मङ्गलायतन विश्वविद्यालय
॥ विश्वं ज्ञाने प्रतिष्ठितम् ॥

MANGALAYATAN
UNIVERSITY

*Learn Today to Lead Tomorrow*

# Partial Differential Equations

**MAL-6113**

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# CONTENTS

# UNIT 1 GEOMETRICAL MEANING OF A DIFFERENTIAL EQUATION

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## 1.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Describe geometrical meaning of a differential equation of first order and first degree
- Explain geometrical meaning of a differential equation of first order and degree higher than one

## 1.1. GEOMETRICAL MEANING OF A DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Let $$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

be a differential equation of first order and first degree.

We know that $\frac{dy}{dx}$ represents the slope of the tangent to the curve of the function $y(x)$.

Let (1) has a solution $y(x)$ passing through a point $A_1(x_1, y_1)$ of the $xy$-plane. Putting the coordinates of the point $A_1(x_1, y_1)$ in (1), we get the value of $\frac{dy}{dx}$ at $A_1(x_1, y_1)$. Let it be denoted by $m_1$. Thus $m_1$ is the slope of the tangent to the curve at $A_1(x_1, y_1)$.

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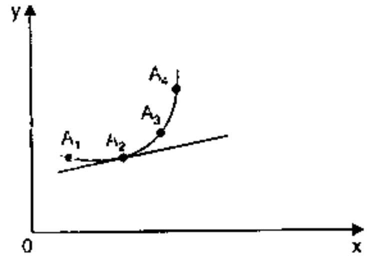


Now a point which moves, subject to the restriction imposed by (1), on passing through $A_1(x_1, y_1)$ must go in the direction $m_1$. Let the point move from $A_1(x_1, y_1)$ in the direction $m_1$ for an infinitesimal small distance, to a point $A_2(x_2, y_2)$. Let the slope of the tangent at $A_2(x_2, y_2)$ be $m_2$. Now let the point move in the direction $m_2$. Let the point move through an infinitesimal small distance in the direction $m_2$ and reach the position $A_3(x_3, y_3)$. In this way, let the point go to a point $A_4(x_4, y_4)$ and so on to the successive points. Hence through the point $A_1(x_1, y_1)$ will pass a curve.

Similarly through every point of the $xy$-plane, there passes a curve, the coordinates of every point of which and the direction of the tangent there at will satisfy the differential equation (1).

Hence equation (1) represents a family of curves such that through every point of the $xy$-plane there passes one curve of the family.

## 1.2. GEOMETRICAL MEANING OF THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Let $$\frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

be a differential equation of first order and first degree.

Since the differential equation (1) is of the first order, therefore its general solution will contain one arbitrary constant. Let the general solution be

$$\phi(x, y, c) = 0 \qquad \ldots(2)$$

It is evident that (2) represents one-parameter family of curves such that through each point of $xy$-plane, there passes one member of the family. If $(x_1, y_1)$ be a point of the $xy$-plane, then putting $x = x_1$ and $y = y_1$ in (2), we have

$$\phi(x_1, y_1, c) = 0.$$

This equation will give one value of $c$ say $c_1$.

Thus through the point $(x_1, y_1)$ in the plane, there will pass one member $\phi(x, y, c_1) = 0$ of the family of curves given by (2). Hence if we put $c = c_1$ in (2), we have $\phi(x, y, c_1) = 0$ which evidently represents a particular member of the family (2) and is a particular solution of the differential equation (1).

**Note.** It may be noted that the general solution (2) contains one parameter $c$ for the first order differential equation (1). Thus the solution of a differential equation of the first order will represent infinitely many curves for different values of the parameter.

## 1.3. GEOMETRICAL MEANING OF A DIFFERENTIAL EQUATION OF FIRST ORDER AND DEGREE HIGHER THAN ONE

Let $$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \ldots(1)$$

be a differential equation of first order and of second degree. Let us take a particular point $(x_1, y_1)$ in the $xy$-plane, then putting $x = x_1$ and $y = y_1$ in (1), there will be two values of $\frac{dy}{dx}$ at this point $(x_1, y_1)$. Therefore, the moving point can pass through each point in two directions and hence two curves of the system (which is the locus of the general solution of (1)) pass through each point of the $xy$-plane.

Let the general solution of equation (1) be

$$\phi(x, y, c) = 0 \qquad ...(2)$$

$\therefore$ (2) must have, therefore, two different values of $c$ for each point and hence $c$ must occur in the solution (2) in the second degree.

In general, if $\quad f\left(x, y, \frac{dy}{dx}\right) = 0$

be a differential equation of first order and of degree $n$ and if its general solution is $\phi(x, y, c) = 0$, then the constant $c$ must occur in the $n$th degree.

**Example 1.** *Give geometrical significance of the solution of the differential equation*

$$\frac{dy}{dx} = -\frac{x}{y}.$$

**Sol.** The given equation is

$$\frac{dy}{dx} = -\frac{x}{y}. \qquad ...(1)$$

(1) $\Rightarrow$ $\qquad x\,dx + y\,dy = 0$

Integrating, we have

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1 \quad \text{or} \quad x^2 + y^2 = c \qquad ...(2)$$

(2) represents a family of concentric circles. (Putting $c = 2c_1$)

Let $(h, k)$ be any point in the $xy$-plane.

$\therefore$ The circle passing through the point $(h, k)$ is $x^2 + y^2 = h^2 + k^2$, which is a particular solution.

Thus the locus of the solution of the differential equation (1) is made of all circles, infinite in number, that have their centres at the origin.

$\therefore$ The geometrical meaning of the solution of equation (1) is that a point $(x, y)$ moving so as to satisfy the equation, moves in a direction perpendicular to the line joining the point $(x, y)$ to the origin, that is, the point moving under the condition (1) describes a circle about the origin as centre.

**Example 2.** *Give geometrical meaning of the solution of the differential equation*

$$\frac{d^2y}{dx^2} = 0.$$

**Sol.** The differential equation is

$$\frac{d^2y}{dx^2} = 0 \qquad ...(1)$$

Integrating (1), we have

$$\frac{dy}{dx} = m, \quad m \text{ being an arbitrary constant.}$$

Again integrating, we have $y = mx + c$ $\qquad ...(2)$

where $c$ is another arbitrary constant.

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(2) is the general solution of differential equation (1).

Now a line through any point (0, $c$) drawn in any direction $m$ is the locus of a particular integral of (1). If we give a particular value $c_1$ to $c$, there will be infinity of lines corresponding to the infinity of values that $m$ can have, and all these lines will be the loci of the integrals. Since to each of the infinity of values that can be given to $c$, there corresponds an infinity of lines (because of the infinity of values that can be given to $m$), therefore, the general solution of the equation represents a doubly infinite system of lines. In other words, the locus of the differential equation (1) consists of a doubly infinite system of lines.

## EXERCISE

*Give geometrical meaning of the solution of the following differential equations :*

1. $\dfrac{dy}{dx} = -\dfrac{y}{x}$
2. $\dfrac{dy}{dx} = m.$

### Answers

1. Infinite number of hyperbolas having coordinate axes as asymptotes.
2. Infinite number of straight lines each having slope $m$.

# UNIT 2 EXACT DIFFERENTIAL EQUATIONS

NOTES

STRUCTURE



## 2.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Define exact differential equations
- Find necessary and sufficient conditions for the equation M$dx$ + N$dy$ = 0 to be exact necessary condition
- Discuss inspection method of solving exact differential equation
- Describe general method of solving exact differential equation
- Find integrating factor

## 2.1. DEFINITION

The differential equation M$dx$ + N$dy$ = 0, where M and N are functions of $x$ and $y$, is called an **exact differential equation** if M$dx$ + N$dy$ is the differential of some function $u$ of $x$ and $y$ *i.e.* if, M$dx$ + N$dy$ = $du$. In such a case, the equation M$dx$ + N$dy$ = 0 reduces to $du = 0$ and so its general solution is $u = c$ *i.e.* $u(x, y) = c$.

**Examples of exact differential equations :**

(*i*) The equation $xdy + ydx = 0$ is exact for $xdy + ydx = d(xy)$. Its general solution is $xy = c$.

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(*ii*) The equation $\frac{1}{x}\,dy - \frac{y}{x^2}\,dx = 0$ is exact, for

$$\frac{1}{x}\,dy - \frac{y}{x^2}\,dx = \frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right).$$

Its general solution is $y/x = c$.

## 2.2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EQUATION Mdx + Ndy = 0 TO BE EXACT

**Necessary condition.** Let $\mathrm{M}dx + \mathrm{N}dy = 0$ be an exact differential equation.

$\therefore$ $\mathrm{M}dx + \mathrm{N}dy$ is the differential of some function, say, $u$ of $x$ and $y$.

$$\therefore \qquad \mathrm{M}dx + \mathrm{N}dy = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$\therefore \qquad \mathrm{M} = \frac{\partial u}{\partial x} \quad \text{and} \quad \mathrm{N} = \frac{\partial u}{\partial y}$$

$$\therefore \qquad \frac{\partial \mathrm{M}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial \mathrm{N}}{\partial x} \qquad \left(\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right)$$

$$\therefore \qquad \frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}.$$

**Sufficient condition.** Let $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$.

Let $v = \int^x \mathrm{M}\,dx$, where $\int^x$ denotes integration w.r.t. $x$ treating $y$ constant.

$$\therefore \qquad \mathrm{M} = \frac{\partial v}{\partial x}$$

Now $$\frac{\partial}{\partial x}\left(\mathrm{N} - \frac{\partial v}{\partial y}\right) = \frac{\partial \mathrm{N}}{\partial x} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \mathrm{N}}{\partial x} - \frac{\partial^2 v}{\partial y \partial x} \qquad \left(\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right)$$

$$= \frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{N}}{\partial x} = 0 \qquad \left(\because \frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}\right)$$

$\therefore$ $\mathrm{N} - \frac{\partial v}{\partial y}$ is a function of $y$ only.

Let $$u = v + \int \left(\mathrm{N} - \frac{\partial v}{\partial y}\right) dy$$

$$\therefore \qquad du = dv + \left(\mathrm{N} - \frac{\partial v}{\partial y}\right) dy = \left(\frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\right) + \left(\mathrm{N} - \frac{\partial v}{\partial y}\right) dy$$

$$= \mathrm{M}dx + \frac{\partial v}{\partial y}\,dy + \mathrm{N}dy - \frac{\partial v}{\partial y}\,dy = \mathrm{M}dx + \mathrm{N}dy$$

$\therefore$ $\mathrm{M}dx + \mathrm{N}dy$ is the differential of the function $u$.

$\therefore$ The equation $\mathrm{M}dx + \mathrm{N}dy = 0$ is exact.

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## 2.3. INSPECTION METHOD OF SOLVING AN EXACT DIFFERENTIAL EQUATION

At times an exact differential equation may be solved by inspection by regrouping the terms of the given equation, by recognizing a certain group of terms of an exact differential. For example :

(*i*) $x dy + y dx = d(xy)$ (*ii*) $x dx + y dy = d\left(\frac{x^2+y^2}{2}\right)$

(*iii*) $\cos y dx - x \sin y dy = d(x \cos y)$ etc.

**Example 1.** *Solve* $y dx + x dy + x^4 dx + \sin y dy = 0$.

**Sol.** We have

$$y\,dx + x\,dy + x^4\,dx + \sin y dy = 0 \qquad ...(1)$$

$\Rightarrow$ $(y + x^4)\,dx + (x + \sin y)\,dy = 0$

Here $M = y + x^4$ and $N = x + \sin y$

$\therefore$ $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The equation (1) is exact.

(1) $\Rightarrow$ $d(xy) + d\left(\frac{x^5}{5}\right) + d(-\cos y) = 0$ $\Rightarrow$ $d\left(xy + \frac{x^5}{5} - \cos y\right) = 0$

$\therefore$ The general solution is $\mathbf{xy + \frac{x^5}{5} - \cos y = c}$.

**Example 2.** *Solve* $(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)dy = 0$.

**Sol.** We have $(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)\,dy = 0$ ...(1)

Here $M = 4x^3y^3 - 2xy$ and $N = 3x^4y^2 - x^2$

$\therefore$ $\frac{\partial M}{\partial y} = 12x^3y^2 - 2x$ and $\frac{\partial N}{\partial x} = 12x^3y^2 - 2x$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\therefore$ The given equation is exact.

(1) $\Rightarrow$ $(4x^3y^3dx + 3x^4y^2dy) - (2xydx + x^2dy) = 0$

$\Rightarrow$ $d(x^4y^3) - d(x^2y) = 0$ $\Rightarrow$ $d(x^4y^3 - x^2y) = 0$

$\therefore$ The general solution is $\mathbf{x^4y^3 - x^2y = c}$.

**Example 3.** *Solve* $(\cos y + y \cos x)\,dx + (\sin x - x \sin y)\,dy = 0$.

**Sol.** We have $(\cos y + y \cos x)\,dx + (\sin x - x \sin y)\,dy = 0$ ...(1)

Here $M = \cos y + y \cos x$ and $N = \sin x - x \sin y$

$\therefore$ $\frac{\partial M}{\partial y} = -\sin y + \cos x$ and $\frac{\partial N}{\partial x} = \cos x - \sin y$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ The given equation is exact.

(1) $\Rightarrow$ $(\cos y\,dx - x\sin y\,dy) + (y\cos x\,dx + \sin x\,dy) = 0$

$\Rightarrow$ $d(x\cos y) + d(y\sin x) = 0 \;\Rightarrow\; d(x\cos y + y\sin x) = 0$

$\therefore$ The general solution is **$x\cos y + y\sin x = c$.**

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## EXERCISE 1

*Solve the following differential equations :*

1. $x\,dx + y\,dy + x^3dx + y^4dy = 0$
2. $x\,dy + y\,dx + \sin x\,dx = 0$
3. $\cos y\,dx - x\sin y\,dy + 5y^4dy = 0$
4. $e^y\,dx + (xe^y + 2y)dy = 0$
5. $e^{2y}dx + 2(xe^{2y} - y)dy = 0$
6. $2x(ye^{x^3} - 1)dx + e^{x^2}dy = 0$
7. $(6x^5y^3 + 4x^3y^5)dx + (3x^6y^2 + 5x^4y^4)\,dy = 0$
8. $(2x^3 + 3y)dx + (3x + y - 1)dy = 0.$

### Answers

1. $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{x^4}{4} + \dfrac{y^5}{5} = c$
2. $xy - \cos x = c$
3. $x\cos y + y^5 = c$
4. $xe^y + y^2 = c$
5. $xe^{2y} - y^2 = c$
6. $ye^{x^2} - x^2 = c$
7. $x^6y^3 + x^4y^5 = c$
8. $\dfrac{1}{2}x^4 + \dfrac{1}{2}y^2 - y + 3xy = c.$

## 2.4. GENERAL METHOD OF SOLVING AN EXACT DIFFERENTIAL EQUATION

Let $M\,dx + N\,dy = 0$ be an exact differential equation

$\therefore$ $M\,dx + N\,dy$ is the differential of some function, say, $u$ of $x$ and $y$

$$\therefore \quad M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\therefore \quad M = \frac{\partial u}{\partial x} \qquad \ldots(1)$$

and
$$N = \frac{\partial u}{\partial y} \qquad \ldots(2)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x} \qquad i.e., \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Also, (1) $\Rightarrow$ $u = \int^x M\,dx + \phi(y)$, where $\phi(y)$ is some function of $y$ only

(2) $\Rightarrow$ $$N = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int^x M\,dx\right) + \frac{d\phi}{dy}$$

$$\Rightarrow \quad \frac{d\phi}{dy} = N - \frac{\partial}{\partial y}\left(\int^x M\,dx\right) = N - \int^x \frac{\partial M}{\partial y}\,dx$$

$$= N - \int^x \frac{\partial N}{\partial x}\,dx \qquad \left(\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$$

$$\therefore \quad u = \int^x M\,dx + \int\left(N - \int^x \frac{\partial N}{\partial x}\,dx\right)dy$$

$$\Rightarrow \qquad u = \int^x M\,dx + \int \begin{pmatrix}\text{terms of N not}\\ \text{containing } x\end{pmatrix} dy$$

$\therefore$ The general solution of the given exact equation is $u = c$

*i.e.,* $\int^x \mathbf{M\,dx} + \int \begin{pmatrix}\textbf{terms of N not}\\ \textbf{containing x}\end{pmatrix} \mathbf{dy = c}$, **where c is an arbitrary constant.**

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**Remark.** If each term of N contain $x$, then the general solution reduces to $\int^x \mathbf{M\,dx = c}$.

