}e^{0z} = \frac{1}{2}\cdot\frac{1}{(0+1)^2}e^{0z} = \frac{1}{2}$$

$\therefore$ $$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 z)e^{-z} + \frac{1}{2}$$

Putting $z = x^2$, we get

$$\mathbf{y = (c_1 + c_2 x^2)\, e^{-x^2} + \frac{1}{2}.}$$

This is the general solution of the given equation.

Example 4. *Solve* $\dfrac{d^2y}{dx^2} - (8e^{2x} + 2)\dfrac{dy}{dx} + 4e^{4x}y = e^{6x}$.

Sol. We have $$\frac{d^2y}{dx^2} - (8e^{2x} + 2)\frac{dy}{dx} + 4e^{4x}y = e^{6x} \qquad ...(1)$$

Here $P = -(8e^{2x} + 2)$, $Q = 4e^{4x}$, $R = e^{6x}$

Let $z = \phi(x)$.

$\therefore$ (1) $\Rightarrow$ $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad ...(2)$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\quad Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2},\quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

Let $\dfrac{dz}{dx} = \sqrt{\pm\dfrac{Q}{a^2}}$. $\therefore$ $\dfrac{dz}{dx} = \sqrt{\pm\dfrac{4e^{4x}}{a^2}}$ We choose '+' sign and $a^2 = 1$.

$\therefore$ $\dfrac{dz}{dx} = 2e^{2x}$ $\therefore$ $z = \int 2e^{2x}\,dx = e^{2x}$ and $\dfrac{d^2z}{dx^2} = 4e^{2x}$.

$\therefore$ $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{4e^{2x} - (8e^{2x} + 2)\,2e^{2x}}{4e^{4x}} = -4,$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{4e^{4x}}{4e^{4x}} = 1,\quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{e^{6x}}{4e^{4x}} = \frac{e^{2x}}{4}.$$

$\therefore$ (2) becomes $\dfrac{d^2y}{dz^2} - 4\dfrac{dy}{dz} + 1\cdot y = \dfrac{e^{2x}}{4}$ or $(D^2 - 4D + 1)\,y = \dfrac{z}{4}$. ...(3)

$\therefore$ The A.E. of (3) is $D^2 - 4D + 1 = 0$. $\therefore$ $D = 2 \pm \sqrt{3}$

$\therefore$ $$\text{C.F.} = c_1 e^{(2-\sqrt{3})z} + c_2 e^{(2+\sqrt{3})z}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1}\cdot\frac{z}{4} = \frac{1}{4}\,(1 - (4D - D^2))^{-1} z$$

$$= \frac{1}{4}\,(1 + 4D - D^2 + ...)\,z = \frac{1}{4}\,(z + 4 - 0 + ...) = \frac{z}{4} + 1$$

$\therefore \quad y = \text{C.F.} + \text{P.I.} = c_1 e^{(2-\sqrt{3})z} + c_2 e^{(2+\sqrt{3})z} + \frac{z}{4} + 1$

Putting $z = e^{2x}$, we get

$$\mathbf{y = c_1 e^{(2-\sqrt{3})e^{2x}} + c_2 e^{(2+\sqrt{3})e^{2x}} + \frac{e^{2x}}{4} + 1.}$$

This is the general solution of the given equation.

Example 5. *Solve* $\frac{d^2y}{dx^2} + (3 \sin x - \cot x)\frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x.$

Sol. We have $\frac{d^2y}{dx^2} + (3 \sin x - \cot x)\frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x$. ...(1)

Here $\quad \text{P} = 3 \sin x - \cot x, \text{Q} = 2 \sin^2 x, \text{R} = e^{-\cos x} \sin^2 x.$

Let $\quad z = \phi(x).$

$\therefore \quad (1) \Rightarrow \frac{d^2y}{dz^2} + \text{P}_1 \frac{dy}{dz} + \text{Q}_1 y = \text{R}_1$ ...(2)

where $\text{P}_1 = \dfrac{\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \text{Q}_1 = \dfrac{\text{Q}}{\left(\frac{dz}{dx}\right)^2}, \quad \text{R}_1 = \dfrac{\text{R}}{\left(\frac{dz}{dx}\right)^2}$

Let $\quad \frac{dz}{dx} = \sqrt{\pm \frac{\text{Q}}{a^2}}. \quad \therefore \quad \frac{dz}{dx} = \sqrt{\pm \frac{2 \sin^2 x}{a^2}}$ We choose '+' sign and $a^2 = 2$.

$\therefore \quad \frac{dz}{dx} = \sin x \quad \therefore \quad z = -\cos x \quad$ and $\quad \frac{d^2z}{dx^2} = \cos x.$

$$\therefore \quad \text{P}_1 = \dfrac{\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (3 \sin x - \cot x) \sin x}{\sin^2 x} = 3,$$

$$\text{Q}_1 = \dfrac{\text{Q}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 \sin^2 x}{\sin^2 x} = 2, \text{R}_1 = \dfrac{\text{R}}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{-\cos x} \sin^2 x}{\sin^2 x} = e^{-\cos x}.$$

$\therefore \quad$ (2) becomes $\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-\cos x} \quad$ or $\quad (\text{D}^2 + 3\text{D} + 2)\, y = e^z.$ ...(3)

$\therefore \quad$ The A.E. of (3) is $\text{D}^2 + 3\text{D} + 2 = 0. \quad \therefore \quad \text{D} = -1, -2$

$\therefore \quad \text{C.F.} = c_1 e^{-z} + c_2 e^{-2z}$

$$\text{P.I.} = \frac{1}{\text{D}^2 + 3\text{D} + 2} e^z = \frac{1}{1 + 3 + 2} e^z = \frac{e^z}{6}.$$

$\therefore \quad y = \text{C.F.} + \text{P.I.} = c_1 e^{-z} + c_2 e^{-2z} + \frac{e^z}{6}$

Putting $z = -\cos x$, we get

$$\mathbf{y = c_1 e^{\cos x} + c_2 e^{2 \cos x} + \frac{e^{-\cos x}}{6}.}$$

This is the general solution of the given equation.

Example 6. *Solve* $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol. Dividing by $(1+x)^2$ the given equation becomes

NOTES

$$\frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{1}{(1+x)^2}y = \frac{4\cos\log(1+x)}{(1+x)^2} \quad ...(1)$$

Here $\quad P = \frac{1}{1+x}, \quad Q = \frac{1}{(1+x)^2}, \quad R = \frac{4\cos\log(1+x)}{(1+x)^2}$.

Let $\quad z = \phi(x)$.

$$\therefore \quad (1) \Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad ...(2)$$

where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Let $\frac{dz}{dx} = \sqrt{\pm\frac{Q}{a^2}}$. $\therefore \quad \frac{dz}{dx} = \sqrt{\mp\frac{1}{a^2(1+x)^2}}$. We choose '+' sign and $a^2 = 1$.

$$\therefore \quad \frac{dz}{dx} = \frac{1}{1+x} \quad \therefore \quad z = \int\frac{dx}{1+x} = \log(1+x) \quad \text{and} \quad \frac{d^2z}{dx^2} = -\frac{1}{(1+x)^2}$$

$$\therefore \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{1}{(1+x)^2} + \frac{1}{1+x}\cdot\frac{1}{1+x}}{\frac{1}{(1+x)^2}} = 0.$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{1}{(1+x)^2}}{\frac{1}{(1+x)^2}} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{4\cos\log(1+x)}{(1+x)^2}}{\frac{1}{(1+x)^2}} = 4\cos\log(1+x).$$

$\therefore$ (2) becomes $\frac{d^2y}{dz^2} + 0\cdot\frac{dy}{dz} + 1\cdot y = 4\cos\log(1+x)$

$$\therefore \quad (D^2+1)\,y = 4\cos z \quad ...(3)$$

$\therefore$ The A.E. of (3) is $\quad D^2 + 1 = 0. \quad \therefore \quad D = \pm i$

$$\therefore \quad \text{C.F.} = c_1\cos z + c_2\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}4\cos z = 4\frac{1}{D^2+1}\cos z = 4\cdot z\frac{1}{2D}\cos z = 2z\sin z$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.} = c_1\cos z + c_2\sin z + 2z\sin z.$$

Putting $z = \log(1+x)$, we get

$$\mathbf{y = c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x).}$$

This is the general solution of the given equation.

EXERCISE 3

Solve the following differential equations by changing the independent variables :

1. $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$

2. $x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + \lambda^2 y = 0$

3. $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}$

4. $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$

5. $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x$

6. $\frac{d^2y}{dx^2} - (1 + 4e^x) \frac{dy}{dx} + 3e^{2x} y = e^{2(x + e^x)}$

7. $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = \cos x - \cos^3 x$

8. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \sin \log (1+x)$

9. $x \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y = x^5$

10. $x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$.

Answers

1. $y = c_1 \cos \sin x + c_2 \sin \sin x$

2. $y = c_1 \cos (\lambda/x) + c_2 \sin (\lambda/x)$

3. $y = c_1 \cos \frac{a}{2x^2} + c_2 \sin \frac{a}{2x^2} + \frac{1}{a^2 x^2}$

4. $y = c_1 \cos \left(2 \log \tan \frac{x}{2}\right) + c_2 \sin \left(2 \log \tan \frac{x}{2}\right)$

5. $y = c_1 e^{\sqrt{2} \sin x} + c_2 e^{-\sqrt{2} \sin x} + \sin^2 x$

6. $y = c_1 e^{e^x} + c_2 e^{3e^x} - e^{2e^x}$

7. $y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x$

8. $y = c_1 \cos \log (1 + x) + c_2 \sin \log (1 + x) - 2 \log (1 + x) \cos \log (1 + x)$

9. $y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}$

10. $y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$.

9.10. METHOD OF SOLVING A LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER BY VARIATION OF PARAMETERS

Let $$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Q y = R \qquad \text{...(1)}$$

be a linear differential equation of the second order where P, Q and R are functions of x.

Let y_1 and y_2 be two linearly independent solutions of the equation

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Q y = 0 \qquad \text{...(2)}$$

$$\therefore \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0 \qquad \text{...(3)}$$

NOTES

$\therefore \quad y = c_1 y_1 + c_2 y_2$ is the general solution of (2) and it is also the complementary function of the solution of (1).

Let $Y = Ay_1 + By_2$, where A and B are functions of x such that

$$A'y_1 + B'y_2 = 0 \quad ...(4) \qquad \text{and} \qquad A'y_1' + B'y_2' = R \qquad ...(5)$$

$$\therefore \qquad Y' = Ay_1' + By_2' + (A'y_1 + B'y_2) = Ay_1' + By_2' \qquad \text{(Using (4))}$$

and
$$Y'' = Ay_1'' + By_2'' + (A'y_1' + B'y_2') = Ay_1'' + By_2'' + R \qquad \text{(Using (5))}$$

Now $Y'' + PY' + QY$

$$= (Ay_1'' + By_2'' + R) + P(Ay_1' + By_2') + Q(Ay_1 + By_2)$$
$$= A(y_1'' + Py_1' + Qy_1) + B(y_2'' + Py_2' + Qy_2) + R = A.0 + B.0 + R = R$$

$\therefore \quad Y = Ay_1 + By_2$ is a particular integral of the equation (1).