**Example 1.** *Show that the equation $(1 + e^{x/y})\,dx + e^{x/y}\,(1 - x/y)\,dy = 0$ is exact, and hence solve it.*

**Sol.** We have $(1 + e^{x/y})dx + e^{x/y}\,(1 - x/y)dy = 0$.

Here $\qquad M = 1 + e^{x/y} \quad$ and $\quad N = e^{x/y}\,(1 - x/y)$

$$\therefore \qquad \frac{\partial M}{\partial y} = e^{x/y} \cdot \left(\frac{-x}{y^2}\right) = -\frac{x}{y^2}\, e^{x/y} \qquad \ldots(1)$$

and

$$\frac{\partial N}{\partial x} = e^{x/y}\,\frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{x/y}\left(-\frac{1}{y}\right) = -\frac{x}{y^2}\, e^{x/y} \qquad \ldots(2)$$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given equation is exact and its general solution is

$$\int^x M dx + \int \text{(terms of N not containing } x)dy = c.$$

$$\Rightarrow \qquad \int^x (1 + e^{x/y})dx = c$$

$$\Rightarrow \qquad \mathbf{x + e^{x/y}\, y = c}.$$ This is the required solution.

**Example 2.** *Solve $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$.*

**Sol.** We have $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$. $\qquad \ldots(1)$

Comparing it with $Mdx + Ndy = 0$,

we have $\qquad M = x^4 - 2xy^2 + y^4 \quad$ and $\quad N = -(2x^2y - 4xy^3 + \sin y)$.

$$\therefore \qquad \frac{\partial M}{\partial y} = -4xy + 4y^3 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{ The equation is exact.}$$

$\therefore$ The general solution is

$$\int^x M dx + \int \text{(terms of N not containing } x)dy = c.$$

$$\Rightarrow \qquad \int^x (x^4 - 2xy^2 + y^4)dx + \int (-\sin y)dy = c$$

$$\Rightarrow \qquad \mathbf{\frac{x^5}{5} - x^2y^2 + y^4x + \cos y = c.}$$

**Example 3.** *Solve $(ax + hy + g)dx + (hx + by + f)dy = 0$ and show that this differential equation represents a family of conics.*

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**Sol.** We have $(ax + hy + g)dx + (hx + by + f)dy = 0$.

Comparing it with $\quad M dx + N dy = 0$,

we get $\quad M = ax + hy + g \quad \text{and} \quad N = hx + by + f$

$$\therefore \quad \frac{\partial M}{\partial y} = h \quad \text{and} \quad \frac{\partial N}{\partial x} = h$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad \therefore \text{ The given equation is exact.}$$

$\therefore$ The general solution is

$$\int^x M dx + \int (\text{terms of N not containing } x) dy = c.$$

$$\int^x (ax + hy + g)dx + \int (by + f)dy = c_1$$

$$\Rightarrow \quad a\frac{x^2}{2} + hyx + gx + \frac{by^2}{2} + fy = c_1$$

$$\Rightarrow \quad ax^2 + by^2 + 2hxy + 2gx + 2fy - 2c_1 = 0$$

$$\Rightarrow \quad \mathbf{ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0}, \text{ where } c = -2c_1.$$

This being a second degree in $x$ and $y$ represents a family of conics.

**Example 4.** *Solve* $x dx + y dy = a^2 \left( \dfrac{x dy - y dx}{x^2 + y^2} \right)$.

**Sol.** We have $\quad x dx + y dy = \dfrac{a^2 x}{x^2 + y^2} dy - \dfrac{a^2 y dx}{x^2 + y^2}$

$$\Rightarrow \quad \left( x + \frac{a^2 y}{x^2 + y^2} \right) dx + \left( y - \frac{a^2 x}{x^2 + y^2} \right) dy = 0$$

Comparing it with $M dx + N dy = 0$, we have

$$M = x + \frac{a^2 y}{x^2 + y^2} \quad \text{and} \quad N = y - \frac{a^2 x}{x^2 + y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 + \left[ \frac{(x^2 + y^2)a^2 - a^2 y \cdot 2y}{(x^2 + y^2)^2} \right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial N}{\partial x} = 0 - \left[ \frac{(x^2 + y^2)a^2 - a^2 x \cdot 2x}{(x^2 + y^2)^2} \right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad \therefore \text{ The given equation is exact.}$$

$\therefore$ The general solution is

$$\int^x M dx + \int (\text{terms of N not containing } x) dy = c.$$

$$\Rightarrow \quad \int \left( x + \frac{a^2 y}{x^2 + y^2} \right) dx + \int y dy = c$$

$$\Rightarrow \quad \int x\,dx + a^2y \int \frac{1}{x^2+y^2}\,dx + \frac{y^2}{2} = c$$

$$\Rightarrow \quad \frac{x^2}{2} + a^2 \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c.$$

**Example 5.** *Solve* $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$.

**Sol.** We have $\quad (x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$

Comparing with $M\,dx + N\,dy = 0$, we get

$$M = x^3 + 3xy^2 \text{ and } N = 3x^2y + y^3$$

$$\therefore \quad \frac{\partial M}{\partial y} = 6xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{ The given equation is exact.}$$

$\therefore$ The general solution is $\int^x M\,dx + \int$ (terms of N not containing $x$) $dy = c$.

$$\Rightarrow \quad \int^x (x^3 + 3xy^2)\,dx + \int y^3\,dy = c$$

$$\Rightarrow \quad \frac{x^4}{4} + 3y^2 \cdot \frac{x^2}{2} + \frac{y^4}{4} = c$$

$$\Rightarrow \quad x^4 + 6x^2y^2 + y^4 = c_1. \qquad \text{(Putting } c_1 = 4c\text{)}$$

## EXERCISE 2

*Solve the following differential equations :*

1. $(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$
2. $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$
3. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$
4. $\cos x\,(\cos x - \sin\alpha\sin y)dx + \cos y\,(\cos y - \sin\alpha\sin x)dy = 0.$
5. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$
6. $x dx + y dy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$
7. $y\sin 2x dx - (y^2 + \cos^2 x)dy = 0$
8. $(x^2 + y^2 + e^x)dx + 2xy dy = 0$
9. $(x^2 - 2xy + 3y^2)dx + (4y^3 + 6xy - x^2)dy = 0$
10. $(x^2 - ay)dx = (ax - y^2)dy$
11. $\sec^2 x\tan y dx + \sec^2 y\tan x dy = 0$
12. $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$

### Answers

1. $a^2x - x^2y - xy^2 - \frac{y^3}{3} = c$
2. $x^2y + xy - x\tan y + \tan y = c$
3. $y(x + \log x) + x\cos y = c$
4. $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) - \sin\alpha\sin x\sin y + \frac{1}{2}\left(y + \frac{\sin 2y}{2}\right) = c$
5. $(e^y + 1)\sin x = c$
6. $x^2 + y^2 - 2\tan^{-1}\frac{x}{y} = c$
7. $y\cos^2 x + \frac{y^3}{3} + c = 0$
8. $\frac{x^3}{3} + xy^2 + e^x = c$
9. $\frac{x^3}{3} - x^2y + 3xy^2 + y^4 + c = 0$
10. $x^3 - 3axy + y^3 = c$
11. $\tan x\tan y = c$
12. $x^4 + 6x^2y^2 + y^4 = c.$

## 2.5. INTEGRATING FACTOR

NOTES

In the foregoing section, we have seen that the equation $Mdx + Ndy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence the equation can be solved. Evidently, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact and therefore cannot be solved. But sometimes, the equation becomes exact if it is multiplied by some function of $x$ and $y$. Such a function of $x$ and $y$ is called an *integrating factor*.

An **integrating factor** of a differential equation is a function of $x$ and $y$ which is such that if the differential equation is multiplied by it, the resulting equation becomes exact. An integrating factor is briefly written as I.F.

We give below some rules for finding integrating factors.

## 2.6. RULE I FOR FINDING INTEGRATING FACTOR

At times an integrating factor may be found by inspection, after regrouping the terms of the given equation, by recognizing a certain group of terms as being a part of an exact differential. For this purpose, the following list of exact differentials would be found very useful.

| S. No. | Group of terms | Integrating factor | Exact differential |
|---|---|---|---|
| 1. | $xdy - ydx$ | $\frac{1}{x^2}$ | $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$ |
| | | $\frac{1}{y^2}$ | $-\frac{ydx - xdy}{y^2} = d\left(-\frac{x}{y}\right)$ |
| | | $\frac{1}{xy}$ | $\frac{dy}{y} - \frac{dx}{x} = d\left(\log \frac{y}{x}\right)$ |
| | | $\frac{1}{x^2+y^2}$ | $\frac{xdy - ydx}{x^2+y^2} = \frac{\frac{xdy - ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ |
| 2. | $x\,dy + y\,dx$ | $\frac{1}{xy}$ | $\frac{dy}{y} + \frac{dx}{x} = d(\log xy)$ |
| | | $\frac{1}{(xy)^n}$, $n \neq 1$ | $\frac{xdy + ydx}{(xy)^n} = d\left(\frac{(xy)^{-n+1}}{-n+1}\right) = d\left(\frac{-1}{(n-1)(xy)^{n-1}}\right)$ |
| | | $\frac{1}{x^2+y^2}$ | $\frac{xdx + ydy}{x^2+y^2} = \frac{1}{2}\cdot\frac{d(x^2+y^2)}{x^2+y^2} = d\left(\frac{1}{2}\log(x^2+y^2)\right)$ |
| 3. | $xdx + ydy$ | $\frac{1}{(x^2+y^2)^n}$, $n \neq 1$ | $\frac{xdx + ydy}{(x^2+y^2)^n} = \frac{1}{2}(x^2+y^2)^{-n}\,d(x^2+y^2) = d\left(\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right)$ |

NOTES

**Example 1.** *Solve* $xdx + ydy + (x^2 + y^2)dy = 0.$

**Sol.** We have $xdx + ydy + (x^2 + y^2)dy = 0$

The last term suggests to have $\frac{1}{x^2 + y^2}$ as an I.F.

Dividing by $x^2 + y^2$, we get, $\frac{xdx + ydy}{x^2 + y^2} + dy = 0$

$$\Rightarrow \quad \frac{1}{x^2 + y^2} \cdot \frac{1}{2} d(x^2 + y^2) + dy = 0$$

$$\Rightarrow \quad \frac{1}{2} d(\log (x^2 + y^2)) + dy = 0$$

$$\Rightarrow \quad d\left(\frac{1}{2} \log (x^2 + y^2) + y\right) = 0.$$

$\therefore$ The general solution is $\mathbf{\frac{1}{2} \log (x^2 + y^2) + y = c.}$

**Example 2.** *Solve* $xdy - ydx - (1 - x^2)dx = 0.$

**Sol.** We have $xdy - ydx - (1 - x^2)\, dx = 0$ ...(1)

The last term suggests to have $\frac{1}{x^2}$ as an I.F.

Dividing (1) by $x^2$, we get $\frac{xdy - ydx}{x^2} - \left(\frac{1}{x^2} - 1\right)dx = 0$

$$\Rightarrow \quad d\left(\frac{y}{x}\right) + d\left(x + \frac{1}{x}\right) = 0$$

$$\Rightarrow \quad d\left(\frac{y}{x} + x + \frac{1}{x}\right) = 0$$

$\therefore$ The general solution is $\frac{y}{x} + x + \frac{1}{x} = c$ or $\mathbf{y + x^2 + 1 = cx.}$

**Example 3.** *Solve* $\frac{dy}{dx} = \frac{y - xy^2 - x^3}{x + x^2y + y^3}.$

**Sol.** We have $\frac{dy}{dx} = \frac{y - xy^2 - x^3}{x + x^2y + y^3}$

$$\Rightarrow \quad (x + x^2y + y^3)dy - (y - xy^2 - x^3)dx = 0$$

$$\Rightarrow \quad (x^3 + xy^2 - y)dx + (y^3 + x^2y + x)dy = 0$$

$$\Rightarrow \quad (x^3 + xy^2)\, dx + (y^3 + x^2y)\, dy + (xdy - ydx) = 0$$

$$\Rightarrow \quad (x^2 + y^2)(xdx + ydy) + (xdy - ydx) = 0$$

The first term suggests to have $\frac{1}{x^2 + y^2}$ as an I.F.

Dividing by $x^2 + y^2$, we get $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$

$$\Rightarrow \quad d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) + \frac{\frac{xdy - ydx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = 0$$

$$\Rightarrow \quad d\left(\frac{x^2 + y^2}{2}\right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) = 0$$

NOTES

$$\Rightarrow \quad d\left(\frac{x^2+y^2}{2}\right)+d\left(\tan^{-1}\frac{y}{x}\right)=0$$

$$\Rightarrow \quad d\left(\frac{x^2+y^2}{2}+\tan^{-1}\frac{y}{x}\right)=0.$$

$\therefore$ The general solution is $\dfrac{x^2+y^2}{2}+\tan^{-1}\dfrac{y}{x}=c$

or $\quad \mathbf{x^2+y^2+2\tan^{-1}\dfrac{y}{x}=c_1}.$ (Putting $c_1=2c$)

**EXERCISE 3**

*Solve the following differential equations :*

1. $ydx-xdy+(x^2+y^2)dx=0$
2. $xdx+ydy+4y^3(x^2+y^2)dy=0$
3. $(x+x^4+2x^2y^2+y^4)dx+ydy=0$
4. $(x^3+xy^2+a^2y)dx+(y^3+yx^2-a^2x)dy=0$
5. $ydx-xdy+(1+x^2)dx+x^2\sin y\,dy=0$
6. $x^2\dfrac{dy}{dx}+xy=\sqrt{1-x^2y^2}$
7. $xdx+ydy=m(xdy-ydx)$
8. $(1+xy)ydx-x(1-xy)dy=0.$

**Answers**

1. $\tan^{-1}\dfrac{y}{x}=x+c$
2. $\dfrac{1}{2}\log(x^2+y^2)+y^4=c$
3. $(x^2+y^2)(2x+c)=1$
4. $x^2+y^2+2a^2\tan^{-1}\dfrac{x}{y}=c$
5. $x^2-y-1-x\cos y=cx$
6. $\sin^{-1}(xy)-\log x=c$
7. $\log(x^2+y^2)-2m\tan^{-1}\dfrac{y}{x}=c$
8. $\log\dfrac{x}{y}-\dfrac{1}{xy}=c.$

## 2.7. RULE II FOR FINDING INTEGRATING FACTOR

**Theorem.** *If in the equation $Mdx+Ndy=0$, M and N are homogeneous functions of the same degree in x and y, i.e., if the equation $Mdx+Ndy=0$ is homogeneous then $\dfrac{1}{Mx+Ny}$ is an integrating factor, provided $Mx+Ny\neq 0$.*

**Proof.** The given differential equation is $Mdx+Ndy=0$. ...(1)

M, N are homogeneous functions of the same degree in $x$ and $y$.

$$Mdx+Ndy=\frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$=\frac{1}{2}\left[(Mx+Ny)\,d(\log xy)+(Mx-Ny)\,d\left(\log\frac{x}{y}\right)\right]$$

Dividing by $Mx+Ny$, we have

$$\frac{Mdx+Ndy}{Mx+Ny}=\frac{1}{2}\left[d(\log xy)+\frac{Mx-Ny}{Mx+Ny}\,d\left(\log\frac{x}{y}\right)\right]$$

NOTES

Since M and N are homogeneous functions of the same degree in $x$ and $y$, $\frac{Mx - Ny}{Mx + Ny}$ is homogeneous and is a function of $\frac{x}{y}$. Let this be denoted by $f\left(\frac{x}{y}\right)$.

$$\therefore \quad \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\, d(\log xy) + \frac{1}{2} f\left(\frac{x}{y}\right) d\left(\log \frac{x}{y}\right).$$

Since $\frac{x}{y} = e^{\log x/y}$, we have

$$f\left(\frac{x}{y}\right) = f\,[e^{\log x/y}] = \phi\left(\log \frac{x}{y}\right), \quad \text{say}$$

$$\therefore \quad \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\, d(\log xy) + \frac{1}{2}\, \phi\left(\log \frac{x}{y}\right) d\left(\log \frac{x}{y}\right),$$

which is an exact differential.

$$\therefore \quad \frac{Mdx + Ndy}{Mx + Ny} = 0 \text{ is an exact differential equation.}$$

$$\therefore \quad \frac{M}{Mx + Ny}\, dx + \frac{N}{Mx + Ny}\, dy = 0$$

is an exact differential equation.

$$\therefore \quad \frac{1}{Mx + Ny} \text{ is an integrating factor of } Mdx + Ndy = 0.$$

**Remark.** If $Mx + Ny = 0$, then $\frac{M}{N} = -\frac{y}{x}$.

$\therefore$ The given differential equation reduces to $ydx - xdy = 0$. For this equation, $\frac{1}{xy}$ is an integrating factor.