$\therefore \quad y = c_1y_1 + c_2y_2 + (Ay_1 + By_2)$ is the general solution of the given differential equation (1).

The functions A and B satisfies the equations

$$A'y_1 + B'y_2 - 0 = 0$$

and
$$A'y_1' + B'y_2' - R = 0.$$

$$\therefore \qquad \frac{A'}{-y_2R + 0} = \frac{B'}{0 + y_1R} = \frac{1}{y_1y_2' - y_1'y_2}$$

$$\therefore \qquad A' = -\frac{y_2R}{y_1y_2' - y_1'y_2} \quad \text{and} \quad B' = \frac{y_1R}{y_1y_2' - y_1'y_2}.$$

A′ and B′ are meaningful because of (3).

Integrating above equations, we get the values of the functions A and B. Arbitrary constants are not used in the values of A and B because $Y = Ay_1 + By_2$ is taken as a particular integral of (1).

$\therefore$ The general solution of the equation (1) is known.

Remark. It is essential to have the coefficient of the second order term in the linear differential equation as unity, otherwise in the above discussion Y will not be a solution of the given equation.

9.11. WORKING RULES OF SOLVING $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Step I. Find two independent solutions of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and call these as y_1 and y_2.

Step II. Let $Y = Ay_1 + By_2$ be the required particular solution of the given equation.

Step III. Solve the equations $A' = -\frac{y_2R}{y_1y_2' - y_1'y_2}$ and $B' = \frac{y_1R}{y_1y_2' - y_1'y_2}$ to find the values of A and B.

Step IV. Write the general solution of the given equation as $y = c_1 y_1 + c_2 y_2 + Ay_1 + By_2$, where c_1 and c_2 are arbitrary constants.

Example 1. *Solve* $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Sol. We have $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

$\Rightarrow \quad (D^2 + 1)y = \operatorname{cosec} x$ ...(1)

$\therefore$ The A.E. of (1) is $D^2 + 1 = 0$. $\therefore \quad D = \pm i$

$\therefore \quad$ C.F. $= c_1 \cos x + c_2 \sin x$

Let P.I. $= A \cos x + B \sin x$ ...(2)

$\therefore \quad A' = -\frac{y_2 R}{y_1 y_2' - y_1' y_2}, \; B' = \frac{y_1 R}{y_1 y_2' - y_1' y_2}$

Here $\quad y_1 = \cos x, \; y_2 = \sin x, \; R = \operatorname{cosec} x$

$\therefore \quad y_1 y_2' - y_1' y_2 = \cos x \cdot \cos x - (-\sin x) \sin x = 1$

$\therefore \quad A' = -\frac{\sin x \operatorname{cosec} x}{1} = -1 \quad$ and $\quad B' = \frac{\cos x \operatorname{cosec} x}{1} = \cot x$

$\therefore \quad A = \int -1\, dx = -x \quad$ and $\quad B = \int \cot x\, dx = \log \sin x$

$\therefore \quad (2) \Rightarrow \quad$ P.I. $= -1 \cdot \cos x + (\log \sin x) \sin x$

Using $\quad y =$ C.F. + P.I., we have

$$\mathbf{y = c_1 \cos x + c_2 \sin x + (\log \sin x) \sin x - \cos x.}$$

This is the general solution of the given equation.

Example 2. *Solve* $\frac{d^2y}{dx^2} + n^2 y = \sec nx$.

Sol. We have $\frac{d^2y}{dx^2} + n^2 y = \sec nx$. ...(1)

$\Rightarrow \quad (D^2 + n^2)y = \sec nx$

$\therefore$ The A.E. is $\quad D^2 + n^2 = 0$. $\therefore \quad D = \pm ni$

$\therefore \quad$ C.F. $= c_1 \cos nx + c_2 \sin nx$.

Let $\quad$ P.I. $= A \cos nx + B \sin nx$ ...(2)

$\therefore \quad A' = -\frac{y_2 R}{y_1 y_2' - y_1' y_2}, \quad B' = \frac{y_1 R}{y_1 y_2' - y_1' y_2}$.

Here $\quad y_1 = \cos nx, \; y_2 = \sin nx, \; R = \sec nx$

$\therefore \quad y_1 y_2' - y_1' y_2 = n \cos nx \cos nx - (-n \sin nx) \sin nx = n$

$\therefore \quad A' = -\frac{\sin nx \sec nx}{n} = -\frac{1}{n} \tan nx$

and $\quad B' = \frac{\cos nx \sec nx}{n} = \frac{1}{n}$

$\therefore \quad A = \int -\frac{1}{n} \tan nx\, dx = -\frac{1}{n} \cdot \frac{\log \sec nx}{n} = \frac{1}{n^2} \log \cos nx$.

$B = \int \frac{1}{n}\, dx = \frac{1}{n} x$

$\therefore \quad (2) \Rightarrow \quad$ P.I. $= \left(\frac{1}{n^2} \log \cos nx\right) \cos nx + \left(\frac{1}{n} x\right) \sin nx$.

NOTES

Using y = C.F. + P.I., we have

$$y = c_1 \cos nx + c_2 \sin nx + \frac{1}{n^2} (\log \cos nx) \cos nx + \frac{1}{n} x \sin nx.$$

This is the general solution of the given equation.

Example 3. *Solve* $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol. We have $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ ...(1)

$\Rightarrow \qquad (D^2 - 1)\, y = \frac{2}{1+e^x}$

$\therefore$ The A.E. is $D^2 - 1 = 0$. $\therefore$ $D = \pm 1$

$\therefore \qquad$ C.F. $= c_1 e^x + c_2 e^{-x}$

Let $\qquad$ P.I. $= A e^x + B e^{-x}$ ...(2)

$\therefore \qquad A' = -\frac{y_2 R}{y_1 y_2' - y_1' y_2}, \; B' = \frac{y_1 R}{y_1 y_2' - y_1' y_2}$

Here $\qquad y_1 = e^x, \quad y_2 = e^{-x}, \quad R = \frac{2}{1+e^x}$

$\therefore \qquad y_1 y_2' - y_1' y_2 = e^x(-e^{-x}) - e^x e^{-x} = -2$

$\therefore \qquad A' = -\frac{e^{-x}\left(\frac{2}{1+e^x}\right)}{-2} = \frac{e^{-x}}{1+e^x}$

and $\qquad B' = \frac{e^x\left(\frac{2}{1+e^x}\right)}{-2} = -\frac{e^x}{1+e^x}$

$\therefore \qquad A = \int \frac{e^{-x}}{1+e^x} dx = \int \frac{1}{e^x(1+e^x)} dx = \int \left[\frac{1}{e^x \cdot 1} + \frac{1}{(-1)(1+e^x)}\right] dx$

$$= \int \left(\frac{1}{e^x} - \frac{1+e^x-e^x}{1+e^x}\right) dx = \int \left(e^{-x} - 1 + \frac{e^x}{1+e^x}\right) dx$$

$$= -e^{-x} - x + \log(1+e^x) = -e^{-x} - \log e^x + \log(1+e^x)$$

$$= \log \frac{1+e^x}{e^x} - e^{-x}$$

$$B = \int -\frac{e^x}{1+e^x} dx = -\log(1+e^x)$$

$\therefore$ (2) $\Rightarrow \qquad$ P.I. $= \left(\log \frac{1+e^x}{e^x} - e^{-x}\right) e^x + (-\log(1+e^x)) e^{-x}$

$$= e^x \log \frac{1+e^x}{e^x} - 1 - e^{-x} \log(1+e^x).$$

Using y = C.F. + P.I., we have

$$y = c_1 e^x + c_2 e^{-x} + e^x \log \frac{1+e^x}{e^x} - 1 - e^{-x} \log(1+e^x).$$

This is the general solution of the given equation.

Example 4. *Solve* $\dfrac{d^2y}{dx^2}+4y=\tan 2x.$

Sol. We have $\quad \dfrac{d^2y}{dx^2}+4y=\tan 2x.$ ...(1)

$\Rightarrow \quad (D^2+4)\,y=\tan 2x$

$\therefore$ The A.E. is $\quad D^2+4=0. \quad \therefore \quad D=\pm 2i$

$\therefore \quad$ C.F $=c_1\cos 2x+c_2\sin 2x$

Let $\quad$ P.I. $=A\cos 2x+B\sin 2x$ ...(2)

$$\therefore \quad A'=-\frac{y_2R}{y_1y_2'-y_1'y_2},\ B'=\frac{y_1R}{y_1y_2'-y_1'y_2}$$

Here $\quad y_1=\cos 2x,\ y_2=\sin 2x,\ R=\tan 2x$

$\therefore \quad y_1y_2'-y_1'y_2=(\cos 2x)2\cos 2x-(-2\sin 2x)\sin 2x=2$

$$\therefore \quad A'=-\frac{\sin 2x\tan 2x}{2}=-\left(\frac{1-\cos^2 2x}{2\cos 2x}\right)=\frac{\cos 2x-\sec 2x}{2}$$

and $$B'=\frac{\cos 2x\tan 2x}{2}=\frac{\sin 2x}{2}$$

$$\therefore \quad A=\int\frac{\cos 2x-\sec 2x}{2}\,dx=\frac{1}{2}\left[\frac{\sin 2x}{2}-\frac{\log(\sec 2x+\tan 2x)}{2}\right]$$

$$=\frac{1}{4}\left(\sin 2x-\log(\sec 2x+\tan 2x)\right)$$

$$B=\int\frac{\sin 2x}{2}\,dx=-\frac{\cos 2x}{4}$$

$$\therefore \quad (2) \Rightarrow \quad \text{P.I.}=\frac{1}{4}\left(\sin 2x-\log(\sec 2x+\tan 2x)\right)\cos 2x-\frac{\cos 2x}{4}\cdot\sin 2x$$

$$=-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x).$$

Using y = C.F. + P.I. we have

$$\mathbf{y=c_1\cos 2x+c_2\sin 2x-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)}$$

This is the general solution of the given equation.

Example 5. *Solve* $(1-x)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=(1-x)^2,\ x\neq 1.$

Sol. Dividing by $1-x$, the given equation becomes

$$\frac{d^2y}{dx^2}+\frac{x}{1-x}\frac{dy}{dx}-\frac{1}{1-x}y=1-x. \qquad ...(1)$$

Here $\quad P=\dfrac{x}{1-x},\ Q=-\dfrac{1}{1-x},\ R=1-x$

$$P + xQ = \frac{x}{1-x} + x\left(-\frac{1}{1-x}\right) = 0$$

$\therefore$ $y_1 = x$ is a solution of

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$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = 0 \quad ...(2)$$

Also $1 + P + Q = 1 + \frac{x}{1-x} + \left(-\frac{1}{1-x}\right) = 0$

$\therefore$ $y_2 = e^x$ is a solution of (2)

Now, $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = (x-1)e^x \neq 0$ $\quad (\because \quad x \neq 1)$

$\therefore$ y_1 and y_2 are linearly independent solutions of (2).

$\therefore$ $y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 e^x$ is the general soultion of (2)

$\therefore$ $y = c_1 x + c_2 e^x$ is the C.F. of the solution of (1).