**Example 1.** *Solve* $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

**Sol.** We have $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$. ...(1)

Here $M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

$$\therefore \quad \frac{\partial M}{\partial y} = x^2 - 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus equation (1) is not exact.

The equation (1) is homogeneous.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying (1) by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \qquad \text{...(2)}$$

This equation is exact.

$\therefore$ The general solution is

$$\int^x \left(\frac{1}{y} - \frac{2}{x}\right) dx - \int -\frac{3}{y}\, dy = c$$

or $$\frac{x}{y} - 2\log x + 3\log y = c$$

NOTES

or $$\frac{x}{y} + \log\frac{y^3}{x^2} = c.$$

## EXERCISE 4

*Solve the following differential equations :*

1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$
2. $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$
3. $(x^4 + y^4)dx - xy^3dy = 0$
4. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$
5. $(x^2 + y^2)\,dx - 2xy\,dy = 0.$

Answers

1. $-\dfrac{x^3}{3y^3} + \log y = c$
2. $\log\dfrac{x^3}{y^2} + \dfrac{y}{x} = c$
3. $y^4 = 4x^4\log x + cx^4$
4. $x^2y^2(y^2 - x^2) = c^2$
5. $x^2 - y^2 = cx.$

## 2.8. RULE III FOR FINDING INTEGRATING FACTOR

**Theorem.** *If the equation $Mdx + Ndy = 0$ is of the form $f(xy)\,ydx + g(xy)\,xdy = 0$, then*

$\dfrac{1}{Mx - Ny}$ *is an integrating factor provided* $Mx - Ny \neq 0$.

**Proof.** The given differential equation is

$$Mdx + Ndy = 0 \qquad \ldots(1)$$

where $M = f(xy)y$ and $N = g(xy)x$.

Now $$Mdx + Ndy = \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[(Mx + Ny)\,d(\log xy) + (Mx - Ny)\,d\left(\log\frac{x}{y}\right)\right]$$

Dividing by $Mx - Ny$, we have

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left[\frac{Mx + Ny}{Mx - Ny}\,d(\log xy) + d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f(xy)yx + g(xy)xy}{f(xy)yx - g(xy)xy}\,d(\log xy) + d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[F(xy)\,d(\log xy) + d\left(\log\frac{x}{y}\right)\right], \quad \text{say}$$

Since $xy = e^{\log xy}$, we have

$$F(xy) = F(e^{\log xy}) = \psi(\log xy), \quad \text{say}$$

$$\therefore \quad \frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left[\psi(\log xy)\, d\,(\log xy) + d\left(\log \frac{x}{y}\right)\right],$$

which is an exact differential.

NOTES

$$\therefore \quad \frac{Mdx + Ndy}{Mx - Ny} = 0 \text{ is an exact differential equation.}$$

$$\therefore \quad \frac{M}{Mx - Ny}\, dx + \frac{N}{Mx - Ny}\, dy = 0 \text{ is an exact differential equation.}$$

$$\therefore \quad \frac{1}{Mx - Ny} \textbf{ is an integrating factor of } Mdx + Ndy = 0.$$

**Example 1.** *Solve* $(xy^2 + 2x^2y^3)\, dx + (x^2y - x^3y^2)\, dy = 0.$

**Sol.** We have $(xy^2 + 2x^2y^3)\, dx + (x^2y - x^3y^2)\, dy = 0.$ ...(1)

Here $M = xy^2 + 2x^2y^3$ and $N = x^2y - x^3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2.$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \quad \therefore \text{ The equation (1) is not exact.}$$

(1) $\Rightarrow$ $(xy + 2(xy)^2)\, y\, dx + (xy - (xy)^2)x\, dy = 0.$

This equation is of the form :

$$f(xy)\, y\, dx + g(xy)\, x\, dy = 0.$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$$

$\therefore$ Multiplying (1) by $\dfrac{1}{3x^3y^3}$, we get

$$\frac{xy^2 + 2x^2y^3}{3x^3y^3}\, dx + \frac{x^2y - x^3y^2}{3x^3y^3}\, dy = 0$$

$$\Rightarrow \quad \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right) dy = 0$$

This equation is exact.

$\therefore$ The general solution is

$$\int^x \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y}\, dy = c$$

or
$$-\frac{1}{-3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or
$$-\frac{1}{xy} + 2\log x - \log y = 3c \quad \text{or} \quad -\frac{1}{xy} + \log\frac{x^2}{y} = k, \text{ where } k = 3c.$$

## EXERCISE 5

*Solve the following differential equations :*

1. $(1 + xy)\, ydx + (1 - xy)\, xdy = 0$
2. $(x^2y^2 + xy + 1)\, ydx + (x^2y^2 - xy + 1)\, xdy = 0$
3. $(x^3y^3 + x^2y^2 + xy + 1)\, ydx + (x^3y^3 - x^2y^2 - xy + 1)\, xdy = 0$
4. $(x^4y^4 + x^2y^2 + xy)\, ydx + (x^4y^4 - x^2y^2 + xy)\, xdy = 0$
5. $(xy \sin xy + \cos xy)\, ydx + (xy \sin xy - \cos xy)\, xdy = 0.$

NOTES

**Answers**

1. $-\frac{1}{xy} + \log\frac{x}{y} = c$
2. $xy - \frac{1}{xy} + \log\frac{x}{y} = c$
3. $x^2y^2 - 1 - 2xy\log y = 2cxy$
4. $\frac{x^2y^2}{2} - \frac{1}{xy} + \log\frac{x}{y} = c$
5. $x \sec xy = cy.$

## 2.9. RULE IV FOR FINDING INTEGRATING FACTOR

**Theorem.** *In the equation $Mdx + Ndy = 0$ if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say, f(x) then $e^{\int f(x)\,dx}$ is an integrating factor.*

**Proof.** The given differential equation is $Mdx + Ndy = 0$ ...(1)

and
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \qquad \text{...(2)}$$

Multiplying (1) by $e^{\int f(x)\,dx}$, we have

$$M(e^{\int f(x)\,dx})\,dx + N\,(e^{\int f(x)\,dx})\,dy = 0 \qquad \text{...(3)}$$

Now (3) will be exact if

$$\frac{\partial}{\partial y}\,[Me^{\int f(x)\,dx}] = \frac{\partial}{\partial x}\,[Ne^{\int f(x)\,dx}]$$

or if $\qquad e^{\int f(x)\,dx}\,\frac{\partial M}{\partial y} = e^{\int f(x)\,dx}\,\frac{\partial N}{\partial x} + Ne^{\int f(x)\,dx}\,f(x) \qquad \left(\because\ \frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)\right)$

or if $\qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + Nf(x)$

or if $\qquad \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, which is true by (2).

$\therefore$ **$e^{\int f(x)\,dx}$ is an integrating factor of $Mdx + Ndy = 0$.**

**Example 1.** *Solve $(x^2 + y^2 + 2x)\,dx + 2ydy = 0$.*

**Sol.** We have $(x^2 + y^2 + 2x)\,dx + 2ydy = 0$. ...(1)

Here $\qquad M = x^2 + y^2 + 2x \quad$ and $\quad N = 2y$

$\therefore \qquad \frac{\partial M}{\partial y} = 2y \quad$ and $\quad \frac{\partial N}{\partial x} = 0$

$\therefore \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \qquad \therefore$ The given equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 = x^0, \text{ a function of } x \text{ only}$$

$$= f(x) \text{ (say)}$$

$\therefore$ $$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^x.$$

$\therefore$ Multiplying (1) by $e^x$, we get

$$(x^2e^x + y^2e^x + 2xe^x)\,dx + 2ye^x\,dy = 0 \qquad ...(2)$$

This equation is exact

$\therefore$ The general solution is

$$\int^x (x^2e^x + y^2e^x + 2xe^x)\,dx = c$$

or $$\int (x^2 + 2x)e^x\,dx + y^2 \int e^x dx = c \quad \text{or} \quad x^2e^x + y^2e^x = c$$

$$\left[\because \int (x^2 + 2x)\,e^x\,dx = \int [\phi(x) + \phi'(x)]\,e^x\,dx = \phi(x)e^x, \text{where } \phi(x) = x^2\right]$$

$\Rightarrow$ $$\mathbf{(x^2 + y^2)\,e^x = c.}$$

## EXERCISE 6

*Solve the following differential equations :*

1. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$
2. $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$
3. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$
4. $(x^3 - 2y^2)dx + 2xy\,dy = 0$
5. $(x^2 + y^2 + 1)\,dx + x(x - 2y)\,dy = 0$
6. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0.$

### Answers

1. $3x^4y + x^4y^3 + x^6 = c$
2. $-\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$
3. $3x^4 + 6x^2y^2 + 4x^3 = c$
4. $x + \frac{y^2}{x^2} = c$
5. $x + y - \frac{1 + y^2}{x} = c$
6. $x^2 - y^2 = cx + 1.$

## 2.10. RULE V FOR FINDING INTEGRATING FACTOR

**Theorem.** *In the equation $Mdx + Ndy = 0$ if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say, f(y) then $e^{\int f(y)\,dy}$ is an integrating factor.*

**Proof.** The given differential equation is $Mdx + Ndy = 0$ ...(1)

and $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \qquad ...(2)$$

Multiplying (1) by $e^{\int f(y)\,dy}$, we have

$$M(e^{\int f(y)\,dy})\,dx + N(e^{\int f(y)\,dy})\,dy = 0 \qquad \text{...(3)}$$

Now (3) will be exact if

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$$\frac{\partial}{\partial y}[M\,e^{\int f(y)\,dy}] = \frac{\partial}{\partial x}[N\,e^{\int f(y)\,dy}]$$

or if $\quad e^{\int f(y)\,dy}\cdot\dfrac{\partial M}{\partial y} + M\,e^{\int f(y)\,dy}\cdot f(y) = e^{\int f(y)\,dy}\,\dfrac{\partial N}{\partial x} \qquad \left(\because\ \dfrac{d}{dy}\displaystyle\int f(y)\,dy = f(y)\right)$

or if $$\frac{\partial M}{\partial y} + M\,f(y) = \frac{\partial N}{\partial x}$$

or if $$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y), \quad \text{which is true by (2).}$$

$\therefore$ **$e^{\int f(y)\,dy}$ is an integrating factor of Mdx + Ndy = 0.**

**Note.** Better form of Rule V is the following :

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M}$ is free from $x$ and $f(y)$ say, then I.F. $= e^{\int -f(y)\,dy}$.

**Example 1.** *Solve* $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$.

**Sol.** We have

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0. \qquad \text{...(1)}$$

Here $\quad M = y^4 + 2y \quad$ and $\quad N = xy^3 + 2y^4 - 4x.$

$\therefore \quad \dfrac{\partial M}{\partial y} = 4y^3 + 2 \quad$ and $\quad \dfrac{\partial N}{\partial x} = y^3 - 4$

$\therefore \quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}. \qquad \therefore$ The given equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = \frac{4y^3 + 2 - y^3 + 4}{y^4 + 2y} = \frac{3(y^3+2)}{y(y^3+2)} = \frac{3}{y},$$

a function of $y$ only $= f(y)$ (say)

$\therefore \quad$ I.F. $= e^{\int -f(y)\,dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \dfrac{1}{y^3}$.

$\therefore$ Multiplying (1) by $1/y^3$, we get

$$\frac{y^4 + 2y}{y^3}\,dx + \frac{xy^3 + 2y^4 - 4x}{y^3}\,dy = 0$$

$$\Rightarrow \qquad \left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0 \qquad \text{...(2)}$$

This equation is exact.

$\therefore$ The general solution is

$$\int^x \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c \quad \text{or} \quad \left(\mathbf{y} + \frac{\mathbf{2}}{\mathbf{y^2}}\right)\mathbf{x} + \mathbf{y^2} = \mathbf{c}.$$

## EXERCISE 7

*Solve the following differential equations :*

1. $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$ 2. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$
3. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

4. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$
5. $3x^2y^2\,dx + 4(x^3y - 3)\,dy = 0.$

**Answers**

1. $e^{6y}(54\,x^2y^2 - 36\,x^3 + 18\,y^2 - 6y + 1) = c$ 2. $x^3y^3 + x^2 = cy$
3. $3x^2y^4 + 6xy^2 + 2y^6 = c$ 4. $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$
5. $x^3y^4 - 4y^3 = c.$

## 2.11. RULE VI FOR FINDING INTEGRATING FACTOR

**Theorem.** *If the equation* $Mdx + Ndy = 0$ *can be expressed as* $x^ay^b\,(mydx + nxdy) + x^cy^d\,(pydx + qxdy) = 0$, *where* $a, b, c, d, m, n, p, q$ *are constants and* $\dfrac{m}{n} \neq \dfrac{p}{q}$, *then* $x^\alpha y^\beta$ *is an integrating factor, where* $\alpha$ *and* $\beta$ *are so chosen that*

$$\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n} \quad \text{and} \quad \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}.$$

**Proof.** The given differential equation is

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0. \qquad \ldots(1)$$

Let $x^\alpha y^\beta$ be an integrating factor of (1).

$\therefore$ Multiplying (1) by $x^\alpha y^\beta$, we get $(mx^{\alpha+a}y^{\beta+b+1}\,dx + nx^{\alpha+a+1}y^{\beta+b}\,dy)$
$+ (px^{\alpha+c}y^{\beta+d+1}\,dx + qx^{\alpha+c+1}y^{\beta+d}\,dy) = 0$

This equation is exact.

$\therefore$ Both parts of the L.H.S. of the equation are exact differentials.

$\therefore$ Using $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, we have

$$\frac{\partial}{\partial y}(mx^{\alpha+a}y^{\beta+b+1}) = \frac{\partial}{\partial x}(nx^{\alpha+a+1}y^{\beta+b})$$

and
$$\frac{\partial}{\partial y}(px^{\alpha+c}y^{\beta+d+1}) = \frac{\partial}{\partial x}(qx^{\alpha+c+1}y^{\beta+d})$$

$\Rightarrow \quad m(\beta + b + 1)\,x^{\alpha+a}y^{\beta+b} = n(\alpha + a + 1)\,x^{\alpha+a}y^{\beta+b}$

and $\quad p(\beta + d + 1)\,x^{\alpha+c}y^{\beta+d} = q(\alpha + c + 1)\,x^{\alpha+c}y^{\beta+d}$

$\Rightarrow \quad m(\beta + b + 1) = n(\alpha + a + 1)$ and $p(\beta + d + 1) = q(\alpha + c + 1)$

$\Rightarrow \quad \dfrac{\mathbf{a+\alpha+1}}{\mathbf{m}} = \dfrac{\mathbf{b+\beta+1}}{\mathbf{n}}$ and $\dfrac{\mathbf{c+\alpha+1}}{\mathbf{p}} = \dfrac{\mathbf{d+\beta+1}}{\mathbf{q}}$.