Let P.I. $= Ax + Be^x$ ...(3)

$\therefore$ $A' = -\frac{y_2 R}{y_1 y_2' - y_1' y_2}$, $B' = \frac{y_1 R}{y_1 y_2' - y_1' y_2}$

$\therefore$ $A' = -\frac{e^x(1-x)}{(x-1)e^x} = 1$, $B' = \frac{x(1-x)}{(x-1)e^x} = -xe^{-x}$

$\therefore$ $A = \int 1\, dx = x$

$$B = \int -x e^{-x}\, dx = -\left[x.\frac{e^{-x}}{-1} - \int 1.\frac{e^{-x}}{-1} dx\right] = x e^{-x} + e^{-x} = e^{-x}(x+1)$$

$\therefore$ (3) $\Rightarrow$ P.I. $= x \,.\, x + e^{-x}(x+1)\; e^x = x^2 + x + 1$.

Using $y =$ C.F. + P.I. we have

$$y = c_1 x + c_2 e^x + x^2 + x + 1.$$

This is the general solution of the given equation.

Example 6. *Verify that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of the equation*

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

Using this information, find the general solution of the equation

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2.$$

Sol. $y = x$ is a solution of

$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$, if $(x^2+1)\,.\,0 - 2x\,.\,1 + 2x = 0$, which is true.

$y = x^2 - 1$ is a solution of $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

if $(x^2+1)\cdot 2 - 2x \cdot 2x + 2(x^2-1) = 0$

or if $2x^2 + 2 - 4x^2 + 2x^2 - 2 = 0$, which is true.

$\therefore$ $y = x$ and $y = x^2 - 1$ are both solution of

$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0. \quad ...(1)$$

Dividing (1) by x^2+1, the functions $y=x$ and $y=x^2-1$ are also solutions of

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$$\frac{d^2y}{dx^2}-\frac{2x}{x^2+1}\frac{dy}{dx}+\frac{2}{x^2+1}y=0. \quad ...(2)$$

Let $\quad y_1=x$ and $y_2=x-1$.

Now $\quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix}=2x^2-x^2+1=x^2+1\neq 0$

$\therefore\quad y_1$ and y_2 are linearly independent solutions of (1) and also of (2).

$\therefore\quad y=c_1y_1+c_2y_2=c_1x+c_2(x^2-1)$ is the general solution of (2).

The equation to be solved is

$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=6(x^2+1)^2. \quad ...(3)$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}-\frac{2x}{x^2+1}\frac{dy}{dx}+\frac{2}{x^2+1}y=6(x^2+1) \quad ...(4)$$

$\therefore\quad$ C.F. of the solution of (4) $=c_1x+c_2(x^2-1)$

Let $\quad$ P.I. of (4) $=Ax+B(x^2-1)$ $\quad ...(5)$

$$\therefore\quad A'=-\frac{y_2R}{y_1y_2'-y_1'y_2},\ B'=\frac{y_1R}{y_1y_2'-y_1'y_2}$$

Here $\quad y_1=x,\ y_2=x^2-1,\ R=6(x^2+1)$

$$A'=-\frac{(x^2-1)6(x^2+1)}{x^2+1}=6(1-x^2),\quad B'=\frac{x\,6(x^2+1)}{x^2+1}=6x$$

$$\therefore\quad A=\int 6(1-x^2)\,dx=6x-2x^3,\quad B=\int 6x\,dx=3x^2$$

$\therefore\quad (5)\ \Rightarrow\ $ P.I. of (4) $=(6x-2x^3)x+3x^2(x^2-1)=x^4+3x^2$

Using $\quad y=$ C.F. + P.I., we have

$$\mathbf{y=c_1x+c_2(x^2-1)+x^4+3x^2.}$$

This is the general solution of the equation (4) and hence of the given equation.

Remark. For applying the method of variation of parameters it is necessary to have the coefficient of $\frac{d^2y}{dx^2}$ as unity.

Example 7. *Solve* $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3.$

Sol. We have $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3.$

$$\Rightarrow\quad \frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=x \quad ...(1)$$

The equation (1) with right side '0' is

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=0 \quad ...(2)$$

NOTES

We find the general solution of (2).

Let $y = xV$. **(Note this step)**

$$\therefore \quad \frac{dy}{dx} = x\frac{dV}{dx} + V \text{ and } \frac{d^2y}{dx^2} = x\frac{d^2V}{dx^2} + 2\frac{dV}{dx}$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (2) we get

$$x\frac{d^2V}{dx^2} + 2\frac{dV}{dx} - \frac{2(1+x)}{x}\left(x\frac{dV}{dx} + V\right) + \frac{2(1+x)}{x^2}xV = 0$$

$$\Rightarrow \quad x\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} = 0 \quad \Rightarrow \quad (D^2 - 2D)V = 0 \quad \ldots(3)$$

The A.E. is $D^2 - 2D = 0$, $\therefore$ $D = 0, 2$

$$\therefore \quad V = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$$

$$\therefore \quad y = xV = x(c_1 + c_2 e^{2x}) = c_1 x + c_2 x e^{2x}$$

Let $y_1 = x$ and $y_2 = xe^{2x}$

$$\therefore \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^{2x} \\ 1 & (1+2x)e^{2x} \end{vmatrix}$$

$$= x(1+2x)e^{2x} - xe^{2x} = 2x^2e^{2x} \neq 0$$

$\therefore$ y_1 and y_2 are independent solutions.

$\therefore$ $y = c_1y_1 + c_2y_2 = c_1x + c_2 xe^{2x}$ is the general solution of (2).

$\therefore$ C.F. of (1) $= c_1 x + c_2 x e^{2x}$

Let $\quad$ P.I. $= Ay_1 + By_2 = Ax + Bxe^{2x}$ $\quad \ldots(4)$

$$\therefore \quad A' = -\frac{y_2 R}{y_1y_2' - y_1'y_2} \qquad B' = \frac{y_1 R}{y_1y_2' - y_1'y_2}$$

$$\therefore \quad A' = -\frac{xe^{2x} \cdot x}{2x^2 e^{2x}} = -\frac{1}{2} \quad \text{and} \quad B' = \frac{x \cdot x}{2x^2 e^{2x}} = \frac{1}{2}e^{-2x}$$

(Here R = x)

$$\therefore \quad A = \int -\frac{1}{2}\,dx = -\frac{x}{2} \quad \text{and} \quad B = \int \frac{1}{2}e^{-2x}\,dx = -\frac{e^{-2x}}{-4} = \frac{1}{4e^{2x}}$$

$$\therefore \quad (4) \quad \Rightarrow \quad \text{P.I.} = -\frac{x}{2} \cdot x - \frac{1}{4e^{2x}} \cdot xe^{2x} = -\frac{x^2}{2} - \frac{x}{4}$$

$\therefore$ Using $y = $ C.F. + P.I., we have

$$\mathbf{y = c_1x + c_2x\, e^{2x} - \frac{x^2}{2} - \frac{x}{4}}.$$

This is the general solution of the given equation.

NOTES

EXERCISE 4

Solve the following differential equations by variation of parameters :

1. $\dfrac{d^2y}{dx^2} + y = \sec x$
2. $\dfrac{d^2y}{dx^2} + 9y = \sec 3x$
3. $\dfrac{d^2y}{dx^2} + n^2 y = \operatorname{cosec} nx$
4. $\dfrac{d^2y}{dx^2} + y = \tan x$
5. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x$
6. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2x$
7. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x \log x$
8. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2 e^x$
9. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \tan x$
10. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = x^{-2} e^{3x}$

Answers

1. $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x$
2. $y = c_1 \cos 3x + c_2 \sin 3x + \dfrac{x}{3} \sin 3x + \dfrac{1}{9} \cos 3x \log \cos 3x$
3. $y = c_1 \cos nx + c_2 \sin nx - \dfrac{x}{n} \cos nx \quad \dfrac{1}{n^2} \sin nx \log \sin nx$
4. $y = c_1 \cos x + c_2 \sin x - \cos x \log (\sec x + \tan x) + 7$
5. $y = c_1 + c_2 e^{2x} - \dfrac{1}{2} e^x \sin x$
6. $y = (c_1 + c_2 x) e^x + 2x + 4$
7. $y = c_1 x + c_2 x^2 - \dfrac{1}{2} x (\log x)^2 - x \log x - x$
8. $y = c_1 x + \dfrac{c_2}{x} + e^x - \dfrac{e^x}{x}$
9. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$
10. $y = (c_1 + c_2 x) e^{3x} - e^{3x} \log x.$

UNIT 10 ORDINARY SIMULTANEOUS DIFFERENTIAL EQUATIONS

NOTES

STRUCTURE

10.0. LEARNING OBJECTIVES

After going through this unit you will be able to:

- Define ordinary simultaneous defferential equations
- Find method of solving simultaneous linear defferential equations with constant coefficients
- Find method of solving simultaneous total defferential equation

10.1. INTRODUCTION

Till now we have been discussing differential equations involving only two variables. In the present chapter, we shall consider differential equations involving more than two variables. If there is one independent variable then the equations are called ordinary differential equations and in case there are more than one independent variable then the equations are called partial differential equations. In the present text, we shall consider equations involving only one independent variable.

NOTES

10.2. ORDINARY SIMULTANEOUS DIFFERENTIAL EQUATIONS

Since the number of dependent variables is more than one, only one such differential equation is not sufficient to find its solution. In the present case, we shall require systems of simultaneous differential equations containing at least two differential equations. We shall study the following types of simultaneous differential equations.

I. Simultaneous linear differential equations with constant coefficients.

II. Simultaneous total differential equations

I. SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

10.3. METHOD OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The number of simultaneous linear differential equations with constant coefficients in a given system must be equal to the number of dependent variables involved in the system

For Example

$$\left.\begin{aligned} \frac{dx}{dt}+\frac{dy}{dt}+2x+y&=0 \\ \frac{dy}{dt}+5x+3y&=0 \end{aligned}\right]$$

is a system of two equations and the number of dependent variables (x and y) is also two. Writing $\frac{d}{dt}$ as D, the above system can also be written as

$$\left.\begin{aligned} (D+2)x+(D+1)y&=0 \\ 5x+(D+3)y&=0 \end{aligned}\right]$$

By using differentiation a set of equations is obtained from which all but one of the dependent variables, say x, can be eliminated. The equation resulting from the elimination of other variables is solved for this variable x Then a relation between a second dependent variable and the independent variable can be deduced, either (1) by the method of eliminating and integration employed in the case of the first variable 'x' or (2) by substituting the value of 'x' in one of the equations involving 'x' and the second variable and the independent variable

The complete solution consists of as many independent relations between the variables as there are dependent variables.

Remark. When the equations are written in the 'D' notation, there is a striking similarity between the procedure used here and the method of solving a system of n equations is n unknowns. This is due to the fact that the operator 'D' may at times be treated as a variable.

10.4. SIMULTANEOUS LINEAR EQUATIONS INVOLVING TWO DEPENDENT VARIABLES

NOTES

Let x, y be two dependent variables and t be the independent variable. A system of simultaneous linear equations with constant coefficients will contain two equations.

Let the simultaneous equations in symbolic form be

$$f_1(D)\,x + g_1(D)\,y = h_1(t) \quad ...(1)$$

$$f_2(D)\,x + g_2(D)\,y = h_2(t). \quad ...(2)$$

Let $\Delta = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix}$

If $\Delta = 0$, then the given system is called a **dependent system.**

We shall consider only those systems for which $\Delta \neq 0$.

Operating (1) by $g_2(D)$, we get

$$g_2(D)\,f_1(D)x + g_2(D)\,g_1(D)\,y = g_2(D)\,h_1(t) \quad ...(3)$$

Operating (2) by $g_1(D)$, we get

$$g_1(D)\,f_2(D)x + g_1(D)\,g_2(D)\,y = g_1(D)\,h_2(t) \quad ...(4)$$

Subtracting (4) from (3), we get

$$(g_2(D)\,f_1(D) - g_1(D)\,f_2(D))\,x = g_2(D)\,h_1(t) - g_1(D)\,h_2(t).$$

In terms of determinants, we have

$$\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} x = \begin{vmatrix} h_1(t) & g_1(D) \\ h_2(t) & g_2(D) \end{vmatrix}$$

Similarly, for y, we have

$$\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} y = \begin{vmatrix} f_1(D) & h_1(t) \\ f_2(D) & h_2(t) \end{vmatrix}$$

The number of arbitrary constants involved in the general solution should be equal to the degree of D in Δ.

Remark. We have $g_1(D)g_2(D) = g_2(D)g_1(D)$ because the linear differential equations are assumed to be with constant coefficients.

Example 1. *Solve the simultaneous equations :*

$$(D - 1)x + Dy = 2t + 1$$

$$(2D + 1)x + 2Dy = t.$$

Sol. We have $\quad (D - 1)x + Dy = 2t + 1 \quad ...(1)$

$$(2D + 1)x + 2Dy = t \quad ...(2)$$

Multiplying (1) by 2 and subtracting (2) from it, we get

$$(2(D - 1) - (2D + 1))x = 4t + 2 - t$$

$$\Rightarrow \quad -3x = 3t + 2 \quad \Rightarrow \quad x = -t - \frac{2}{3}$$

To find y, (1) $\Rightarrow$ $Dy = 2t + 1 - (D - 1)x = 2t + 1 - (D - 1)\left(-t - \frac{2}{3}\right) = t + \frac{4}{3}$.

$$\therefore \quad y = \int\left(t + \frac{4}{3}\right)dt + c_1 = \frac{t^2}{2} + \frac{4}{3}t + c_1$$

$\therefore$ The general solution of the given system is

$$\mathbf{x = -t - \frac{2}{3}, \; y = \frac{1}{2}t^2 + \frac{4}{3}t + c_1.}$$

Remark. Here $\Delta = \begin{vmatrix} D-1 & D \\ 2D+1 & 2D \end{vmatrix} = (D-1)2D - (2D+1)D = -3D.$

Since, the degree of D in Δ is 1, the general solution will have one arbitrary constant.

Example 2. *Solve the simultaneous equations :*

$$\frac{dx}{dt} - 7x + y = 0$$

$$\frac{dy}{dt} - 2x - 5y = 0.$$

Sol. We have

$$Dx - 7x + y = 0 \qquad \ldots(1)$$

$$Dy - 2x - 5y = 0 \qquad \ldots(2)$$

(1) $\Rightarrow$ $(D-7)x + y = 0$ ...(3)

(2) $\Rightarrow$ $-2x + (D-5)y = 0$ ...(4)

Operating on (3) by D − 5 and subtracting (4) from it, we get

$$((D-5)(D-7)+2)x = 0 \;\Rightarrow\; (D^2 - 12D + 37)x = 0$$

$\therefore$ The A.E. is $D^2 - 12D + 37 = 0.$ $\therefore$ $D = 6 \pm i$

$\therefore$ $x = e^{6t}(c_1 \cos t + c_2 \sin t)$

To find y, (1) $\Rightarrow$ $y = 7x - Dx$

$\therefore$ $y = 7e^{6t}(c_1 \cos t + c_2 \sin t) - 6e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t)$

$\therefore$ $y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$

$\therefore$ The general solution of the given system is

$$\mathbf{x = e^{6t}(c_1 \cos t + c_2 \sin t), \; y = e^{6t}((c_1 - c_2)\cos t + (c_1 + c_2)\sin t).}$$

Remark. For the above system :

$$\Delta = \begin{vmatrix} D-7 & 1 \\ -2 & D-5 \end{vmatrix} = (D-7)(D-5) + 2 = D^2 - 12D + 37.$$

Since, the degree of D in Δ is 2, the general solution will have 2 arbitrary constants.

Alternative method

Using determinants,

$$\Delta = \begin{vmatrix} D-7 & 1 \\ -2 & D-5 \end{vmatrix} = (D-7)(D-5) + 2 = D^2 - 12D + 37$$

For x $\Delta x = \begin{vmatrix} 0 & 1 \\ 0 & D-5 \end{vmatrix} = 0 - 0 = 0$

$\Rightarrow$ $(D^2 - 12D + 37)x = 0$

Example 3. *Solve simultaneously the differential equations :*

$$\frac{dx}{dt} + 4x + 3y = t$$

$$\frac{dy}{dt} + 2x + 5y = e^t.$$

NOTES

Sol. We have

$$Dx + 4x + 3y = t \quad ...(1)$$

$$Dy + 2x + 5y = e^t \quad ...(2)$$

(1) $\Rightarrow$ $(D + 4)x + 3y = t$ ...(3)

(2) $\Rightarrow$ $2x + (D + 5)y = e^t$ ...(4)

Operating on (3) by D + 5 and multiplying (4) by 3 and subtracting, we get

$$(D + 5)(D + 4)x - 6x = (D + 5)t - 3e^t$$

$$\Rightarrow \quad (D^2 + 9D + 14)x = -3e^t + 5t + 1$$

$\therefore$ The A.E. is $D^2 + 9D + 14 = 0$. $\therefore$ $D = -2, -7$

$\therefore$ $\text{C.F.} = c_1e^{-2t} + c_2e^{-7t}$

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14}(-3e^t + 5t + 1)$$

$$= -3\left(\frac{1}{D^2 + 9D + 14}e^t\right) + \frac{5}{14}\left(1 + \frac{9D + D^2}{14}\right)^{-1} t + \frac{1}{D^2 + 9D + 14}e^{0t}$$

$$= -3\left(\frac{1}{1 + 9 + 14}e^t\right) + \frac{5}{14}\left(1 - \frac{9}{14}D + \ldots\right)t + \frac{1}{0 + 0 + 14}e^{0t}$$

$$= -\frac{1}{8}e^t + \frac{5}{14}\left(t - \frac{9}{14}\right) + \frac{1}{14} = -\frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196}$$

$$\therefore \quad x = \text{C.F.} + \text{P.I.} = c_1e^{-2t} + c_2e^{-7t} - \frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196} \quad ...(5)$$

(1) $\Rightarrow$ $3y = t - \dfrac{dx}{dt} - 4x = t - \left[-2c_1e^{-2t} - 7c_2e^{-7t} - \dfrac{1}{8}e^t + \dfrac{5}{14} - 0\right]$

$$- 4\left[c_1e^{-2t} + c_2e^{-7t} - \frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196}\right]$$

$$= -2c_1e^{-2t} + 3c_2e^{-7t} - \frac{3}{7}t + \frac{5}{8}e^t + \frac{27}{98}$$

$$\therefore \quad y = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} - \frac{1}{7}t + \frac{5}{24}e^t + \frac{9}{98}$$

$\therefore$ The general solution of the given system is

$$\mathbf{x} = c_1e^{-2t} + c_2e^{-7t} - \frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196},$$

$$\mathbf{y} = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} + \frac{5}{24}e^t - \frac{1}{7}t + \frac{9}{98}.$$

Remark. For the above system

$$\Delta = \begin{vmatrix} D + 4 & 3 \\ 2 & D + 5 \end{vmatrix} = (D + 4)(D + 5) - 6 = D^2 + 9D + 14$$

Since, the degree of D in Δ is 2, the general solution will have two arbitrary constants.

Example 4. *Solve the simultaneous equations* $\dfrac{dx}{dt} + y = \sin t$, $\dfrac{dy}{dt} + x = \cos t$ *given that* $x = 2$ *and* $y = 0$ *when* $t = 0$.

NOTES

Sol. Given equations are

$$Dx + y = \sin t \qquad ...(1)$$

$$x + Dy = \cos t \qquad ...(2)$$

Operating on (1) by D and subtracting (2) from it, we get

$$D^2x - x = D \sin t - \cos t \quad \Rightarrow \quad (D^2 - 1)\, x = 0$$

$\therefore$ The A.E. is $D^2 - 1 = 0$, $\therefore$ $D = \pm 1$

$$\therefore \qquad x = c_1 e^t + c_2 e^{-t} \qquad ...(3)$$

To find y. (1) $\Rightarrow$ $y = \sin t - D\, x$

$$\therefore \qquad y = \sin t - c_1 e^t + c_2 e^{-t} \qquad ...(4)$$

Now $x = 2$, $y = 0$ when $t = 0$.

$\therefore$ (3) $\Rightarrow$ $2 = c_1 + c_2$

(4) $\Rightarrow$ $0 = -c_1 + c_2$

Solving we get $c_1 = 1$, $c_2 = 1$.

$\therefore$ The required solution is $\mathbf{x = e^t + e^{-t}, y = \sin t - e^t + e^{-t}}$.

Example 5. *Solve the simultaneous equations :*

$$\frac{d^2x}{dt^2} + 4x + 5y = t^2, \quad \frac{d^2y}{dt^2} + 5x + 4y = t + 1.$$

Sol. We have

$$(D^2 + 4)x + 5y = t^2 \qquad ...(1)$$

$$5x + (D^2 + 4)y = t + 1 \qquad ...(2)$$

Operating (1) by $D^2 + 4$ and multiplying (2) by 5 and subtracting, we get

$$((D^2 + 4)^2 - 25)x = (D^2 + 4)t^2 - 5(t + 1)$$

$$\Rightarrow \qquad (D^2 - 1)(D^2 + 9)x = 4t^2 - 5t - 3.$$

$\therefore$ The A.E. is $(D^2 - 1)(D^2 + 9) = 0$. $\therefore$ $D = \pm 1, \pm 3i$

$$\therefore \qquad \text{C.F.} = c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 8D^2 - 9}(4t^2 - 5t - 3) = \frac{1}{-9}\left(1 - \frac{8D^2 + D^4}{9}\right)^{-1}\left(4t^2 - 5t - 3\right)$$

$$= -\frac{1}{9}\left(1 + \frac{8}{9}D^2 + \frac{D^4}{9} + \ldots\ldots\right)(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left((4t^2 - 5t - 3) + \frac{8}{9}(8)\right) = -\frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$$

$$\therefore \qquad x = c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right) \qquad ...(3)$$

$$(1) \Rightarrow \qquad 5y = t^2 - \frac{d^2x}{dt^2} - 4x \qquad ...(4)$$

$$(3) \Rightarrow \qquad \frac{dx}{dt} = c_1 e^t - c_2 e^{-t} - 3c_3 \sin 3t + 3c_4 \cos 3t - \frac{1}{9}(8t - 5)$$

$$\therefore \qquad \frac{d^2x}{dt^2} = c_1 e^t + c_2 e^{-t} - 9c_3 \cos 3t - 9c_4 \sin 3t - \frac{8}{9}$$

$$\therefore \quad (4) \Rightarrow \quad 5y = t^2 - c_1 e^t - c_2 e^{-t} + 9c_3 \cos 3t + 9c_4 \sin 3t + \frac{8}{9}$$

$$-4\left(c_1 e^t + c_2 e^{-t} + c_3 \cos 3t - c_4 \sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}\right)$$

$$= -5c_1 e^t - 5c_2 e^{-t} + 5c_3 \cos 3t + 5c_4 \sin 3t + \frac{25}{9}t^2 - \frac{20}{9}t + \frac{220}{81}$$

$$\therefore \quad y = -c_1 e^t - c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t + \frac{5}{9}t^2 - \frac{4}{9}t + \frac{44}{81}$$

∴ The general solution of the given system is

$$\mathbf{x = c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right),}$$

$$\mathbf{y = -c_1 e^t - c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right).}$$

Remark. For the above system $\Delta = \begin{vmatrix} D^2+4 & 5 \\ 5 & D^2+4 \end{vmatrix} = (D^2+4)^2 - 25 = D^4 + 8D^2 - 9.$

Since, the degree of D in Δ is 4, the general solution will have four arbitrary constants.