Solving these equations, we get the values of $\alpha$ and $\beta$ and hence of the integrating factor $x^\alpha y^\beta$ is known.

**Example 1.** *Solve* $(2x^2y^2 + y)\,dx = (x^3y - 3x)\,dy$.

**Sol.** We have $\quad (2x^2y^2 + y)\,dx + (-x^3y + 3x)\,dy = 0. \qquad \ldots(1)$

Here $\quad M = 2x^2y^2 + y \quad$ and $\quad N = -x^3y + 3x.$

$\therefore \quad \dfrac{\partial M}{\partial y} = 4x^2y + 1 \quad$ and $\quad \dfrac{\partial N}{\partial x} = 3 - 3x^2y$

Now $\quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}. \quad \therefore \quad$ The given equation is not exact.

**NOTES**

We write (1) in the form

$$x^2y(2ydx - xdy) + x^0y^0(1ydx + 3xdy) = 0 \qquad ...(2)$$

Comparing it with

$$x^ay^b(mydx + nxdy) + x^cy^d\,(pydx + qxdy) = 0, \text{ we have}$$

$$a = 2,\ b = 1,\ m = 2,\ n = -1,\ c = 0,\ d = 0,\ p = 1,\ q = 3$$

Also $\quad \dfrac{m}{n} = \dfrac{2}{-1} = -2 \quad$ and $\quad \dfrac{p}{q} = \dfrac{1}{3} \quad \therefore \quad \dfrac{m}{n} \neq \dfrac{p}{q}$.

Let I.F. $= x^\alpha y^\beta$

$\therefore \quad \dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n} \quad$ and $\quad \dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q}$.

$\Rightarrow \quad \dfrac{2+\alpha+1}{2} = \dfrac{1+\beta+1}{-1} \quad$ and $\quad \dfrac{0+\alpha+1}{1} = \dfrac{0+\beta+1}{3}$

$\Rightarrow \quad -\alpha - 3 = 4 + 2\beta \quad$ and $\quad 3\alpha + 3 = \beta + 1$

$\Rightarrow \quad \alpha + 2\beta + 7 = 0 \quad$ and $\quad 3\alpha - \beta + 2 = 0.$

Solving these equations, we have

$$\alpha = -11/7 \quad \text{and} \quad \beta = -19/7.$$

$\therefore \quad$ I.F. $= x^{-11/7}\,y^{-19/7}$ .

Multiplying (1) by $x^{-11/7}\,y^{-19/7}$, the equation (1) becomes

$$(2x^{3/7}y^{-5/7} + x^{-11/7}\,y^{-12/7})\,dx - (x^{10/7}\,y^{-12/7} - 3x^{-4/7}\,y^{-19/7})dy = 0 \qquad ...(3)$$

This equation is exact.

$\therefore$ The general solution is

$$\int^x (2x^{3/7}\,y^{-5/7} + x^{-11/7}\,y^{-12/7})\,dx = c$$

or $\quad 2y^{-5/7}\,\dfrac{x^{10/7}}{10/7} + y^{-12/7}\,\dfrac{x^{-4/7}}{-4/7} = c$

or $\quad \dfrac{7}{5}\,x^{10/7}\,y^{-5/7} - \dfrac{7}{4}\,x^{-4/7}\;y^{-12/7} = c.$

**Alternative method of finding I.F.**

The given equation is

$$x^2y(2y\,dx - x\,dy) + x^0y^0(1.ydx + 3xdy) = 0 \qquad ...(1)$$

Let $x^\alpha\,y^\beta$ be an integrating factor of (1).

Multiplying (1) by $x^\alpha y^\beta$, we get

$$x^{\alpha+2}\,y^{\beta+1}(2ydx - xdy) + x^\alpha\,y^\beta\,(ydx + 3x\,dy) = 0$$

$\Rightarrow \quad (2x^{\alpha+2}\,y^{\beta+2}\,dx - x^{\alpha+3}\,y^{\beta+1}\,dy) + (x^\alpha y^{\beta+1}\,dx + 3x^{\alpha+1}\,y^\beta\,dy) = 0$

This equation is exact.

$\therefore$ Both parts of the L.H.S. of the equation are exact differentials.

$\therefore$ Using $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, we have

$$\frac{\partial}{\partial y}(2x^{\alpha+2}\,y^{\beta+2}) = \frac{\partial}{\partial x}(-\,x^{\alpha+3}\,y^{\beta+1}) \quad \text{and} \quad \frac{\partial}{\partial y}(x^\alpha\,y^{\beta+1}) = \frac{\partial}{\partial x}(3x^{\alpha+1}\,y^\beta)$$

$\Rightarrow \quad 2(\beta+2)x^{\alpha+2}\,y^{\beta+1} = -\,(\alpha+3)x^{\alpha+2}\,y^{\beta+1} \quad$ and $\quad (\beta+1)x^\alpha\,y^\beta = 3(\alpha+1)x^\alpha y^\beta$

$\Rightarrow \quad 2(\beta+2) = -\,(\alpha+3) \quad$ and $\quad \beta+1 = 3(\alpha+1)$

$\Rightarrow \quad \alpha + 2\beta + 7 = 0 \quad$ and $\quad 3\alpha - \beta + 2 = 0$

Solving these equations, we get

$$\alpha = -11/7 \quad \text{and} \quad \beta = -19/7.$$

$$\therefore \qquad \text{I.F} = x^{-11/7}\, y^{-19/7}.$$

## EXERCISE 8

*Solve the following differential equations :*

**1.** $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

**2.** $(2ydx + 3xdy) + 2xy\,(3ydx + 4xdy) = 0$

**3.** $(3xy + 2y^3)dx + (4x^2 + 6xy^2)dy = 0$

**4.** $x(4ydx + 2xdy) + y^3\,(3ydx + 5xdy) = 0$

**5.** $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0.$

### Answers

**1.** $-x^{-3/2}\, y^{3/2} + 6x^{1/2}y^{1/2} = c$

**2.** $x^2y^3 + 2x^3y^4 = c$

**3.** $x^3y^4 + x^2y^6 = c$

**4.** $x^4y^3 + x^3y^5 = c$

**5.** $5x^{-36/13}\, y^{24/13} - 12x^{-10/13}\, y^{-15/13} = c.$

NOTES

# UNIT 3 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

**STRUCTURE**



## 3.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Discuss equations solvable for $p$, $y$ and $x$
- Describe clairaut's equation

## 3.1. INTRODUCTION

So far, we have discussed differential equations of the first order and first degree. Now we shall study differential equations of the first order and degree higher than the first. For convenience, we denote $\frac{dy}{dx}$ by $p$.

A differential equation of the first order and $n$th degree is of the form

$$p^n + P_1(x, y)p^{n-1} + P_2(x, y)p^{n-2} + \ldots\ldots + P_n(x, y) = 0 \qquad \ldots(1)$$

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In the various cases which follow, the problem is reduced to that of solving one or more equations of the first order and first degree.

## 3.2. EQUATIONS SOLVABLE FOR p

Resolving the left hand side of (1) into $n$ linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)]\ \ldots\ldots\ [p - f_n(x, y)] = 0.$$

$\Rightarrow \quad p - f_1(x, y) = 0, \quad p - f_2(x, y) = 0, \ldots\ldots \text{ and } p - f_n(x, y) = 0.$

Each of these equations is of the first order and first degree and can be solved by the methods already discussed.

If the solutions of the above $n$ component equations are

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots\ldots \text{ and } F_n(x, y, c) = 0$$

then the general solution of (1) is given by

$$\mathbf{F_1(x, y, c), F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0.}$$

**Example 1.** *Solve* $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0.$

**Sol.** The given equation is $x^2p^2 + xyp - 6y^2 = 0$, where $p = \frac{dy}{dx}$.

$$\Rightarrow \quad (xp + 3y)(xp - 2y) = 0$$

The component equations are

$$\Rightarrow \quad xp + 3y = 0 \quad \ldots(1) \quad \text{and} \quad xp - 2y = 0 \quad \ldots(2)$$

$$(1) \Rightarrow \quad x\frac{dy}{dx} + 3y = 0 \quad \Rightarrow \quad \frac{dy}{y} + 3\frac{dx}{x} = 0$$

Integrating, $\log y + 3 \log x = \log c$ or $x^3y - c = 0.$

$$(2) \Rightarrow \quad x\frac{dy}{dx} - 2y = 0 \quad \Rightarrow \quad \frac{dy}{y} - 2\frac{dx}{x} = 0$$

Integrating, $\log y - 2 \log x = \log c$ or $\frac{y}{x^2} = c$ or $y - cx^2 = 0.$

$\therefore$ The general solution of the given equation is

$$\mathbf{(x^3y - c)(y - cx^2) = 0.}$$

**Example 2.** *Solve* $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0.$

**Sol.** Solving the given equation for $p$, we have

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{(2y^2 - 3x^2) \pm (3x^2 + 2y^2)}{2xy} = \frac{2y}{x}, \quad -\frac{3x}{y}$$

Now $\quad p = \frac{2y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2y}{x} \quad$ or $\quad \frac{dy}{y} - \frac{2dx}{x} = 0$

Integrating, $\log y - 2 \log x = \log c$ or $\frac{y}{x^2} = c$ or $y - cx^2 = 0.$

Also $\quad p = -\frac{3x}{y} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{3x}{y} \quad$ or $\quad ydy + 3xdx = 0$

Integrating, $\frac{y^2}{2} + \frac{3x^2}{2} = k$ or $y^2 + 3x^2 = c.$

(Putting $c = 2k$)

$\therefore$ The general solution of the given equation is

$$\mathbf{(y - cx^2)(y^2 + 3x^2 - c) = 0.}$$

**Example 3.** *Solve* $p^2 + 2py \cot x = y^2.$

**Sol.** The given equation can be written as

$$(p + y\cot x)^2 = y^2(1 + \cot^2 x). \qquad \text{(Adding } y^2\cot^2 x \text{ to both sides)}$$

or $$p + y\cot x = \pm y \operatorname{cosec} x.$$

$\therefore$ The component equations are

$$p = y(-\cot x + \operatorname{cosec} x) \quad \ldots(1)$$

**NOTES**

and $$p = y(-\cot x - \operatorname{cosec} x) \quad \ldots(2)$$

(1) $\Rightarrow$ $$p = y\left(\frac{1-\cos x}{\sin x}\right) \Rightarrow \frac{dy}{dx} = y\tan\frac{1}{2}x$$

$\Rightarrow$ $$\frac{dy}{y} = \tan\frac{1}{2}x\,dx$$

Integrating, $$\log y = 2\log\sec\frac{1}{2}x + \log c$$

$\Rightarrow$ $$y = c\sec^2\frac{1}{2}x \quad \Rightarrow \quad y\cos^2\frac{1}{2}x - c = 0.$$

(2) $\Rightarrow$ $$p = -y\left(\frac{1+\cos x}{\sin x}\right) \Rightarrow \frac{dy}{dx} = -y\cot\frac{1}{2}x$$

$\Rightarrow$ $$\frac{dy}{y} = -\cot\frac{1}{2}x\,dx$$

Integrating. $$\log y = -2\log\sin\frac{1}{2}x + c$$

$\Rightarrow$ $$y = \frac{c}{\sin^2\frac{1}{2}x} \quad \Rightarrow \quad y\sin^2\frac{1}{2}x - c = 0.$$

$\therefore$ The general solution of the given equation is

$$\left(\mathbf{y\cos^2\frac{x}{2} - c}\right)\left(\mathbf{y\sin^2\frac{x}{2} - c}\right) = \mathbf{0}.$$

**Example 4.** *Solve* $x^2\left(\dfrac{dy}{dx}\right)^2 - 2xy\dfrac{dy}{dx} + 2y^2 - x^2 = 0.$

**Sol.** Solving the given equation for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{xy \pm \sqrt{x^2y^2 - 2x^2y^2 + x^4}}{x^2}$$

$$= \frac{xy \pm x\sqrt{x^2 - y^2}}{x^2} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x} \quad \ldots(1)$$

This is a homogeneous equation.

Let $$y = vx. \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) $\Rightarrow$ $$v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x}$$

NOTES

$$\Rightarrow \qquad x\frac{dv}{dx} = \pm\sqrt{1-v^2} = \frac{dv}{\sqrt{1-v^2}} = \pm\frac{dx}{x}$$

Integrating, we get $\sin^{-1} v = \pm \log x + c$

$\Rightarrow \qquad \sin^{-1}(y/x) = \log x + c, \quad \sin^{-1}(y/x) = -\log x + c$

$\therefore$ The general solution of the given equation is

$$\mathbf{(\sin^{-1} y/x - \log x - c)(\sin^{-1} y/x + \log x - c) = 0.}$$

**Example 5.** *Solve* $p^3 - p(x^2 + xy + y^2) + xy(x + y) = 0$.

**Sol.** We have $p^3 - p(x^2 + xy + y^2) + xy(x + y) = 0$ ...(1)

$p = x$ is a root of (1) if $x^3 - x(x^2 + xy + y^2) + xy(x + y) = 0$

or if $0 = 0$, which is true.

$\therefore$ $p = x$ is a root of (1).

Dividing L.H.S. of (1) by $p - x$, we get

$$(p - x)(p^2 + px - xy - y^2) = 0 \qquad ...(2)$$

$$p^2 + px - xy - y^2 = 0 \quad \Rightarrow \quad p = \frac{-x \pm \sqrt{x^2 + 4xy + 4y^2}}{2}$$

$$\therefore \qquad p = \frac{-x \pm (x + 2y)}{2} = y, -x - y$$

$\therefore$ (2) $\Rightarrow$

$$p = x \qquad ...(3)$$

$$p = y \qquad ...(4)$$

$$p = -x - y \qquad ...(5)$$

$$(3) \Rightarrow \frac{dy}{dx} = x \quad \Rightarrow \quad y = \frac{x^2}{2} + c \quad \Rightarrow \quad x^2 - 2y + 2c = 0$$

$$(4) \Rightarrow \frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = dx \quad \Rightarrow \quad \log y = x + c$$

$$\Rightarrow \quad \log y - x - c = 0$$

$$(5) \Rightarrow \frac{dy}{dx} = -(x + y) \qquad ...(6)$$

Let $z = x + y$ $\therefore$ $\dfrac{dz}{dx} = 1 + \dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

$\therefore$ (6) $\Rightarrow$ $\dfrac{dz}{dx} - 1 = -z$ or $\dfrac{dz}{1-z} = dx$

$\Rightarrow \qquad -\log(1 - z) = x + c \quad \Rightarrow \quad x + \log(1 - x - y) + c = 0.$

$\therefore$ The general solution of (1) is

$$\mathbf{(x^2 - 2y + 2c)(\log y - x - c)(x + \log(1 - x - y) + c) = 0.}$$

**Example 6.** *Solve* $p^4 - (x + 1 + 2y)p^3 + (x + 2y + 2xy)p^2 - 2xy\,p = 0$.

**Sol.** We have

$$p^4 - (x + 1 + 2y)p^3 + (x + 2y + 2xy)p^2 - 2xy\,p = 0 \qquad ...(1)$$

0 and 1 are roots of (1).

NOTES

Dividing L.H.S. of (1) by $p(p-1)$, we get

$$p(p-1)(p^2-(x+2y)p+2xy)=0 \qquad ...(2)$$

$$\Rightarrow \qquad p(p-1)(p-x)(p-2y)=0$$

$$\Rightarrow \qquad p=0 \qquad ...(3)$$

$$p-1=0 \qquad ...(4)$$

$$p-x=0 \qquad ...(5)$$

and $$p-2y=0 \qquad ...(6)$$

$$(3) \Rightarrow \frac{dy}{dx}=0 \quad \Rightarrow \quad y=c \quad \Rightarrow \quad y-c=0.$$

$$(4) \Rightarrow \frac{dy}{dx}=1 \quad \Rightarrow \quad y=x+c \quad \Rightarrow \quad y-x-c=0.$$

$$(5) \Rightarrow \frac{dy}{dx}=x \quad \Rightarrow \quad y=\frac{x^2}{2}+c \quad \Rightarrow \quad 2y-x^2-2c=0.$$

$$(6) \Rightarrow \frac{dy}{dx}=2y \quad \Rightarrow \quad \frac{dy}{y}=2dx \quad \Rightarrow \quad \log y=2x+c$$

$$\Rightarrow \quad \log y-2x-c=0$$

$\therefore$ The general solution of (1) is

$$\mathbf{(y-c)(y-x-c)(2y-x^2-2c)(\log y-2x-c)=0.}$$

## EXERCISE 1

*Solve the following differential equations :*

1. $p^2-7p+12=0$
2. $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$
3. $yp^2+(x-y)p-x=0$
4. $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$
5. $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$
6. $p^2-2p\sinh x-1=0$
7. $p(p+y)=x(x+y)$
8. $4y^2p^2+2pxy(3x+1)+3x^3=0$
9. $xp^2-(y-x)p-y=0$
10. $xyp^2+p(3x^2-2y^2)-6xy=0$
11. $\left(1-y^2+\frac{y^4}{x^2}\right)\left(\frac{dy}{dx}\right)^2-\frac{2y}{x}\frac{dy}{dx}+\frac{y^2}{x^2}=0$
12. $\left(1-y^2-\frac{y^4}{x^2}\right)\left(\frac{dy}{dx}\right)^2-\frac{2y}{x}\frac{dy}{dx}+\frac{y^2}{x^2}=0$
13. $p^3(x+2y)+3p^2(x+y)+(2x+y)p=0$
14. $x^2p^3+yp^2(1+x^2y)+py^3=0$
15. $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0.$

### Answers

1. $(y-3x-c)(y-4x-c)=0$
2. $(y^2-x^2-c)(y-cx)=0$
3. $(y-x-c)(x^2+y^2-c)=0$
4. $(xy-c)(x^2y-c)=0$
5. $(xy-c)(x^2-y^2-c)=0$
6. $(y-e^x-c)(y-e^{-x}-c)=0$
7. $\left(y-\frac{x^2}{2}-c\right)(\log(1-x-y)+x+c)=0$
8. $(y^2+x^3-c)(y^2+\frac{1}{2}x^2-c)=0$
9. $(y+x-c)(y-cx)=0$
10. $(y-cx^2)(3x^2+y^2-c)=0$

11. $\left(\log \frac{x+\sqrt{x^2-y^2}}{y} + y - c\right)\left(\log \frac{x+\sqrt{x^2-y^2}}{y} - y - c\right) = 0$

12. $\left(\log \frac{x+\sqrt{x^2+y^2}}{y} + y - c\right)\left(\log \frac{x+\sqrt{x^2+y^2}}{y} - y - c\right) = 0$

13. $(y-c)(y+x-c)(xy+x^2+y^2-c)=0$ 14. $(y-c)(xy-cy-1)(y-ce^{1/x})=0$

15. $(x^3-3y+c)(e^{x^2/2}-cy)(xy+cy+1)=0.$

## 3.3. EQUATIONS SOLVABLE FOR y

If the given equation is solvable for $y$, we can express $y$ explicitly in terms of $x$ and $p$. Thus, the equations of this type can be put as

$$y = f(x, p). \quad ...(1)$$

Differentiating (1) w.r.t. $x$, we get

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx} = F\left(x, p, \frac{dp}{dx}\right), \text{ say}$$

$$\Rightarrow \quad p = F\left(x, p, \frac{dp}{dx}\right) \quad ...(2)$$

Equation (2) is a differential equation of first order in $p$ and $x$.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

The elimination of $p$ from (1) and (3) gives the general solution of (1).

**Note.** If $p$ cannot be easily eliminated, then we solve equations (1) and (3) for $x$ and $y$ to get

$$x = \phi_1(p, c),\ y = \phi_2(p, c).$$

These two relations together constitute the general solution of the given equation with $p$ as a parameter.

**Example 1.** *Solve* $y + px = x^4p^2$

**Sol.** We have $y = -px + x^4p^2$. ...(1)

Differentiating both sides w.r.t. $x$, we get

$$p = \left(-p.1 - x\frac{dp}{dx}\right) + \left(4x^3p^2 + 2x^4p\frac{dp}{dx}\right)$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

or $$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Discarding the factor $(1 - 2px^3)$ because it does not contain $\frac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{p} + 2\frac{dx}{x} = 0.$$

Integrating $\log p + 2\log x = \log c$

or $$\log px^2 = \log c \quad \text{or} \quad px^2 = c \quad \text{or} \quad p = \frac{c}{x^2}.$$

Putting this value of $p$ in (1), we get $y = -\left(\frac{c}{x^2}\right)x + x^4\left(-\frac{c}{x^2}\right)^2$ or $\mathbf{y = -\frac{c}{x} + c^2}$.

This is the required general solution.

NOTES

**Example 2.** *Solve* $y = 2px - p^2$

**Sol.** We have $\quad y = 2px - p^2$. ...(1)

Differentiating both sides w.r.t. $x$, we get

$$p = 2p.1 + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} \quad \text{or} \quad p + (2x - 2p)\frac{dp}{dx} = 0$$

or $\quad p\dfrac{dx}{dp} + 2x - 2p = 0 \quad \text{or} \quad \dfrac{dx}{dp} + \dfrac{2}{p}x = 2.$ ...(2)

This is a linear equation.

$\therefore \quad \text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$

$\therefore$ The solution of (2) is

$$x\,(\text{I.F.}) = \int 2(\text{I.F.})\,dp + c \quad \text{or} \quad xp^2 = \int 2p^2 dp + c$$

or $\quad xp^2 = \dfrac{2}{3}p^3 + c \quad \text{or} \quad x = \dfrac{2}{3}p + cp^{-2}$ ...(3)

Putting this value of $x$ in (1), we have

$$y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2 \quad \text{or} \quad y = \frac{4}{3}p^2 + 2cp^{-1} \qquad ...(4)$$

$\therefore$ The general solution of the given question is

$$\mathbf{x = \frac{2}{3}p + cp^{-2},\ y = \frac{4}{3}p^2 + 2cp^{-1}}$$

where $p$ is a parameter.

**Example 3.** *Solve* $y = p\sin p + \cos p$.

**Sol.** We have $\quad y = p\sin p + \cos p$.

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \left(p\cos p\frac{dp}{dx} + \sin p\frac{dp}{dx}\right) - \sin p\frac{dp}{dx}$$

$$\Rightarrow \quad p = p\cos p\frac{dp}{dx} \quad \Rightarrow \quad \cos p\,dp = dx$$

Integrating, we get $\quad \sin p = x + c$.

$\therefore \quad x = \sin p - c$

$\therefore$ The general solution of the given equation is $\mathbf{x = \sin p - c,\ y = p\sin p + \cos p}$, where $p$ is a parameter.

**Example 4.** *Solve* $p^3 + p = e^y$.

**Sol.** We have $\quad p^3 + p = e^y$

$\Rightarrow \quad \log(p^3 + p) = \log e^y \quad \Rightarrow \quad \log(p^3 + p) = y$

$\Rightarrow \quad y = \log p + \log(1 + p^2)$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{p}\frac{dp}{dx} + \frac{1}{1+p^2}\cdot 2p\frac{dp}{dx}$$

$$\Rightarrow \quad p = \left(\frac{1}{p} + \frac{2p}{1+p^2}\right)\frac{dp}{dx} \quad \Rightarrow \quad \left(\frac{1}{p^2} + \frac{2}{1+p^2}\right)dp = dx$$

NOTES

Integrating, we get

$$-1/p + 2\tan^{-1} p = x + c$$

$\therefore$ $$x = -1/p + 2\tan^{-1} p - c.$$

$\therefore$ The general solution of the given equation is

$$\mathbf{x = -1/p + 2\tan^{-1} p - c,\ y = \log p + \log(1 + p^2),}$$

where $p$ is a parameter.

**Example 5.** *Solve* $p^3 + \lambda p^2 = a(y + \lambda x)$.

**Sol.** We have $p^3 + \lambda p^2 = a(y + \lambda x)$.

$$\Rightarrow \quad y = \frac{p^3}{a} + \frac{\lambda p^2}{a} - \lambda x \qquad ...(1)$$

Differentiating (1) w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{3p^2}{a}\frac{dp}{dx} + \frac{2\lambda p}{a}\frac{dp}{dx} - \lambda \Rightarrow p = \left(\frac{3p^2 + 2\lambda p}{a}\right)\frac{dp}{dx} - \lambda$$

$$\Rightarrow \quad a(p + \lambda) = (3p^2 + 2\lambda p)\frac{dp}{dx} \quad \Rightarrow \quad \frac{3p^2 + 2\lambda p}{a(p + \lambda)}dp = dx$$

$$\Rightarrow \quad \left(\frac{3}{a}p - \frac{\lambda}{a} + \frac{\lambda^2}{a(p + \lambda)}\right)dp = dx$$

Integrating, we get

$$\frac{3p^2}{2a} - \frac{\lambda}{a}p + \frac{\lambda^2}{a}\log(p + \lambda) = x + c$$

$$\Rightarrow \quad x = \frac{3p^2}{2a} - \frac{\lambda}{a}p + \frac{\lambda^2}{a}\log(p + \lambda) - c.$$

Putting the value of $x$ in (1), we get

$$y = \frac{p^3}{a} + \frac{\lambda p^2}{a} - \frac{3\lambda p^2}{2a} + \frac{\lambda^2 p}{a} - \frac{\lambda^3}{a}\log(p + \lambda) + c\lambda$$

$$\Rightarrow \quad y = \frac{p^3}{a} - \frac{\lambda}{2a}p^2 + \frac{\lambda^2 p}{a} - \frac{\lambda^3}{a}\log(p + \lambda) + c\lambda.$$

$\therefore$ The general solution of the given equation is

$$\mathbf{x = \frac{3p^2}{2a} - \frac{\lambda}{a}p + \frac{\lambda^2}{a}\log(p + \lambda) - c,}$$

$$\mathbf{y = \frac{p^3}{a} - \frac{\lambda}{2a}p^2 + \frac{\lambda^2}{a}p - \frac{\lambda^3}{a}\log(p + \lambda) + c\lambda,}$$

where $p$ is a parameter.

**Example 6.** *Solve* $x = yp + p^2$.

**Sol.** We have $x = yp + p^2 \quad \Rightarrow \quad y = \frac{x}{p} - p$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{p}\cdot 1 + x\left(-\frac{1}{p^2}\right)\frac{dp}{dx} - \frac{dp}{dx}$$

$$\Rightarrow \quad p - \frac{1}{p} = -\left(\frac{x}{p^2} + 1\right)\frac{dp}{dx}$$

NOTES

$$\Rightarrow \qquad p^3 - p = -(x + p^2)\frac{dp}{dx}$$

$$\Rightarrow \qquad (p^3 - p)\frac{dx}{dp} + x = -p^2$$ [Note this step]

$$\Rightarrow \qquad \frac{dx}{dp} + \frac{x}{p^3 - p} = \frac{-p}{p^2 - 1} \qquad ...(1)$$

(1) is a linear equation.

$$\text{I.F.} = e^{\int \frac{1}{p^3 - p} dp}$$

Now
$$\int \frac{dp}{p^3 - p} = \int \frac{dp}{p(p-1)(p+1)}$$

$$= \int \left[\frac{1}{p(-1)(1)} + \frac{1}{(1)(p-1)(2)} + \frac{1}{(-1)(-2)(p+1)}\right] dp$$

$$= \int \left[-\frac{1}{p} + \frac{1}{2(p-1)} + \frac{1}{2(p+1)}\right] dp$$

$$= -\log p + \frac{1}{2}\log(p-1) + \frac{1}{2}\log(p+1) = \log\frac{\sqrt{p^2-1}}{p}.$$

$$\text{I.F.} = e^{\log\frac{\sqrt{p^2-1}}{p}} = \frac{\sqrt{p^2-1}}{p}$$

$\therefore$ The solution of (1) is

$$x \cdot \frac{\sqrt{p^2-1}}{p} = \int -\frac{p}{p^2-1} \cdot \frac{\sqrt{p^2-1}}{p} dp + c$$

$$\Rightarrow \qquad \frac{x\sqrt{p^2-1}}{p} = -\log(p + \sqrt{p^2-1}) + c$$

$$\Rightarrow \qquad x = -\frac{p}{\sqrt{p^2-1}}\log(p + \sqrt{p^2-1}) + \frac{cp}{\sqrt{p^2-1}}$$

Putting the value of $x$ in the given equation, we get

$$y = -\frac{1}{\sqrt{p^2-1}}\log(p + \sqrt{p^2-1}) + \frac{c}{\sqrt{p^2-1}} - p$$

$\therefore$ The general solution of the given equation is

$$\mathbf{x} = -\frac{\mathbf{p}}{\sqrt{\mathbf{p}^2-1}}\log(\mathbf{p} + \sqrt{\mathbf{p}^2-1}) + \frac{\mathbf{cp}}{\sqrt{\mathbf{p}^2-1}},$$

$$\mathbf{y} = -\frac{1}{\sqrt{\mathbf{p}^2-1}}\log(\mathbf{p} + \sqrt{\mathbf{p}^2-1}) + \frac{\mathbf{c}}{\sqrt{\mathbf{p}^2-1}} - \mathbf{p},$$

where $p$ is a parameter.

**NOTES**

**Example 7.** *Solve* $y = xp^2 - \dfrac{1}{p}$.

**Sol.** We have $$y = xp^2 - \frac{1}{p} \qquad \text{...(1)}$$

Differentiating (1) w.r.t. $x$, we get

$$\frac{dy}{dx} = x \; 2p \, \frac{dp}{dx} + p^2 \cdot 1 + \frac{1}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \quad p = \left(2xp + \frac{1}{p^2}\right)\frac{dp}{dx} + p^2$$

$$\Rightarrow \quad p - p^2 = \frac{2xp^3 + 1}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \quad p^3 - p^4 = (2xp^3 + 1)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dx}{dp} = \frac{2xp^3 + 1}{p^3(1-p)} = \frac{2x}{1-p} + \frac{1}{p^3(1-p)}$$

$$\Rightarrow \quad \frac{dx}{dp} + \left(\frac{2}{p-1}\right)x = \frac{1}{p^3(1-p)} \qquad \text{...(2)}$$

(2) is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p-1}\,dp} = e^{2\log(p-1)} = e^{\log(p-1)^2} = (p-1)^2$$

$\therefore$ The solution of (2) is

$$x \cdot (p-1)^2 = \int \frac{(p-1)^2}{p^3(1-p)}\,dp + c$$

$$\Rightarrow \quad x(p-1)^2 = \int \left(\frac{1}{p^3} - \frac{1}{p^2}\right)dp + c$$

$$\Rightarrow \quad x(p-1)^2 = -\frac{1}{2p^2} + \frac{1}{p} + c$$

$$\Rightarrow \quad x(p-1)^2 = \frac{2p-1}{2p^2} + c$$

$$\Rightarrow \quad x = \frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2}$$

Putting the value of $x$ in (1), we get

$$y = p^2\left[\frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2}\right] - \frac{1}{p}$$

$$= \frac{2p-1}{2(p-1)^2} + \frac{cp^2}{(p-1)^2} - \frac{1}{p}$$

$\therefore$ The general solution of the given equation is

$$x = \frac{2p-1}{2p^2(p-1)^2} + \frac{c}{(p-1)^2},$$

$$y = \frac{2p-1}{2(p-1)^2} + \frac{cp^2}{(p-1)^2} - \frac{1}{p}.$$

where $p$ is a parameter.