Example 6. *Solve the simultaneous equations :* $\frac{dx}{dt} = 2y, \ \frac{dy}{dt} = 2z, \ \frac{dz}{dt} = 2x$.

Sol. We have

$$Dx - 2y + 0z = 0 \quad \ldots(1)$$

$$0x + Dy - 2z = 0 \quad \ldots(2)$$

$$2x - 0y - Dz = 0 \quad \ldots(3)$$

Using determinants, $\Delta = \begin{vmatrix} D & -2 & 0 \\ 0 & D & -2 \\ 2 & 0 & -D \end{vmatrix} = -D^3 + 8$

The degree of D in Δ is 3.

∴ The general solution will have three arbitrary constants.

For x $\quad \Delta \cdot x = \begin{vmatrix} 0 & -2 & 0 \\ 0 & D & -2 \\ 0 & 0 & -D \end{vmatrix}$

$$\Rightarrow \quad (-D^3 + 8)x = 0$$

∴ The A.E. is $-D^3 + 8 = 0$. ∴ $D = 2, -1 \pm \sqrt{3}\,i$

$$\therefore \quad x = c_1 e^{2t} + e^{-t}(c_2 \cos \sqrt{3}\,t + c_3 \sin \sqrt{3}\,t)$$

$$(1) \Rightarrow \quad 2y = \frac{dx}{dt} = 2c_1 e^{2t} - e^{-t}(c_2 \cos \sqrt{3}\,t + c_3 \sin \sqrt{3}\,t)$$

$$+ e^{-t}(-\sqrt{3}\,c_2 \sin \sqrt{3}\,t + \sqrt{3}\,c_3 \cos \sqrt{3}\,t)$$

$$= 2c_1 e^{2t} - e^{-t}\,[(c_2 - \sqrt{3}\,c_3) \cos \sqrt{3}\,t + (c_3 + \sqrt{3}\,c_2) \sin \sqrt{3}\,t]$$

$$\therefore \quad y = c_1 e^{2t} - \frac{1}{2}e^{-t}\,[(c_2 - \sqrt{3}\,c_3) \cos \sqrt{3}\,t + (c_3 + \sqrt{3}\,c_2) \sin \sqrt{3}\,t]$$

$$(2) \Rightarrow \quad 2z = \frac{dy}{dt} = 2c_1 e^{2t} + \frac{1}{2}e^{-t}\,[(c_2 - \sqrt{3}\,c_3) \cos \sqrt{3}\,t + (c_3 + \sqrt{3}\,c_2) \sin \sqrt{3}\,t]$$

$$- \frac{1}{2}e^{-t}\,[-(\sqrt{3}\,c_2 - 3c_3) \sin \sqrt{3}\,t + (\sqrt{3}\,c_3 + 3c_2) \cos \sqrt{3}\,t]$$

NOTES

$= 2c_1 e^{2t} - e^{-t} [(c_2 + \sqrt{3} c_3) \cos \sqrt{3} t + (c_3 - \sqrt{3} c_2) \sin \sqrt{3} t]$

$\therefore \quad z = c_1 e^{2t} - \frac{1}{2} e^{-t} [(c_2 + \sqrt{3} c_3) \cos \sqrt{3} t + (c_3 - \sqrt{3} c_2) \sin \sqrt{3} t].$

$\therefore$ The general solution of the given system is

$$\mathbf{x = c_1 e^{2t} + e^{-t} (c_2 \cos \sqrt{3} t + c_3 \sin \sqrt{3} t),}$$

$$\mathbf{y = c_1 e^{2t} - \frac{1}{2} e^{-t} [(c_2 - \sqrt{3} c_3) \cos \sqrt{3} t + (c_3 + \sqrt{3} c_2) \sin \sqrt{3} t],}$$

$$\mathbf{z = c_1 e^{2t} - \frac{1}{2} e^{-t} [(c_2 + \sqrt{3} c_3) \cos \sqrt{3} t + (c_3 - \sqrt{3} c_2) \sin \sqrt{3} t].}$$

Example 7. *Solve the simultaneous equations :* $t \frac{dx}{dt} + y = 0, \quad t \frac{dy}{dt} + x = 0$, *given that* $x(1) = 1, y(-1) = 0$.

Sol. We have $$t \frac{dx}{dt} + y = 0 \quad \text{...(1)}$$

$$t \frac{dy}{dt} + x = 0 \quad \text{...(2)}$$

Differentiating (1) w.r.t. t, we get

$$t \frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0 \quad \text{...(3)}$$

Multiplying (3) by t, we get

$$t^2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} + t \frac{dy}{dt} = 0 \quad \text{...(4)}$$

$$(4) - (2) \Rightarrow \quad t^2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} - x = 0 \quad \text{...(5)}$$

(5) is a homogeneous linear equation.

Let $z = \log t. \quad \therefore \quad t = e^z$

$$\therefore \quad t \frac{d}{dt} = D, \quad t^2 \frac{d^2}{dt^2} = D(D-1), \text{ where } D = \frac{d}{dz}$$

$\therefore \quad (5) \Rightarrow \quad D(D-1)x + Dx - x = 0$

$\Rightarrow \quad (D^2 - D + D - 1)x = 0$

$\Rightarrow \quad (D^2 - 1)x = 0 \quad \text{...(6)}$

The A.E., of (6) is $D^2 - 1 = 0 \quad \therefore \quad D = \pm 1$

$\therefore$ The general solution of (6) is

$$x = c_1 e^z + c_2 e^{-z}$$

$\therefore$ The general solution of (5) is

$$x = c_1 t + \frac{c_2}{t}$$

NOTES

(1) $\Rightarrow$ $$y = -t\frac{dx}{dt} = -t\left[c_1 - \frac{c_2}{t^2}\right] = -c_1 t + \frac{c_2}{t}$$

$\therefore$ The general solution of the given system is

$$x = c_1 t + \frac{c_2}{t} \qquad ...(7)$$

$$y = -c_1 t + \frac{c_2}{t} \qquad ...(8)$$

$$x(1) = 1 \Rightarrow c_1 + c_2 = 1$$

$$y(-1) = 0 \Rightarrow c_1 - c_2 = 0$$

$\therefore$ $$c_1 = \frac{1}{2} \text{ and } c_2 = \frac{1}{2}$$

$\therefore$ The required solution is

$$x = \frac{1}{2}t + \frac{1/2}{t} \quad y = -\frac{1}{2}t + \frac{1/2}{t}$$

or $$x = \frac{1}{2}\left(t + \frac{1}{t}\right), \; y = \frac{1}{2}\left(-t + \frac{1}{t}\right).$$

EXERCISE 1

Solve the following systems of simultaneous linear differential equations :

1. $\frac{dx}{dt} = \lambda y, \; \frac{dy}{dt} = \lambda x$
2. $\frac{dx}{dt} - 7x + y = 0, \; \frac{dy}{dt} - 2x - 5y = 0$
3. $\frac{dx}{dt} + 5x + y = e^t, \; \frac{dy}{dt} - x + 3y = e^{2t}$
4. $\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 2y = 3e^t, \; 3\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 4e^{2t}$
5. $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t \; \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$
6. $\frac{dx}{dt} = ax + by, \; \frac{dy}{dt} = a'x + b'y$
7. $\frac{dx}{dt} + 4x + 3y = t^2 \; \frac{dy}{dt} + 2x + 5y = e^{2t}$
8. $\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t, \; \frac{dx}{dt} + 4\frac{dy}{dt} = 3y$
9. $\frac{d^2x}{dt^2} + y = \sin t, \; \frac{d^2y}{dt^2} + x = \cos t$
10. $\frac{d^2x}{dt^2} - 3x - 4y = 0, \; \frac{d^2y}{dt^2} + x + y = 0.$

Answers

1. $x = c_1 e^{\lambda t} + c_2 e^{-\lambda t}, \; y = c_1 e^{\lambda t} - c_2 e^{-\lambda t}$
2. $x = e^{6t}(c_1 \cos t + c_2 \sin t), \; y = e^{6t}((c_1 - c_2)\cos t + (c_1 + c_2)\sin t)$
3. $x = (c_1 + c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t} \; y = -(c_1 + c_2 + c_2 t)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}$

NOTES

4. $x = c_1 e^{-6t/5} + \frac{1}{2} e^{2t} - \frac{3}{11} e^t \quad y = -8c_1 e^{-6t/5} + c_2 e^{-t} + \frac{15}{22} e^t$

5. $x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3 \cos t \quad y = c_1 (\sqrt{2} + 1) e^{\sqrt{2}t} + c_2 (1 - \sqrt{2}) e^{-\sqrt{2}t} + 2 \sin t$

6. $x = c_1 e^{mt} + c_2 e^{nt}, y = \frac{c_1}{b} e^{mt} (m - a) + \frac{c_2}{b} e^{nt} (n - a),$

where $m = \frac{a + b' + \sqrt{(a - b')^2 + 4a'b}}{2}$ and $n = \frac{a + b' - \sqrt{(a - b')^2 + 4a'b}}{2}$

7. $x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{1}{14}\left(5t^2 - \frac{31}{7} t + \frac{209}{98}\right) - \frac{1}{12} e^{2t}$

$y = -\frac{2}{3} c_1 e^{-2t} + c_2 e^{-7t} - \frac{1}{7}\left(t^2 - \frac{9}{7} t + \frac{67}{98}\right) + \frac{1}{6} e^{2t}$

8. $x = (c_1 + c_2 t)e^t + c_3 e^{-3t/2} - t \quad y = (c_2 (3 - t) - c_1) e^t - \frac{1}{6} c_3 e^{-3t/2}$

9. $x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t)$

$y = - c_1 e^t - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}(t + 2)(\sin t - \cos t)$

10. $x = (c_1 + c_2 t) e^{-t} + (c_3 + c_4 t) e^t \quad y = -\frac{1}{2} (c_1 + c_2 + c_2 t) e^{-t} - \frac{1}{2} (c_3 - c_4 + c_4 t) e^t.$

II. SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

10.5. TOTAL DIFFERENTIAL EQUATION

Let P, Q, R be functions of three variables x, y and z. A differential equation of the form P dx + Q dy + R dz = 0 is called a **total differential equation.**

It can be proved that a total differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \qquad ...(1)$$

is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

For example, consider the total differential equation

$$(y + 3z)dx + (x + 2z)dy + (3x + 2y)dz = 0 \qquad ...(2)$$

Here $\quad P = y + 3z,\ Q = x + 2z,\ R = 3x + 2y$

It can be checked that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} = 3.$$

NOTES

$\therefore$ The equation (2) is exact and it can be written as

$$(y\,dx + x\,dy) + 2(y\,dz + z\,dy) + 3(x\,dz + z\,dx) = 0$$

$$\Rightarrow \quad d(xy) + d(2yz) + d(3zx) = 0 \quad \Rightarrow \quad d(xy + 2yz + 3zx) = 0$$

Integrating, we get $xy + 2yz + 3zx = c$

This is the general solution of the total differential equation (2)

There are other methods also for solving a total differential equation of the form (1).

In the present chapter, we shall study the method of solving a system of two simultaneous total differential equations

10.6. METHOD OF SOLVING SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

We shall consider systems of two simultaneous total differential equations in three variables.

Let
$$P_1dx + Q_1dy + R_1dz = 0 \qquad \text{...(1)}$$
$$P_2dx + Q_2dy + R_2dz = 0 \qquad \text{...(2)}$$

be a system of two simultaneous total differential equations, where P_1, Q_1, R_1, P_2, Q_2, R_2 are all functions of x, y and z. Solving (1) and (2) for dx, dy, dz, we get

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

$$\Rightarrow \quad \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} \qquad \text{...(3)}$$

where $X = Q_1R_2 - Q_2R_1$, $Y = R_1P_2 - R_2P_1$, $Z = P_1Q_2 - P_2Q_1$

Thus, a system of two simultaneous total equations (1), (2) can always be put in the form (3).

The equations (3) are said to be completely solved when we get a solution of the form $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$, where u_1 and u_2 are two independent solutions of the equation (3). The functions u_1 and u_2 are said to be independent if $\frac{u_1}{u_2}$ is not merely a constant.

There are number of rules of solving differential equation (3), depending upon the nature of the functions X, Y and Z.

Remark. In case any denominator of the equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ is zero, then the corresponding numerator is also taken equal to zero.

For example, if $X = 0$, then we take $dx = 0$ *i.e.*, x = constant.

10.7. TYPE I

In this type we shall consider the solution of systems of equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ in which the equality of two fractions of the given system gives an equation involving only two variables. This equation is solved by usual methods. Same method is repeated by equating two other fractions of the given system.

Example 1. *Solve the following systems of simultaneous differential equations :*

(*i*) $\dfrac{dx}{y^2}=\dfrac{dy}{x^2}=\dfrac{dz}{x^2y^2z^2}$ (*ii*) $\dfrac{z\,dx}{y^2z}=\dfrac{dy}{xz}=\dfrac{dz}{y^2}$

(*iii*) $\dfrac{dx}{z}=\dfrac{dy}{0}=\dfrac{dz}{x}$ (*iv*) $\dfrac{dx}{x^2+2y^2}=-\dfrac{dy}{xy}=\dfrac{dz}{xz}$.

Sol. (*i*) We have

$$\frac{dx}{y^2}=\frac{dy}{x^2}=\frac{dz}{x^2y^2z^2} \qquad ...(1)$$

Taking first two fractions of (1), we get $x^2\,dx=y^2\,dy$.

Integrating, we get $\dfrac{x^3}{3}=\dfrac{y^3}{3}+c_1$ or $x^3-y^3=3c_1$.

Taking last two fractions of (1), we get $\dfrac{dy}{x^2}=\dfrac{dz}{x^2y^2z^2}$

$\Rightarrow$ $y^2dy=z^{-2}dz$

Integrating, we get $\dfrac{y^3}{3}=\dfrac{z^{-1}}{-1}+c_2$ or $\dfrac{y^3}{3}+\dfrac{1}{z}=c_2$ or $y^3z+3=3c_2z$

$\therefore$ The general solution of the given system is

$$\mathbf{x^3-y^3=3c_1,\ y^3z+3=3c_2\,z.}$$

(*ii*) We have $$\frac{z\,dz}{y^2z}=\frac{dy}{xz}=\frac{dz}{y^2} \qquad ...(1)$$

Taking the first two fractions of (1), we get

$$x^2dx=y^2dy \quad \text{or} \quad 3x^2dx-3y^2dy=0 \qquad ...(2)$$

Integrating (2), we have $x^3-y^3=c_1$

Taking the first and last fractions of (1), we get

$$xdx=zdz \quad \text{or} \quad 2xdx-2zdz=0 \qquad ...(3)$$

Integrating (3), we have $x^2-z^2=c_2$

$\therefore$ The general solution of the given system is

$$\mathbf{x^3-y^3=c_1,\ x^2-z^2=c_2.}$$

(*iii*) We have $$\frac{dx}{z}=\frac{dy}{0}=\frac{dz}{x} \qquad ...(1)$$

Second fraction of (1) implies $dy=0$

$\therefore$ $y=c_1$

Taking the first and third fractions of (1), we get

$$xdx=zdz \quad \text{or} \quad 2xdx-2zdz=0 \qquad ...(2)$$

Integrating (2), we have $x^2-z^2=c_2$

$\therefore$ The general solution of the given system is $\mathbf{y=c_1,\ x^2-z^2=c_2}$.

(*iv*) We have $$\frac{dx}{x^2+2y^2}=-\frac{dy}{xy}=\frac{dz}{xz} \qquad ...(1)$$

NOTES

Taking the first two fractions of (1), we get $\frac{dx}{x^2+2y^2} = -\frac{dy}{xy}$.

$\Rightarrow \quad \frac{dx}{dy} = -\frac{x^2+2y^2}{xy} \quad \Rightarrow \quad 2x\frac{dx}{dy} = -\frac{2x^2}{y} - 4y$

$$\Rightarrow \quad 2x\frac{dx}{dy} + x^2\left(\frac{2}{y}\right) = -4y \qquad ...(2)$$

Let $\quad z = x^2$.

$\therefore \quad (2) \quad \Rightarrow \quad \frac{dz}{dy} + z\left(\frac{2}{y}\right) = -4y$

This is a linear differential equation of order one.

$$\text{I.F.} = e^{\int \frac{2}{y} dy} = e^{2 \log y} = y^2$$

$\therefore \quad zy^2 = \int (-4y)y^2\, dy + c_1 \quad \text{or} \quad x^2y^2 + y^4 = c_1$

Taking the last two fractions of (1), we get $\frac{dy}{y} + \frac{dz}{z} = 0$ ...(3)

Integrating (3), we get

$\log y + \log z = \log c_2 \quad \text{or} \quad yz = c_2$

$\therefore$ The general solution of the given system is

$\mathbf{x^2y^2 + y^4 = c_1, \; yz = c_2}$.

10.8. TYPE II

In this type we shall consider the solution of system of equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ in which the equality of two fractions of the given system gives an equation involving only two variables. This equation is solved by usual methods. The solution of this equation is used to eliminate an undesired variable from the equation obtained by equating two other fractions of the given system. This equation is solved and the first arbitrary constant is also eliminated.

Example 2. *Solve the following systems of simultaneous differential equations :*

(*i*) $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{z(xy - 2x^2)}$

(*ii*) $\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \tan(y - 2x)}$

(*iii*) $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$

(*iv*) $\frac{dx}{xz(z^2 + xy)} = -\frac{dy}{yz(z^2 + xy)} = \frac{dz}{x^4}$

Sol. (*i*) We have $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{z(xy - 2x^2)}$ ...(1)

Taking first two fractions of (1), we get $\frac{dx}{xy} = \frac{dy}{y^2}$

$\Rightarrow$ $$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get $\log x = \log y + \log c_1$

$\therefore$ $$x = c_1 y$$

Taking last two fractions of (1), we get $\frac{dy}{y^2} = \frac{dz}{z(xy - 2x^2)}$ ...(2)

We eliminate x from (2)

(2) $\Rightarrow$ $$\frac{dy}{y^2} = \frac{dz}{z(c_1 y \cdot y - 2c_1^2 y^2)} \qquad (\because \quad x = c_1 y)$$

$\Rightarrow$ $$dy = \frac{dz}{z(c_1 - 2c_1^2)} \quad \Rightarrow \quad (c_1 - 2c_1^2)\, dy = \frac{dz}{z}$$

Integrating, we get $(c_1 - 2c_1^2)\, y = \log z + c_2$

$\Rightarrow$ $$\left(\frac{x}{y} - \frac{2x^2}{y^2}\right) y = \log z + c_2 \quad \Rightarrow \quad x - \frac{2x^2}{y} = \log z + c_2.$$

$\therefore$ The general solution of the given system is

$$\mathbf{x = c_1 y,\ x - \frac{2x^2}{y} = \log z + c_2.}$$

(*ii*) We have $$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \tan(y - 2x)} \qquad ...(1)$$

Taking the first two fractions of (1), we get

$$dy - 2dx = 0 \qquad ...(2)$$

Integrating (2), we have $y - 2x = c_1$ ...(3)

Taking the last two fractions of (1) and using (3), we have

$$\frac{dy}{2} - \frac{dz}{5z + \tan c_1} = 0 \qquad ...(4)$$

Integrating (4), we have $\frac{1}{2} y - \frac{1}{5} \log | 5z + \tan c_1 | = c_2$

or $$5y - 2 \log | 5z + \tan(y - 2x) | = 10c_2$$

$\therefore$ The general solution of the given system is

$$\mathbf{y - 2x = c_1,\ 5y - 2 \log | 5z + \tan(y - 2x) | = 10\, c_2}.$$

(*iii*) We have $$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)} \qquad ...(1)$$

Taking the first two fractions of (1), we get

$$xdx = ydy \quad \text{or} \quad 2xdx - 2ydy = 0 \qquad ...(2)$$

Integrating (2), we have $x^2 - y^2 = c_1$ ...(3)

NOTES

Taking the last two fractions of (1) and using (3), we have

$$ydy - \frac{dz}{c_1 z^2} = 0 \qquad ...(4)$$

Integrating (4), we have $\frac{y^2}{2} - \frac{1}{c_1}\left(\frac{z^{-1}}{-1}\right) = c_2$

$$\Rightarrow \qquad \frac{y^2}{2} + \frac{1}{z(x^2 - y^2)} = c_2$$

∴ The general solution of the given system is

$$\mathbf{x^2 - y^2 = c_1,\ \frac{y^2}{2} + \frac{1}{z(x^2 - y^2)} = c_2.}$$

(*ii*) We have $\frac{dx}{xz(z^2 + xy)} = -\frac{dy}{yz(z^2 + xy)} = \frac{dz}{x^4}$ ...(1)

Taking the first two fractions of (1), we get $\frac{dx}{x} + \frac{dy}{y} = 0$ ...(2)

Integrating (2), we have

$$\log x + \log y = \log c_1 \quad \text{or} \quad xy = c_1 \qquad ...(3)$$

Taking the first and third fractions of (1) and using (3), we get

$$\frac{dx}{z(z^2 + c_1)} = \frac{dz}{x^3} \quad \text{or} \quad x^3 dx - (z^3 + c_1 z)dz = 0 \qquad ...(4)$$

Integrating (4), we have

$$\frac{x^4}{4} - \left(\frac{z^4}{4} + \frac{c_1 z^2}{2}\right) = c_2 \quad \text{or} \quad x^4 - z^4 - 2xyz^2 = 4c_2$$

∴ The general system of the given system is

$$\mathbf{xy = c_1,\ x^4 - z^4 - 2xyz^2 = 4c_2.}$$

10.9. TYPE III

In this type, we shall consider the solution of system of equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ by using the formula :

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{X_1 dx + Y_1 dy + Z_1 dz}{X_1 X + Y_1 Y + Z_1 Z}$$

where X_1, Y_1, Z_1, are some functions of x, y and z. If for some choice of X_1, Y_1, Z_1 the sum $X_1 X + Y_1 Y + Z_1 Z$ is zero, then we have $X_1\ dx + Y_1\ dy + Z_1\ dz = 0$.