## EXERCISE 2

NOTES

*Solve the following differential equations :*

1. $xp^2 - 2yp + ax = 0$
2. $y - 2px = \tan^{-1}(xp^2)$
3. $16x^2 + 2p^2y - p^3x = 0$
4. $y = x + 2\tan^{-1}p$
5. $y = 3x + \log p$
6. $x - yp = ap^2$
7. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$
8. $y = 1 + xp^3$
9. $xp^2 - 2yp - ax = 0$
10. $y = 2p + \sqrt{1+p^2}$
11. $x^2 + p^2x = yp$
12. $y = p\tan p + \log\cos p$
13. $y = x(1+p) + p^2$
14. $y = 2px + \phi(xp^2)$.

### Answers

1. $2y = cx^2 + \frac{a}{c}$
2. $y = 2\sqrt{cx} + \tan^{-1}c$
3. $16 + 2c^2y - c^3x^2 = 0$
4. $x + c = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p,\ y = x + 2\tan^{-1}p$
5. $y = 3x + \log\frac{3}{1-ce^{3x}}$
6. $x = \frac{p}{\sqrt{p^2-1}}(c - a\log(p+\sqrt{p^2-1})),\ y = \frac{1}{\sqrt{p^2-1}}(c - a\log(p+\sqrt{p^2-1})) - ap$
7. $y = c^2 + 2\sqrt{cx}$
8. $x^{2/3} = (y-1)^{2/3} + c$
9. $2cy = c^2x^2 + a$
10. $x = 2\log p + \log\left(p+\sqrt{1+p^2}\right) + c\quad y = 2p + \sqrt{1+p^2}$
11. $x = c\sqrt{p} - \frac{1}{3}p^2,\ y = \frac{c}{3}p^{3/2} - \frac{2}{9}p^3 + c^2$
12. $x = \tan p + c,\ y = p\tan p + \log\cos p$
13. $x = 2(1-p) + ce^{-p},\ y = 2 + p^2 + (1+p)ce^{-p}$
14. $y = 2\sqrt{cx} + \phi(c)$

## 3.4. EQUATIONS SOLVABLE FOR x

If the given equation is solvable for $x$, we can express $x$ explicitly in terms of $y$ and $p$. Thus, the equations of this type can be put as

$$x = f(y, p). \qquad ...(1)$$

Differentiating (1) w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F\left(y, p, \frac{dp}{dy}\right), \text{ say}$$

$$\Rightarrow \qquad \frac{1}{p} = F\left(x, p, \frac{dp}{dy}\right) \qquad ...(2)$$

Equation (2) is a differential equation of first order in $p$ and $y$.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

The elimination of $p$ from (1) and (3) gives the general solution of (1).

**Note.** If $p$ cannot be easily eliminated, then we solve equations (1) and (3) for $x$ and $y$ to get

$$\mathbf{x = \phi_1(p, c), \; y = \phi_2(p, c).}$$

These two relations together constitute the general solution of the given equation with $p$ as a parameter.

**Example 1.** *Solve* $x = y + a \log p$.

**Sol.** We have $\qquad x = y + a \log p$. ...(1)

Diff. w.r.t. $y$, we get

$$\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dp}{dy}.$$

$$\Rightarrow \quad \frac{1}{p} - 1 = \frac{a}{p}\frac{dp}{dy} \quad \Rightarrow \quad 1 - p = a\frac{dp}{dy} \quad \Rightarrow \quad dy = \frac{a}{1-p}dp$$

Integrating, we get $y = -a \log(1-p) + c$

Putting the value of $y$ in (1), we get

$$x = -a \log(1-p) + c + a \log p \quad \text{or} \quad x = a \log\frac{p}{1-p} + c.$$

$\therefore$ The general solution of the given equation is

$$\mathbf{x = a \log\frac{p}{1-p} + c, \; y = -a \log(1-p) + c,}$$

where $p$ is a parameter.

**Example 2.** *Solve* $xp^3 = a + bp$.

**Sol.** We have $\qquad x = \dfrac{a}{p^3} + \dfrac{b}{p^2}$.

Diff. w.r.t. $y$, we get

$$\frac{dx}{dy} = \left(-\frac{3a}{p^4} - \frac{2b}{p^3}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} = -\frac{1}{p^4}(3a + 2bp)\frac{dp}{dy} \quad \Rightarrow \quad dy = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)dp$$

Integrating, we get $y = -\dfrac{3ap^{-2}}{-2} - \dfrac{2bp^{-1}}{-1} + c \quad \Rightarrow \quad y = \dfrac{3a}{2p^2} + \dfrac{2b}{p} + c$ ...(1)

Using given equation and (1), the general solution of the given equation is

$$\mathbf{x = \frac{a}{p^3} + \frac{b}{p^2}, \; y = \frac{3a}{2p^2} + \frac{2b}{p} + c}$$

where $p$ is a parameter.

**Example 3.** *Solve* $y = 2px + y^2p^3$

**Sol.** We have $\qquad y = 2px + y^2p^3$ ...(1)

$$\Rightarrow \quad 2px = y - y^2p^3 \quad \Rightarrow \quad x = \frac{1}{2}\left(\frac{y}{p} - y^2p^2\right) \quad \text{...(2)}$$

**NOTES**

Differentiating w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2y^2 p\frac{dp}{dy}\right)$$

$$\Rightarrow \quad \frac{1}{p} = \frac{1}{2p} - yp^2 - \left(\frac{y}{2p^2} + y^2 p\right)\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} + yp^2 + \left(\frac{y}{2p^2} + y^2 p\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \quad \frac{p}{y}\left(\frac{y}{2p^2} + y^2 p\right) + \left(\frac{y}{2p^2} + y^2 p\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \quad \left(\frac{y}{2p^2} + y^2 p\right)\left(\frac{p}{y} + \frac{dp}{dy}\right) = 0$$

$$\Rightarrow \quad \frac{p}{y} + \frac{dp}{dy} = 0 \quad \Rightarrow \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating $\quad \log y + \log p = \log c \quad$ or $\quad py = c \quad$ or $\quad p = \dfrac{c}{y}$

Putting this value of $p$ in (1), we have

$$y = 2\left(\frac{c}{y}\right)x + y^2\left(\frac{c}{y}\right)^3 \quad \text{or} \quad \mathbf{y^2 = 2cx + c^3}.$$

This is the general solution of the given equation.

**Example 4.** *Solve* $\quad p = \tan\left(x - \dfrac{p}{1+p^2}\right)$

**Sol.** We have $\quad p = \tan\left(x - \dfrac{p}{1+p^2}\right) \qquad ...(1)$

$$\Rightarrow \quad \tan^{-1} p = x - \frac{p}{1+p^2} \quad \Rightarrow \quad x = \tan^{-1} p + \frac{p}{1+p^2}. \qquad ...(2)$$

Differentiating w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2).1 - p.2p}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} = \left(\frac{1}{1+p^2} + \frac{1-p^2}{(1+p^2)^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy} \quad \Rightarrow \quad dy = \frac{2p}{(1+p^2)^2}\,dp$$

Integrating, $\quad y = \displaystyle\int (1+p^2)^{-2}\, 2p\, dp + c = \frac{(1+p^2)^{-1}}{-1} + c$

or $\quad y = c - \dfrac{1}{1+p^2}. \qquad ...(3)$

Using given equation and (3), the general solution of the given equation is

$$\mathbf{x = \tan^{-1}p + \frac{p}{1+p^2}}, \qquad \mathbf{y = c - \frac{1}{1+p^2}},$$

where $p$ is a parameter.

## EXERCISE 3

*Solve the following differential equations :*

**1.** $y = 3px + 6p^2y^2$ **2.** $y = 2px + p^2y$

**3.** $p^3 - 4xyp + 8y^2 = 0$ **4.** $y^2 \log y = xyp + p^2$

**5.** $x = y + p^2$ **6.** $yp^2 - 2xp + y = 0$

**7.** $p^3 - 2xyp + 4y^2 = 0$ **8.** $4x = py(p^2 - 3)$.

### Answers

**1.** $y^3 = 3cx + 6c^2$ **2.** $y^2 = 2cx + c^2$ **3.** $64y = c(4x - c)^2$

**4.** $\log y = cx + c^2$ **5.** $x = c - 2p - 2\log(p-1),\ y = c - p^2 - 2p - 2\log(p-1)$

**6.** $y^2 = 2cx - c^2$ **7.** $2y = c(x - c)^2$

**8.** $x = \dfrac{cp(p^2-3)}{4(p^2-4)^{9/10}(p^2+1)^{3/5}},\ y = \dfrac{c}{(p^2-4)^{9/10}(p^2+1)^{3/5}}$.

## 3.5. CLAIRAUT'S EQUATION

An equation of the form

$$y = px + f(p) \qquad ...(1)$$

is known as **Clairaut's equation.**

Differentiating (1) w.r.t. $x$, we get

$$p = \left(p + x\frac{dp}{dx}\right) + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$ because it does not contain $\dfrac{dp}{dx}$, we have $\dfrac{dp}{dx} = 0$.

Integrating $\qquad p = c$

Putting $\qquad p = c$ in (1), the required solution is $\mathbf{y = cx + f(c)}$.

Thus, the solution of Clairaut's equation is obtained by writing $c$ for $p$ in the equation.

**Example 1.** *Solve* $(y - px)(p - 1) = p$.

**Sol.** We have $\quad (y - px)(p - 1) = p \qquad ...(1)$

$$\Rightarrow \qquad y - px = \frac{p}{p-1} \quad \Rightarrow \quad y = px + \frac{p}{p-1} \qquad ...(2)$$

(2) is of the form $y = px + f(p)$ with $f(p) = \dfrac{p}{p-1}$

$\therefore$ (2) is a Clairaut's equation.

$\therefore$ The general solution of the given equation is $\mathbf{y = cx + \dfrac{c}{c-1}}$.

(By replacing $p$ by $c$)

**Example 2.** *Solve* $\sin px \cos y = \cos px \sin y + p$.

**Sol.** We have $\quad \sin px \cos y - \cos px \sin y = p$.

NOTES

$\Rightarrow \quad \sin(px - y) = p \quad \Rightarrow \quad px - y = \sin^{-1} p$

$\Rightarrow \quad y = px - \sin^{-1} p.$

This is a Clairaut's equation.

$\therefore$ Its general solution is $\mathbf{y = cx - \sin^{-1} c}$. (By replacing $p$ by $c$)

**Example 3.** *Solve* $(y - px)^2 = 1 + p^2.$

**Sol.** We have $(y - px)^2 = 1 + p^2.$

$$\Rightarrow \quad y - px = \pm\sqrt{1+p^2} \quad \Rightarrow \quad y = px \pm \sqrt{1+p^2} \qquad \text{...(1)}$$

Taking +ve sign, (1) $\Rightarrow \quad y = px + \sqrt{1+p^2}$

This is a Clairaut's equation and its solution is $y = cx + \sqrt{1+c^2}$. (By replacing $p$ by $c$)

$$\Rightarrow \quad y - cx - \sqrt{1+c^2} = 0. \qquad \text{...(2)}$$

Taking –ve sign, (1) $\Rightarrow \quad y = px - \sqrt{1+p^2}$

This is a Clairaut's equation and its solution is $y = cx - \sqrt{1+c^2}$.

$$\Rightarrow \quad y - cx + \sqrt{1+c^2} = 0. \qquad \text{...(3)}$$

Using (2) and (3), the general solution of the given equation is

$$(y - cx - \sqrt{1+c^2})(y - cx + \sqrt{1+c^2}) = 0 \quad \text{or} \quad \mathbf{(y - cx)^2 = 1 + c^2.}$$

**Example 4.** *Solve* $p^2x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0.$

**Sol.** We have $p^2x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0.$

$$\therefore \quad p = \frac{-2y + 2xy + x - 2 \pm \sqrt{(2y - 2xy - x + 2)^2 - 4x(x-2)(y^2+y)}}{2x(x-2)}$$

$$= \frac{-2y + 2xy + x - 2 \pm (2y - x + 2)}{2x(x-2)} = \frac{y}{x-2}, \frac{y+1}{x}.$$

$$p = \frac{y}{x-2} \quad \Rightarrow \quad y = px - 2p \qquad \text{...(1)}$$

$$p = \frac{y+1}{x} \quad \Rightarrow \quad y = px - 1 \qquad \text{...(2)}$$

(1) is a Clairaut's equation.

$\therefore$ Its solution is $y = cx - 2c$ or $y - cx + 2c = 0.$

(2) is also a Clairaut's equation.

$\therefore$ Its solution is $y = cx - 1$ or $y - cx + 1 = 0.$

$\therefore$ The general solution of the given equation is

$$\mathbf{(y - cx + 2c)(y - cx + 1) = 0.}$$

**Note.** Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

**Example 5.** *Solve* $e^{4x}(p-1)+e^{2y}p^2=0.$

**Sol.** We have $\quad e^{4x}(p-1)+e^{2y}p^2=0 \qquad ...(1)$

Let $\quad X=e^{2x}$ and $Y=e^{2y}$*

$$\therefore \quad \frac{dX}{dx}=2e^{2x} \quad \text{and} \quad \frac{dY}{dx}=2e^{2y}\frac{dy}{dx}=2e^{2y}p$$

$$\therefore \quad \frac{dY}{dX}=\frac{dY}{dx}\Big/\frac{dX}{dx}=\frac{2e^{2y}p}{2e^{2x}}=\frac{e^{2y}}{e^{2x}}p=\frac{Y}{X}p$$

Let $\quad P=\dfrac{dY}{dX} \quad \therefore \quad P=\dfrac{Y}{X}p \quad$ or $\quad p=\dfrac{X}{Y}P.$

$$\therefore \quad (1) \Rightarrow X^2\left(\frac{X}{Y}P-1\right)+Y\cdot\frac{X^2}{Y^2}P^2=0 \quad \text{or} \quad XP-Y+P^2=0$$

or $\quad Y=PX+P^2$

This is a Clairaut's equation.

$\therefore$ **Its solution is** $\quad Y=cX+c^2 \qquad$ **(By replacing P by c)**

$\Rightarrow \quad \mathbf{e^{2y}=ce^{2x}+c^2.}$

This is the general solution of the given equation.

**Example 6.** *Solve* $(px-y)(py+x)=2p.$

**Sol.** We have $\quad (px-y)(py+x)=2p \qquad ...(1)$

Let $\quad X=x^2 \quad$ and $\quad Y=y^2.$

$$\therefore \quad \frac{dX}{dx}=2x \quad \text{and} \quad \frac{dY}{dx}=2yp.$$

$$\therefore \quad \frac{dY}{dX}=\frac{2yp}{2x}=\frac{y}{x}p$$

Let $\quad P=\dfrac{dY}{dX}$

$$\therefore \quad P=\frac{y}{x}p \quad \text{or} \quad p=\frac{\sqrt{X}}{\sqrt{Y}}P.$$

$$\therefore \quad (1) \Rightarrow \left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{Y}+\sqrt{X}\right)=2\frac{\sqrt{X}}{\sqrt{Y}}P$$

or $\quad (PX-Y)(P+1)=2P \quad$ or $\quad PX-Y=\dfrac{2P}{P+1} \quad$ or $\quad Y=PX-\dfrac{2P}{P+1}$

This is a Clairaut's equation.

$\therefore$ Its solution is $\quad Y=cX-\dfrac{2c}{c+1} \qquad$ **(By replacing P by c)**

$$\Rightarrow \quad \mathbf{y^2=cx^2-\frac{2c}{c+1}.}$$

This is the general solution of the given equation.