We integrate this equation to get one relation between x, y and z. The functions X, Y, Z, are called **multipliers**. By using different set of multipliers or by using two fractions of the given system, we find another independent solution of the given system.

NOTES

Example 3. *Solve the following systems of simultaneous differential equations :*

(*i*) $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$

(*ii*) $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{y(z^2-x^2)}=\dfrac{dz}{z(x^2-y^2)}$

(*iii*) $\dfrac{dx}{x(y^2+z)}=\dfrac{dy}{-y(x^2+z)}=\dfrac{dz}{z(x^2-y^2)}$

(*iv*) $\dfrac{dx}{y^2+z^2}=\dfrac{dy}{-xy}=\dfrac{dz}{-xz}$

(*v*) $\dfrac{dx}{z^2-2yz-y^2}=\dfrac{dy}{xy+xz}=\dfrac{dz}{xy-xz}$.

Sol. (*i*) We have $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$ ...(1)

Choosing l, m, n as multiplier, each fraction of (1)

$$=\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{ldx+mdy+ndz}{0}.$$

$\therefore \quad ldx+mdy+ndz=0$

Integrating, we get $lx+my+nz=c_1$

Choosing x, y, z as multiplier, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

$$=\frac{xdx+ydy+zdz}{0}.$$

$\therefore \quad xdx+ydy+zdz=0$

Integrating, we get $\frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2=c_2 \quad\Rightarrow\quad x^2+y^2+z^2=2c_2$.

$\therefore$ The general solution of the given system is

$$\mathbf{lx+my+nz=c_1,\ x^2+y^2+z^2=2c_2}.$$

(*ii*) We have $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{y(z^2-x^2)}=\dfrac{dz}{z(x^2-y^2)}$ ...(1)

Taking $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers, each fraction of (1)

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(y^2-z^2)+(z^2-x^2)+(x^2-y^2)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$\therefore \quad \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

NOTES

Integrating, we get $\log|x| + \log|y| + \log|z| = \log c_1$

or $\quad |xyz| = c_1 \quad$ or $\quad xyz = \pm c_1 \quad$ or $\quad xyz = c_2 \qquad$ (By putting $c_2 = \pm c_1$)

Taking x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0 \quad$ or $\quad 2xdx + 2ydy + 2zdz = 0$

Integrating, we get $\quad x^2 + y^2 + z^2 = c_3$

$\therefore\;$ The general solution of the given system is $\mathbf{xyz = c_2,\ x^2 + y^2 + z^2 = c_3}$.

(*iii*) We have $\displaystyle \frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$ ...(1)

Taking $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + (-1)\,dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\therefore \quad 2xdx + 2ydy - 2dz = 0$

Integrating, we get $x^2 + y^2 - 2z = c_1$ ...(2)

Taking $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multiplier, each fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get $\log|x| + \log|y| + \log|z| = \log c_2$

or $\quad |xyz| = c_2 \quad$ or $\quad xyz = \pm c_2 \quad$ or $\quad xyz = c_3 \qquad$ (By putting $c_3 = \pm c_2$)

$\therefore\;$ The general solution of the given equation is $\mathbf{x^2 + y^2 - 2z = c_1,\ xyz = c_3}$.

(*iv*) We have $\quad \displaystyle \frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$ ...(1)

Taking the last two fractions of (1), we get $\dfrac{dy}{y} - \dfrac{dz}{z} = 0$

Integrating, we have $\log|y| - \log|z| = \log c_1$

or $\quad \left|\dfrac{y}{z}\right| = c_1 \quad$ or $\quad \dfrac{y}{z} = \pm c_1 \quad$ or $\quad y = c_2 z \qquad$ ... (2) (By putting $c_2 = \pm c_1$)

Taking x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0 \quad \text{or} \quad 2xdx + 2ydy + 2zdz = 0$

Integrating, we have $x^2 + y^2 + z^2 = c_3$

$\therefore$ The general solution of the given system is

$$\mathbf{y = c_2 z, \quad x^2 + y^2 + z^2 = c_3.}$$

NOTES

(v) We have $\dfrac{dx}{z^2 - 2yz - y^2} = \dfrac{dy}{xy + xz} = \dfrac{dz}{xy - xz}$ ...(1)

Taking the last two fractions of (1), we get

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\Rightarrow \quad \frac{dy}{y+z} = \frac{dz}{y-z}$$

$$\Rightarrow \quad (y - z)dy = (y + z)dz$$

$$\Rightarrow \quad ydy - zdz - (z\,dy + ydz) = 0$$

Integrating, we get

$$\frac{y^2}{2} - \frac{z^2}{2} - yz = c_1$$

$\Rightarrow \quad y^2 - z^2 - 2yz = c_2$ (Putting $c_2 = 2c_1$)

Choosing x, y, z as multiplies, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(z^2 - 2yz - y^2) + y(xy + xz) + z\,(xy - xz)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_3$$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_4$ (Putting $c_4 = 2c_3$)

$\therefore$ The general solution of the given system is

$$\mathbf{y^2 - z^2 - 2yz = c_2, \; x^2 + y^2 + z^2 = c_4.}$$

10.10. TYPE IV

In this type we shall consider the solution of system of equations $\dfrac{dx}{X} = \dfrac{dy}{Y} = \dfrac{dz}{Z}$ by using the formula :

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{X_1 dx + Y_1 dy + Z_1 dz}{X_1 X + Y_1 Y + Z_1 Z}.$$

NOTES

where X_1, Y_1, Z_1 are some functions of x, y and z. If for some choice of X_1, Y_1, Z_1, the sum $X_1\,dx + Y_1\,dy + Z_1\,dz$ is the exact differential of a factor of $X_1X + Y_1Y + Z_1Z$, then the quotient $\frac{X_1dx + Y_1dy + Z_1\,dz}{X_1X + Y_1Y + Z_1Z}$ is equated with a suitable fraction of given equations to get one solution of the given equations. By using different set of multipliers or by using two fractions of the given equations, we find another independent solution of the given equations.

Example 4. *Solve the following systems of simultaneous equations :*

(*i*) $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$

(*ii*) $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

(*iii*) $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z + \frac{1}{z}}$

(*iv*) $\frac{dx}{y^2 + yz + z^2} = \frac{dy}{z^2 + zx + x^2} = \frac{dz}{x^2 + xy + y^2}$

Sol. (*i*) We have $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$ ...(1)

Taking 1, 1, 1 and x, y, z as multipliers, each fraction of (1)

$$= \frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{x^2 - yz + y^2 - zx + z^2 - xy} = \frac{xdx + ydy + zdz}{x^3 - xyz + y^3 - xyz + z^3 - xyz}$$

$$\Rightarrow \quad \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} = \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)}$$

$$\Rightarrow \quad (x+y+z)d(x+y+z) = \frac{1}{2}\,(2xdx + 2ydy + 2zdz)$$

$$\Rightarrow \quad (x+y+z)\,d(x+y+z) - \frac{1}{2}\,d(x^2 + y^2 + z^2) = 0$$

Integrating, we get $\quad \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2 + y^2 + z^2) = c_1$

or $\quad xy + yz + zx = c_1$

Taking 1, – 1, 0 and 0, 1, – 1 as multipliers, each fraction of (1)

$$= \frac{dx - dy + 0}{(x^2 - yz) - (y^2 - zx) + 0} = \frac{0 + dy - dz}{0 + (y^2 - zx) - (z^2 - xy)}$$

$$\Rightarrow \quad \frac{dx - dy}{x^2 - y^2 + z(x-y)} = \frac{dy - dz}{y^2 - z^2 + x(y-z)}$$

$$\Rightarrow \quad \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(y+z+x)} \Rightarrow \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

NOTES

Integrating, we get $\log|x-y| - \log|y-z| = \log c_2$

or $$\left|\frac{x-y}{y-z}\right| = c_2 \quad \text{or} \quad \frac{x-y}{y-z} = \pm c_2 \quad \text{or} \quad x-y = c_3(y-z).$$

(By putting $c_3 = \pm c_2$)

∴ The general solution of the given system is

$$\mathbf{xy + yz + zx = c_1,\ x - y = c_3(y - z).}$$

(*ii*) We have $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad ...(1)$$

Taking 1, −1, 0 and 0, 1, −1 as multipliers, each fraction of (1)

$$= \frac{dx - dy + 0}{(y+z)-(z+x)+0} = \frac{0 + dy - dz}{0 + (z+x) - (x+y)} \qquad ...(2)$$

$$\Rightarrow \quad \frac{dx - dy}{-(x-y)} = \frac{dy - dz}{-(y-z)} \quad \Rightarrow \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

Integrating, we get $\log|x-y| - \log|y-z| = \log c_1$

$$\Rightarrow \quad \left|\frac{x-y}{y-z}\right| = c_1 \quad \text{or} \quad \frac{x-y}{y-z} = \pm c_1 \quad \text{or} \quad x - y = c_2(y-z)$$

(By putting $c_2 = \pm c_1$)

Taking 1, −1, 0 and 1, 1, 1 as multipliers, each fraction of (1)

$$\frac{dx - dy + 0}{(y+z)-(z+x)+0} = \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)}$$

$$\Rightarrow \quad \frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{or} \quad 2\frac{d(x-y)}{x-y} + \frac{d(x+y+z)}{x+y+z} = 0$$

Integrating, we get $2\log|x-y| + \log|x+y+z| = \log c_3$

$$\Rightarrow \quad (x-y)^2\,|x+y+z| = c_3 \quad \Rightarrow \quad (x-y)^2(x+y+z) = \pm c_3$$

$$\Rightarrow \quad (x-y)^2(x+y+z) = c_4. \qquad \text{(By putting } c_4 = \pm c_3\text{)}$$

∴ The general solution of the given system is

$$\mathbf{x - y = c_2(y - z),\ (x - y)^2(x + y + z) = c_4.}$$

(*iii*) We have $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z + \frac{1}{z}} \qquad ...(1)$$

Taking 1, 1, 0 and 1, −1, 0 as multipliers, each fraction of (1)

$$= \frac{dx + dy + 0}{\cos(x+y) + \sin(x+y) + 0} = \frac{dx - dy + 0}{\cos(x+y) - \sin(x+y) + 0} \qquad ...(2)$$

(1) and (2) $$\Rightarrow \quad \frac{zdz}{z^2+1} = \frac{d(x+y)}{\cos(x+y) + \sin(x+y)}$$

$$\Rightarrow \quad \frac{1}{2}\cdot\frac{2zdz}{z^2+1} = \frac{dt}{\cos t + \sin t} = \frac{dt}{\sqrt{2}\sin(t + \pi/4)}, \text{ where } t = x + y.$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\,\frac{2z}{z^2+1}\,dz - \operatorname{cosec}\left(t + \frac{\pi}{4}\right)dt = 0$$

NOTES

Integrating, we get $\frac{1}{\sqrt{2}} \log |z^2 + 1| - \log \tan \frac{1}{2}\left(t + \frac{\pi}{4}\right) = \log c_1$

$$\Rightarrow \quad \frac{(z^2+1)^{1/\sqrt{2}}}{\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right)} = c_1$$

Also, (2) $\Rightarrow$ $\frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)}\, d(x+y) = d(x-y)$

$$\Rightarrow \quad \frac{\cos t - \sin t}{\cos t + \sin t}\, dt - d(x-y) = 0, \text{ where } t = x+y.$$

Integrating, we get $\log |\cos t + \sin t| - (x-y) = \log c_2$

$$\Rightarrow \quad |\cos t + \sin t|\, e^{y-x} = c_2$$

$$\Rightarrow \quad (\cos(x+y) + \sin(x+y))e^{y-x} = \pm c_2$$

$$\Rightarrow \quad (\cos(x+y) + \sin(x+y))e^{y-x} = c_3 \qquad (\text{By putting } c_3 = \pm c_2)$$

$\therefore$ The general solution of the given system is

$$\frac{(\mathbf{z^2+1})^{1/\sqrt{2}}}{\tan\left(\frac{\mathbf{x+y}}{2} + \frac{\pi}{8}\right)} = c_1,\ (\cos(\mathbf{x+y}) + \sin(\mathbf{x+y}))\, e^{y-x} = c_3.$$

(*iv*) We have $\frac{dx}{y^2 + yz + z^2} = \frac{dy}{z^2 + zx + x^2} = \frac{dz}{x^2 + xy + y^2}$ ...(1)

Taking $1, -1, 0$ and $0, 1, -1$ as multipliers, each fraction of (1)

$$= \frac{dx - dy + 0}{(y^2 + yz + z^2) - (z^2 + zx + x^2) + 0} = \frac{0 + dy - dz}{0 + (z^2 + zx + x^2) - (x^2 + xy + y^2)}$$

$$\Rightarrow \quad \frac{dx - dy}{y^2 - x^2 + yz - zx} = \frac{dy - dz}{z^2 - y^2 + zx - xy}$$

$$\Rightarrow \quad \frac{dx - dy}{(y-x)(x+y+z)} = \frac{dy - dz}{(z-y)(z+y+x)}$$

$$\Rightarrow \quad \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} \quad \Rightarrow \quad \frac{d(y-z)}{y-z} - \frac{d(x-y)}{x-y} = 0$$

Integrating, we get $\log |y-z| - \log |x-y| = \log c_1$

$$\Rightarrow \quad \left|\frac{y-z}{x-y}\right| = c_1 \quad \text{or} \quad \frac{y-z}{x-y} = \pm c_1$$

or $\quad y - z = c_2(x-y) \qquad (\text{By putting } c_2 = \pm c_1)$

Taking $1, -1, 0$ and $1, 0, -1$ as multipliers, each fractions of (1)

$$= \frac{dx - dy}{(y^2 + yz + z^2) - (z^2 + zx + x^2)} = \frac{dx - dz}{(y^2 + yz + z^2) - (x^2 + xy + y^2)}$$

$$\Rightarrow \quad \frac{dx - dy}{y^2 - x^2 + yz - zx} = \frac{dx - dz}{z^2 - x^2 + yz - xy}$$

$$\Rightarrow \quad \frac{dx - dy}{(y-x)(y+x+z)} = \frac{dx - dz}{(z-x)(z+x+y)}$$

$$\Rightarrow \quad \frac{dx - dy}{-(x-y)} = \frac{dx - dz}{-(x-z)} \Rightarrow \frac{d(x-z)}{x-z} - \frac{d(x-y)}{x-y} = 0$$

Integrating, we get $\log | x - z | - \log | x - y | = \log c_3$

$$\Rightarrow \quad \left| \frac{x-z}{x-y} \right| = c_3 \quad \text{or} \quad \frac{x-z}{x-y} = \pm c_3$$

or $\quad x - z = c_4(x - y)$. (By putting $c_4 = \pm c_3$)

∴ The general solution of the given system is

$$y - z = c_2(x - y), \quad x - z = c_4 (x - y)$$

NOTES

EXERCISE 2

Solve the following systems of simultaneous differential equations :

1. (i) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ (ii) $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

(iii) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sin x}$ (iv) $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$

(v) $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ (vi) $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

(vii) $\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{dz}{z-c}$ (viii) $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

2. (i) $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot (y - 3x)}$ (ii) $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

(iii) $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin (y + 2x)}$ (iv) $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{x+y}$

(v) $\frac{dx}{-xy^2} = \frac{dy}{y^3} = \frac{dz}{axz}$ (vi) $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

(vii) $\frac{dx}{x+y} = -\frac{dy}{x+y} = \frac{dz}{z}$ (viii) $\frac{dx}{z} = -\frac{dy}{z} = \frac{dz}{z^2 + (x+y)^2}$

3. (i) $\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$ (ii) $\frac{yzdx}{y-z} = \frac{zxdy}{z-x} = \frac{xydz}{x-y}$

(iii) $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$ (iv) $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$

(v) $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ (vi) $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$

(vii) $\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$ (viii) $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2 + y^2}$

NOTES

4. (i) $\dfrac{dx}{x^2-y^2-z^2}=\dfrac{dy}{2xy}=\dfrac{dz}{2xz}$ (ii) $\dfrac{dx}{1+y}=\dfrac{dy}{1+x}=\dfrac{dz}{z}$

(iii) $\dfrac{dx}{x^2-y^2-yz}=\dfrac{dy}{x^2-y^2-zx}=\dfrac{dz}{z(x-y)}$ (iv) $\dfrac{dx}{xz}=\dfrac{dy}{yz}=\dfrac{dz}{xy}$

(v) $\dfrac{dx}{x(x+y)}=-\dfrac{dy}{y(x+y)}=-\dfrac{dz}{(x-y)(2x+2y+z)}$

(vi) $\dfrac{dx}{y(x+y)+az}=\dfrac{dy}{y(x+y)-az}=\dfrac{dz}{z(x+y)}$

(vii) $\dfrac{dx}{x}=\dfrac{dy}{z}=\dfrac{dz}{-y}$ (viii) $\dfrac{dx}{x^2+y^2}=\dfrac{dy}{2xy}=\dfrac{dz}{(x+y)z}$

Answers

1. (i) $x=c_1y,\ \ y=c_2z$ (ii) $\sin x=c_1\sin y,\ \sin y=c_2\sin z$

(iii) $x-y=c_1,\ z+\cos x=c_2$ (iv) $bx-ay=c_1\ \ cy-bz=c_2$

(v) $x^2-y^2=c_1,\ x^2-z^2=c_2$ (vi) $\dfrac{1}{y}-\dfrac{1}{x}=c_1,\ \ \dfrac{1}{z}-\dfrac{1}{y}=c_2$

(vii) $x-a=c_1(y-b),\ y-b=c_2(z-c)$ (viii) $x^3+y^3=c_1,\ yz-y^2=c_2$

2. (i) $y-3x=c_1,\ x-\log|z+\cot(y-3x)|=c_2$

(ii) $x=c_1y,\ xy-z^2=2c_2$

(iii) $y+2x=c_1,\ x^3\sin(y+2x)-z=c_2$ (iv) $x+y=c_1\ \ 2(x+y)x-z^2=c_2$

(v) $xy=c_1,\ \log|z|+\dfrac{ax}{3y^2}+c_2=0$ (vi) $x=c_1y,\ x-\log\left|z-\dfrac{2x}{y}\right|=c_2$

(vii) $x+y=c_1,\ x-(x+y)\log|z|=c_2$ (viii) $x+y=c_1,\ y+\dfrac{1}{2}\log(z^2+(x+y)^2)=c_2$

3. (i) $ax^2+by^2+cz^2=2c_1,\ a^2x^2+b^2y^2+c^2z^2=2c_2$

(ii) $x+y+z=c_1,\ xyz=c_2$

(iii) $x+y+z=c_1,\ x^2+y^2+z^2=2c_2$ (iv) $x^3+y^2+z^2=2c_1,\ x=c_2yz$

(v) $x+y+z=c_1,\ xyz=c_2$ (vi) $xyz=c_1,\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}+c_2=0$

(vii) $x+y-\log|z|=c_1,\ (x^2+y^2)e^{-2\tan^{-1}(y/x)}=e^{-2c_2}$

(viii) $x^2-y^2-z^2=2c_1,\ 2xy-z^2=2c_2$

4. (i) $y=c_1z,\ x^2+y^2+z^2=c_2z$ (ii) $(1+x)^2-(1+y)^2=2c_1,\ x+y+2=c_2z$

(iii) $z-x+y=c_1,\ x^2-y^2=c_2z^2$ (iv) $xy-z^2=c_1,\ x=c_2y$

(v) $xy=c_1,\ (x+y)(x+y+z)=c_2$ (vi) $x+y=c_1z,\ x^2-y^2-2az=c_2$

(vii) $x=c_1e^{\tan^{-1}(y/z)},\ y^2+z^2=2c_2$ (viii) $x+y=c_1z,\ y=c_2(x^2-y^2)$.

NOTES

Hints

3. (*iv*) Try x, y, z and $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers.

(*viii*) Try $x, -y, -z$ and $y, x, -z$ as multipliers.

4. (*i*) Try x, y, z as multipliers. (*ii*) Try 1, 1, 0 as multipliers.

(*iii*) Try 1, − 1, 0 and $x, -y$, 0 as multipliers. (*iv*) Try $1/x$, $1/y$, 0 as multipliers.

(*v*) Try 1, 1, 0 and 1, 1, 1 as multipliers. (*vi*) Try 1, 1, 0 and $x, -y$, 0 as multipliers.

(*vii*) We have $\dfrac{dx}{x} = \dfrac{0 + z\,dy - y\,dz}{z^2 + y^2} = \dfrac{\frac{1}{z}dy + \left(-\frac{y}{z^2}\right)dz}{1 + \left(\frac{y}{z}\right)^2} = \dfrac{d(y/z)}{1 + (y/z)^2}$

$\therefore \quad \log|x| - \tan^{-1}\dfrac{y}{z} = \log c.$

(*viii*) Try 1, 1, 0 and 1, − 1, 0 as multipliers.