---

***Why this step.** In problems involving $e^{lx}$ and $e^{my}$, use the substitutions $X=e^{kx}$ and $y=e^{ky}$, where $k$ is the H.C.F. of $l$ and $m$.

NOTES

**Example 7.** *Solve* $(px^2 + y^2)(px + y) = (p + 1)^2$ *by using the substitutions* $X = x + y$ *and* $Y = xy$.

**Sol.** Given equation is $\quad (px^2 + y^2)(px + y) = (p + 1)^2 \qquad ...(1)$

We have $\quad X = x + y \quad$ and $\quad Y = xy$

$$\therefore \quad \frac{dX}{dx} = 1 + p \quad \text{and} \quad \frac{dY}{dx} = y + xp$$

$$\therefore \quad \frac{dY}{dX} = \frac{y + xp}{1 + p}$$

Let $\quad P = \dfrac{dY}{dX}. \quad \therefore \quad P = \dfrac{xp + y}{1 + p}.$

(1) $\Rightarrow \quad (px^2 + y^2 + xyp + xy - xyp - xy)(px + y) = (p + 1)^2 \quad$ [Note this step]

$$\Rightarrow \quad [(px + y)(x + y) - (p + 1)xy](px + y) = (p + 1)^2$$

$$\Rightarrow \quad (px + y)^2 (x + y) - (p + 1)xy(px + y) = (p + 1)^2$$

$$\Rightarrow \quad \frac{(px + y)^2 (x + y)}{(p + 1)^2} - \frac{xy(px + y)}{p + 1} = 1$$

$$\Rightarrow \quad (x + y)\left(\frac{px + y}{p + 1}\right)^2 - xy\left(\frac{px + y}{p + 1}\right) = 1$$

$$\Rightarrow \quad XP^2 - YP = 1 \quad \Rightarrow \quad YP = XP^2 - 1 \quad \Rightarrow \quad Y = PX - \frac{1}{P}.$$

This is a Clairaut's equation.

$\therefore$ Its solution is $\quad Y = cX - \dfrac{1}{c}. \qquad$ (By replacing P by c)

$$\Rightarrow \quad \mathbf{xy = c(x + y) - \frac{1}{c}}$$

This is the general solution of the given equation.

## EXERCISE 4

*Solve the following differential equations (Q. No. 1–10) :*

1. $y = xp + \dfrac{a}{p}$ $\qquad$ 2. $y = px + \sqrt{a^2p^2 + b^2}$ $\qquad$ 3. $p = \log(px - y)$
4. $p = \sin(y - px)$ $\qquad$ 5. $p^2(x^2 - 1) - 2pxy + y^2 - 1 = 0$ $\qquad$ 6. $e^{3x}(p - 1) + p^3e^{2y} = 0$
7. $x^2(y - px) = yp^2$ $\qquad$ 8. $(y + px)^2 = x^2p$ $\qquad$ 9. $y = 3px + 6y^2p^2$
10. $p^2 \cos^2 y + p \sin x \cos x \cos y - \sin y \cos^2 x = 0.$
11. Reduce the equation $y^2(y - xp) = x^4p^2$ to Clairaut's equation by using the substitutions $x = \dfrac{1}{u}, y = \dfrac{1}{v}$ and hence solve the equation.

**NOTES**

### Answers

| | | |
|---|---|---|
| **1.** $y = cx + \frac{a}{c}$ | **2.** $y = cx + \sqrt{a^2c^2 + b^2}$ | **3.** $y = cx - e^c$ |
| **4.** $y = cx + \sin^{-1} c$ | **5.** $(y - cx)^2 = 1 + c^2$ | **6.** $e^y = ce^x + c^2$ |
| **7.** $y^2 = cx^2 + c^2$ | **8.** $xy = cx - c^2$ | **9.** $y^3 = cx + \frac{2}{3}c^2$ |
| **10.** $\sin y = c \sin x + c^2$ | **11.** $x = cy + c^2xy$. | |

### Hints

| | |
|---|---|
| **7.** Try $X = x^2, Y = y^2$ | **8.** Try $X = x, Y = xy$ |
| **9.** Try $X = x, Y = y^3$ | **10.** Try $X = \sin x, Y = \sin y$. |

# UNIT 4 SINGULAR SOLUTIONS

NOTES

STRUCTURE



## 4.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Define discriminant relation
- Find the singular solution

## 4.1. INTRODUCTION

We know that an ordinary differential equation of order one admitting solution has one arbitrary constant in its general solution. We can also find its particular solutions by giving different values to the arbitrary constant in its general solution. If a differential equation of order one is of degree greater than one then there may exist a solution of this equation which is neither general nor particular. Such a solution cannot be derived from the general solution by giving some particular value to the arbitrary constant.

In the present chapter, we shall discuss the possibility of such solutions.

## 4.2. DISCRIMINANT RELATION

Let $f(X) = 0$ be an equation in the variable X. The relation obtained by eliminating X from the equations $f(X) = 0$ and $\frac{df}{dX} = 0$ is called the **X-discriminant relation** of the equation $f(X) = 0$. This relation gives the required condition for the given equation $f(X) = 0$ to have multiple roots.

For example, let $f(X) = aX^2 + bX + c$

$$\therefore \qquad \frac{df}{dX} = 2aX + b.$$

$$\frac{df}{dX} = 0 \quad \Rightarrow \quad 2aX + b = 0$$

$$\Rightarrow \qquad X = -\frac{b}{2a}$$

$\therefore$ Putting $X = -\frac{b}{2a}$ in $f(X) = 0$, we get

$$a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = 0 \quad \text{or} \quad b^2 - 2b^2 + 4ac = 0$$

or $$b^2 - 4ac = 0.$$

$\therefore$ The X-discriminant relation of the equation

$aX^2 + bX + c = 0$ is $b^2 - 4ac = 0$.

**Verification :** $f(X) = 0 \Rightarrow X = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ If $b^2 - 4ac = 0$, then the equation $f(X) = 0$ has multiple roots *i.e.*, equal roots $-\frac{b}{2a}, -\frac{b}{2a}$.

## 4.3. THE p-DISCRIMINANT RELATION

Let $f(x, y, p) = 0$ ...(1)

be a differential equation of order one and degree greater than one.

By definition, *p–discriminant relation* of (1) is obtained by eliminating $p$ between the equations

$$f(x, y, p) = 0 \quad \text{and} \quad \frac{\partial f}{\partial p} = 0$$

Thus, the *p-discriminant relation* represents the locus, for each point of which $f(x, y, p) = 0$ has equal values of $p$.

## 4.4. THE c-DISCRIMINANT RELATION

Let $\phi(x, y, c) = 0$ ...(1)

be a one parameter family of curves and of at least second degree in $c$.

By definition, *c-discriminant relation* of (1) is obtained by eliminating $c$ between the equations

$$\phi(x, y, c) = 0$$

and $$\frac{\partial \phi}{\partial c} = 0$$

Thus, the *c-discriminant relation* represents the locus, for each point of which $\phi(x, y, c) = 0$ has equal values of $c$.

## 4.5. THE ENVELOPE OF A FAMILY OF CURVES

NOTES

Let $\phi(x, y, c) = 0$ be a one parameter family of curves and of at least second degree in $c$. This family represents infinitely many curves of same kind. Let the values of $c$ be arranged in order of magnitude. The successive $c$'s thus differing by infinitesimal amounts. Suppose that all these curves are drawn. Curves corresponding to two consecutive values of $c$ are called **consecutive curves** and their point of intersection is called an **ultimate point of intersection**. The limiting position of these points of intersection includes the envelope of the given family of curves. Since in the limit the $c$'s for two consecutive curves become equal and the $c$-discriminant relation represents the locus of points for which $\phi(x, y, c)$ will have equal values of $c$, the envelope is a part of the locus of the $c$-discriminant relation. In differential calculus, we prove that at any point of the envelope, the latter is touched by some curve of the family $\phi(x, y, c) = 0$.

$\therefore$ At every point of the envelope, $x$, $y$ and $p$ have the same value for the envelope and the curve of the family that touches it there.

## 4.6. SINGULAR SOLUTION

Let $f(x, y, p) = 0$ be a differential equation of order one and of degree greater than one.

Let $\phi(x, y, c) = 0$ be the general solution of this differential equation. Let $E(x, y) = 0$ be the equation of the envelope of the family $\phi(x, y, c) = 0$. We have seen that $E(x, y) = 0$ is a part of the c-discriminant relation of the family $\phi(x, y, c) = 0$. Also, at every point of the envelope, $x$, $y$ and $p$ have the same value for the envelope and the curve of the family $\phi(x, y, c) = 0$ that touches it there.

$\therefore$ The equation $E(x, y) = 0$ of the envelope represents a solution of the differential equation $f(x, y, p) = 0$.

This solution is called the **singular solution** of the differential equation $f(x, y, p) = 0$. This solution does not contain any arbitrary constant and is not deducible from the general solution $\phi(x, y, c) = 0$ by giving a particular value to the arbitrary constant in it.

We can also show that the equation of the envelope $E(x, y) = 0$ *i.e.*, the singular solution of the equation $f(x, y, p) = 0$ is also a part of the $p$-discriminant relation of the equation $f(x, y, p) = 0$. This is because at the ultimate points of intersection of consecutive curves of the members of the family $\phi(x, y, c) = 0$, the $p$'s for the intersecting curves become equal. Thus the locus of the points where $p$'s have equal roots will include the envelope.

$\therefore$ The $p$-discriminant relation of $f(x, y, p) = 0$ contains the equation of the envelope *i.e.*, the singular solution of the equation $f(x, y, p) = 0$.

$\therefore$ The singular solution is contained in both $p$ and $c$ discriminant relations.

**Remark.** The $p$ and $c$ discriminant relations may sometimes represent other loci beside the envelope *i.e.*, they may contain other equations beside the equation of the singular solution. These loci are called **Tac locus, Nodal locus** and **Cuspidal locus**. Generally the equations of the tac locus, nodal locus and cuspidal locus do not satisfy the equation of the differential equation. This is why, these are called **extraneous loci**. The detail regarding these loci is beyond the scope of this book.

NOTES

**Working rules for finding singular solution**

**Step I.** Find the general solution of the given differential equation.

**Step II.** Find the $p$-discriminant relation of the given differential equation and write it in the form $ET^2C = 0$, where E, T, C are some expressions in $x$ and $y$.

**Step III.** Find the $c$-discriminant relation of the general solution and write it in the form $EN^2C^3 = 0$, where N is some expression in $x$ and $y$.

**Step IV.** Consider the equation $E = 0$. If this equation satisfies the given differential equation then this constitutes the singular solution of the given differential equation. $T = 0$, $N = 0$ and $C = 0$ respectively gives the equation of Tac locus, Nodal locus and Cuspidal locus.

**Remarks 1.** In case $f(x, y, p) = 0$ is quadratic in $p$ then it is advisable to find its $p$-discriminant relation by equating the discriminant of $f(x, y, p)$ to zero. Similar argument also work for $c$-discriminant relation.

**2.** If the $p$-discriminant relation and the $c$-discriminant relation are same, then the common equation gives $E = 0$.

**Example 1.** *Solve the differential equation* $y = px + \frac{a}{p}$ *and obtain its singular solution.*

**Sol.** We have $\quad y = px + \frac{a}{p}$. ...(1)

This is a Clairaut's equation.

$\therefore$ By replacing $p$ by $c$, its general solution is

$$y = cx + \frac{a}{c} \quad i.e. \quad cy = c^2x + a.$$

(1) $\Rightarrow$ $\quad py = p^2x + a \Rightarrow xp^2 - yp + a = 0$

Let $\quad f(x, y, p) = xp^2 - yp + a$

This is quadratic in $p$.

$\therefore$ $\quad$ Disc. = 0 $\quad\Rightarrow\quad y^2 - 4ax = 0$ ...(2)

Let $\quad \phi(x, y, c) = c^2x + a - cy \quad\Rightarrow\quad \phi(x, y, c) = xc^2 - yc + a$

This is quadratic in $c$.

$\therefore$ $\quad$ Disc. = 0 $\quad\Rightarrow\quad y^2 - 4ax = 0$ ...(3)

Using (2), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$$(y^2 - 4ax) \cdot 1^2 \cdot 1 = 0.$$

Using (3), the $c$-discriminant relation ($EN^2C^3 = 0$) can be written as

$$(y^2 - 4ax) \cdot 1^2 \cdot 1^3 = 0.$$

$\therefore$ $\quad E = 0 \quad\Rightarrow\quad y^2 - 4ax = 0$ ...(4)

(4) $\Rightarrow$ $\quad 2y\frac{dy}{dx} - 4a = 0 \quad\Rightarrow\quad 2yp = 4a \quad\Rightarrow\quad p = \frac{2a}{y}$

Also, $\quad y^2 - 4ax = 0 \quad\Rightarrow\quad x = \frac{y^2}{4a}$.

Putting the values of $p$ and $x$ in (1), we get $y = \frac{2a}{y} \cdot \frac{y^2}{4a} + a \cdot \frac{y}{2a}$ or $y = y$, which is true.

$\therefore \quad y^2 - 4ax = 0$ satisfies the given equation.

$\therefore \quad \mathbf{y^2 - 4ax = 0}$ is the singular solution of the given equation.

**Example 2.** *Investigate the equation $y = px - 2p^2$ for singular solution.*

NOTES

**Sol.** We have $y = px - 2p^2$ ...(1)

This is a Clairaut's equation.

Replacing $p$ by $c$, the general solution of (1) is

$$y = cx - 2c^2 \quad i.e., \quad y - cx + 2c^2 = 0 \qquad ...(2)$$

Let $\quad f(x, y, p) = y - px + 2p^2. \quad \therefore \quad \frac{\partial f}{\partial p} = -x + 4p$

$\therefore \quad f(x, y, p) = 0 \quad \Rightarrow \quad y - px + 2p^2 = 0$ ...(3)

and $\quad \frac{\partial f}{\partial p} = 0 \quad \Rightarrow \quad -x + 4p = 0$ ...(4)

(4) $\Rightarrow \quad p = \frac{x}{4}$

Putting the value of $p$ in (3), we get

$$y - \left(\frac{x}{4}\right)x + 2\left(\frac{x}{4}\right)^2 = 0$$

$\Rightarrow \quad y - \frac{x^2}{4} + \frac{x^2}{8} = 0 \quad \Rightarrow \quad y - \frac{x^2}{8} = 0 \quad \Rightarrow \quad x^2 - 8y = 0$ ...(5)

Let $\quad \phi(x, y, c) = \quad y - cx + 2c^2$

Eliminating $c$ between $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$, we get $\quad x^2 - 8y = 0$ ...(6)

Using (5), the $p$-discriminant relation ($\text{ET}^2 \text{C} = 0$) can be written as

$(x^2 - 8y) \ 1^2 \,.\, 1 = 0.$

Using (6), the $c$-discriminant relation ($\text{EN}^2\text{C}^3 = 0$) can be written as

$(x^2 - 8y).\ 1^2 \,.\, 1^3 = 0.$

$\therefore \quad \text{E} = 0 \quad \Rightarrow \quad x^2 - 8y = 0$ ...(7)

(7) $\Rightarrow \ y = \frac{x^2}{8}$ and thus $p = \frac{x}{4}$. Putting the values of $y$ and $p$ in (1), we get $\frac{x^2}{8} = \left(\frac{x}{4}\right) x - 2\left(\frac{x}{4}\right)^2$, which is true.

$\therefore \quad x^2 - 8y = 0$ satisfies the given equation.

$\therefore \quad \mathbf{x^2 - 8y = 0}$ is the singular solution of the given equation.

**Example 3.** *Obtain the singular solution of the equation $p^2 + y^2 = 1$, $p \geq 0$. Also interpret the result geometrically.*

**Sol.** We have $\quad p^2 + y^2 = 1.$ ...(1)

$\Rightarrow \quad p = \sqrt{1 - y^2} \qquad (\because \quad p \geq 0)$

$\Rightarrow \quad \frac{dy}{\sqrt{1 - y^2}} = dx$

Integrating, we get

$$\sin^{-1} y = x + c. \Rightarrow y = \sin(x + c) \quad \text{or} \quad y - \sin(x + c) = 0. \quad ...(2)$$

This is the general solution.

NOTES

Let $f(x, y, p) = p^2 + y^2 - 1 \quad \therefore \quad \frac{\partial f}{\partial p} = 2p$

$\therefore \quad f(x, y, p) = 0 \quad \Rightarrow \quad p^2 + y^2 - 1 = 0$ ...(3)

and $\frac{\partial f}{\partial p} = 0 \quad \Rightarrow \quad 2p = 0$ ...(4)

Eliminating $p$ between (3) and (4), we get $0 + y^2 - 1 = 0$ *i.e.,* $y^2 - 1 = 0$. ...(5)

Let $\phi(x, y, c) = y - \sin(x + c). \quad \therefore \quad \frac{\partial \phi}{\partial c} = -\cos(x + c)$

$\therefore \quad \phi(x, y, c) = 0 \quad \Rightarrow \quad y - \sin(x + c) = 0$ ...(6)

and $\frac{\partial \phi}{\partial c} = 0 \quad \Rightarrow \quad -\cos(x + c) = 0$ ...(7)

We have $\sin^2(x + c) + \cos^2(x + c) = 1$

$\therefore \quad y^2 + 0^2 = 1 \quad$ *i.e.,* $\quad y^2 - 1 = 0$ ...(8)

Using (5), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$(y^2 - 1)\ 1^2 .\ 1 = 0.$

Using (8), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$(y^2 - 1)\ 1^2 .\ 1^3 = 0.$

$\therefore \quad E = 0 \quad \Rightarrow \quad y^2 - 1 = 0$ ...(9)

(9) $\Rightarrow \quad y = \pm 1$ and thus $p = 0$.

Putting the values of $y$ and $p$ in (1), we get $0 + 1 = 1$, which is true.

$\therefore \quad y^2 - 1 = 0$ satisfies the given equation.

$\therefore \quad$ **$y^2 - 1 = 0$** is the singular solution of the given equation. The equation $y = \sin(x + c)$ represents the family of sine curves lying between $y = 1$ and $y = -1$. The equation $y^2 - 1 = 0$ represents the lines $y = 1$ and $y = -1$. The lines $y = 1, y = -1$ represent the envelope of the family of sine curves.

**Example 4.** *Find the general solution and singular solution of the differential equation* $4p^2(x - 2) = 1$

**Sol.** We have $\quad 4p^2(x - 2) = 1.$ ...(1)

$$\Rightarrow \quad p^2 = \frac{1}{4(x-2)} \quad \Rightarrow \quad p = \pm \frac{1}{2\sqrt{x-2}}$$

$\therefore \quad p = \frac{1}{2\sqrt{x-2}}$ ...(2)

and $\quad p = -\frac{1}{2\sqrt{x-2}}$ ...(3)

Integrating (2), we get

$$y = \sqrt{x-2} + c \quad \text{or} \quad y - \sqrt{x-2} - c = 0.$$

Integrating (3), we get

$$y = -\sqrt{x-2} + c \quad \text{or} \quad y + \sqrt{x-2} - c = 0$$

NOTES

$\therefore$ The general solution of (1) is

$$(y - \sqrt{x-2} - c)(y + \sqrt{x-2} - c) = 0 \quad \text{or} \quad \mathbf{(y - c)^2 - (x - 2) = 0.}$$

Let $f(x, y, p) = 4p^2 (x - 2) - 1$. $f(x, y, p)$ is quadratic in $p$.

$\therefore$ Disc. $= 0 \Rightarrow (0)^2 - 4 \,.\, 4(x - 2)(-1) = 0 \Rightarrow x - 2 = 0$ ...(4)

Let $\phi(x, y, c) = (y - c)^2 - (x - 2)$

$\Rightarrow \phi(x, y, c) = c^2 - 2yc + (y^2 - x + 2)$

$\phi(x, y, c)$ is quadratic in $c$.

$\therefore$ Disc. $= 0 \Rightarrow 4y^2 - 4.1(y^2 - x + 2) = 0$

$\Rightarrow 4(x - 2) = 0 \Rightarrow x - 2 = 0$ ...(5)

Using (4), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$(x - 2) \,.\, 1^2 \,.\, 1 = 0.$

Using (5), the $c$-discriminant relation ($EN^2C^3 = 0$) can be written as

$(x - 2) \cdot 1^2 \cdot 1^3 = 0.$

$\therefore$ $E = 0 \Rightarrow x - 2 = 0$ ...(6)

$x - 2 = 0 \Rightarrow x = 2$

$\Rightarrow \dfrac{dx}{dy} = 0 \Rightarrow \dfrac{1}{p} = 0$

$\therefore$ $x - 2 = 0 \Rightarrow 4(x - 2) = \dfrac{1}{p^2} \Rightarrow 4p^2(x - 2) = 0$

$\therefore$ $x - 2 = 0$ satisfies the given equation

$\therefore$ $\mathbf{x - 2 = 0}$ is the singular solution of the given equation.

**Example 5.** *Find the general and singular solutions of the equation* $xp^2 - 2yp + 4x = 0$.

**Sol.** We have $xp^2 - 2yp + 4x = 0$.

$$\Rightarrow \quad y = \frac{xp^2 + 4x}{2p} \Rightarrow y = \frac{px}{2} + \frac{2x}{p} \quad ...(1)$$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(p \,.\, 1 + x\frac{dp}{dx}\right) + 2\left(\frac{1}{p} \,.\, 1 - \frac{x}{p^2}\frac{dp}{dx}\right)$$

$$\Rightarrow \quad p = \frac{p}{2} + \frac{x}{2}\frac{dp}{dx} + \frac{2}{p} - \frac{2x}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{p}{2} - \frac{2}{p} = \left(\frac{x}{2} - \frac{2x}{p^2}\right)\frac{dp}{dx} \Rightarrow \frac{p^2 - 4}{2p} = x\left(\frac{p^2 - 4}{2p^2}\right)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{p^2 - 4}{2p} = \frac{x}{p}\left(\frac{p^2 - 4}{2p}\right)\frac{dp}{dx} \Rightarrow 1 = \frac{x}{p}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} = \frac{dx}{x}$$

NOTES

Integrating, we get

$$\log p = \log x + \log c \quad \Rightarrow \quad p = cx.$$

Putting the value of $p$ in (1), we get

$$y = \frac{cx \cdot x}{2} + \frac{2x}{cx}$$

$$\Rightarrow \qquad y = \frac{cx^2}{2} + \frac{2}{c} \quad \Rightarrow \quad 2cy = c^2x^2 + 4 \quad \Rightarrow \quad \mathbf{2cy - c^2x^2 - 4 = 0.}$$

This represents the general solution of (1).

Let $\qquad f(x, y, p) = xp^2 - 2yp + 4x$

$f(x, y, p)$ is quadratic in $p$.

$\therefore \qquad$ Disc. $= 0 \quad \Rightarrow \quad 4y^2 - 4 \,.\, x.\, 4x = 0 \quad \Rightarrow \quad y^2 - 4x^2 = 0 \qquad$ ...(2)

Let $\qquad \phi(x, y, c) = 2cy - c^2 x^2 - 4 \quad \Rightarrow \quad \phi(x, y, c) = -c^2x^2 + 2cy - 4$

$\therefore \qquad \phi(x, y, c)$ is quadratic in $c$.

$\therefore \qquad$ Disc. $= 0 \quad \Rightarrow \quad 4y^2 - 4(-x^2)(-4) = 0 \quad \Rightarrow \quad y^2 - 4x^2 = 0 \qquad$ ...(3)

Using (2), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$(y^2 - 4x^2) \cdot 1^2 \cdot 1 = 0.$

Using (3), the $c$-discriminant relation ($EN^2C^3 = 0$) can be written as

$(y^2 - 4x^2) \cdot 1^2 \cdot 1^3 = 0$

$\therefore \qquad E = 0 \quad \Rightarrow \quad y^2 - 4x^2 = 0. \qquad$ ...(4)

$$(4) \quad \Rightarrow \quad 2y\frac{dy}{dx} - 8x = 0 \quad \Rightarrow \quad p = \frac{4x}{y}$$

Putting the value of $p$ in (1), we get

$$x \cdot \frac{16x^2}{y^2} - 2y \cdot \frac{4x}{y} + 4x = 0 \quad \text{or} \quad 16x^3 - 4xy^2 = 0 \quad \text{or} \quad 4x(4x^2 - y^2) = 0,$$

which is true, because $y^2 - 4x^2 = 0$.

$\therefore \quad y^2 - 4x^2 = 0$ satisfies the given equation.

$\therefore \quad \mathbf{y^2 - 4x^2 = 0}$ is the singular solution of the given equation.

**Example 6.** *Find the singular solution of the equation* $xp^2 - (x - a)^2 = 0$.

**Sol.** We have $\qquad xp^2 - (x - a)^2 = 0. \qquad$ ...(1)

$$(1) \quad \Rightarrow \quad p^2 = \frac{(x-a)^2}{x} \quad \Rightarrow \quad p = \pm\frac{x-a}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\left(\sqrt{x} - \frac{a}{\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \sqrt{x} - \frac{a}{\sqrt{x}} \quad \Rightarrow \quad y = \frac{2x^{3/2}}{3} - 2a\sqrt{x} + c$$

$$\frac{dy}{dx} = -\left(\sqrt{x} - \frac{a}{\sqrt{x}}\right) \quad \Rightarrow \quad y = -\frac{2x^{3/2}}{3} + 2a\sqrt{x} + c.$$

$\therefore \quad$ General solution of (1) is

$$\left(y - \frac{2x^{3/2}}{3} + 2a\sqrt{x} - c\right)\left(y + \frac{2x^{3/2}}{3} - 2a\sqrt{x} - c\right) = 0$$

NOTES

$$\Rightarrow \quad (y-c)^2 - \left(\frac{2x^{3/2}}{3} - 2a\sqrt{x}\right)^2 = 0$$

$$\Rightarrow \quad (y-c)^2 - \frac{4x}{9}(x-3a)^2 = 0.$$

Let $\quad f(x, y, p) = xp^2 - (x-a)^2$

$f(x, y, p)$ is quadratic in $p$.

$\therefore \quad$ Disc. $= 0 \quad \Rightarrow \quad 0^2 - 4 \cdot x \cdot -(x-a)^2 = 0$

$$\Rightarrow \quad x(x-a)^2 = 0 \qquad \text{...(2)}$$

Let $\quad \phi(x, y, c) = (y-c)^2 - \frac{4x}{9}(x-3a)^2$

$$\Rightarrow \quad \phi(x, y, c) = c^2 - 2cy + \left(y^2 - \frac{4x}{9}(x-3a)^2\right)$$

$\therefore \quad \phi(x, y, c)$ is quadratic in $c$.

$$\text{Disc.} = 0 \quad \Rightarrow \quad 4y^2 - 4.1.\left(y^2 - \frac{4x}{9}(x-3a)^2\right)$$

$$= \frac{4x}{9}(x-3a)^2 = 0 \quad \Rightarrow \quad x(x-3a)^2 = 0 \qquad \text{...(3)}$$

Using (2), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$x.(x-a)^2 \cdot 1 = 0.$

Using (3), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$x \cdot (x-3a)^2 \cdot 1^3 = 0.$

$$\therefore \quad E = 0 \quad \Rightarrow \quad x = 0. \qquad \text{...(4)}$$

$x = 0$ satisfies the given equation when written in the form $\left(\frac{dx}{dy}\right)^2 = \frac{x}{(x-a)^2}$.

$\therefore$ **x = 0 is the singular solution of the given equation.**

**Remark.** $T = 0 \Rightarrow x - a = 0$ and $N = 0 \Rightarrow x - 3a = 0$.

$\therefore$ The equations of Tac-locus and Nodal locus are respectively $x - a = 0$ and $x - 3a = 0$.

**Example 7.** *Find the singular solution of the differential equation* $yp^2 - 2xp + y = 0$.

**Sol.** We have $yp^2 - 2xp + y = 0$.

$$\Rightarrow \quad x = \frac{yp}{2} + \frac{y}{2p} \qquad \text{...(1)}$$

Differentiating w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(p.1 + y\frac{dp}{dy}\right) + \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}\right)$$

$$\Rightarrow \quad \frac{1}{p} = \frac{p}{2} + \frac{y}{2}\frac{dp}{dy} + \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} - \frac{p}{2} = \left(\frac{y}{2} - \frac{y}{2p^2}\right)\frac{dp}{dy} \quad \Rightarrow \quad -\left(\frac{p}{2} - \frac{1}{2p}\right) = \frac{y}{p}\left(\frac{p}{2} - \frac{1}{2p}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -1 = \frac{y}{p}\frac{dp}{dy} \quad \Rightarrow \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

NOTES

Integrating, we get $\log y + \log p = \log c$.

$\Rightarrow \quad yp = c \quad \Rightarrow \quad p = \frac{c}{y}$

Putting the value of $p$ in (1), we get

$$x = \frac{y}{2}\cdot\frac{c}{y} + \frac{y}{2}\cdot\frac{y}{c} \quad \Rightarrow \quad x = \frac{c}{2} + \frac{y^2}{2c} \quad \Rightarrow \quad y^2 - 2cx + c^2 = 0.$$

This is the general solution of (1)

Let $\quad f(x, y, p) = yp^2 - 2xp + y.$

This is quadratic in $p$.

$\therefore \quad \text{Disc.} = 0 \Rightarrow 4x^2 - 4 \cdot y \cdot y = 0 \Rightarrow x^2 - y^2 = 0$ ...(2)

Let $\quad \phi(x, y, c) = y^2 - 2cx + c^2$

$\Rightarrow \quad \phi(x, y, c) = c^2 - 2xc + y^2$

This is quadratic in $c$.

$\therefore \quad \text{Disc.} = 0 \Rightarrow 4x^2 - 4 \cdot 1 \cdot y^2 = 0 \Rightarrow x^2 - y^2 = 0$ ...(3)

Using (2), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$(x^2 - y^2) \cdot 1^2 \cdot 1 = 0.$

Using (3), the $c$-discriminant relation ($EN^2C^3 = 0$) can be written as

$(x^2 - y^2) \cdot 1^2 \cdot 1^3 = 0.$

$\therefore \quad E = 0 \quad \Rightarrow \quad x^2 - y^2 = 0$ ...(4)

(3) $\Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad p = \frac{x}{y}$.

Putting the value of $p$ in (1), we get

$$x = \frac{y}{2}\cdot\frac{x}{y} + \frac{y}{2}\cdot\frac{y}{x}$$

or $\quad x = \frac{x}{2} + \frac{x^2}{2x}$, which is true. $[\because \; x^2 - y^2 = 0]$

$\therefore \quad x^2 - y^2 = 0$ satisfies the given equation.

$\therefore \quad \mathbf{x^2 - y^2 = 0}$ is the singular solution of the given equation.

**Example 8.** *Examine and investigate for singular solution of the differential equation :*

$$(8p^3 - 27)\,x = 12p^2\,y.$$

**Sol.** We have $\quad (8p^3 - 27)x = 12p^2y$ ...(1)

(1) $\Rightarrow \quad y = \frac{8p^3x - 27x}{12p^2} \quad \Rightarrow \quad y = \frac{2}{3}px - \frac{9x}{4p^2}$ ...(2)

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{2}{3}\left(p \cdot 1 + x\frac{dp}{dx}\right) - \frac{9}{4}\left(\frac{1}{p^2} - \frac{2x}{p^3}\frac{dp}{dx}\right)$$

$$\Rightarrow \quad p = \frac{2}{3}p + \frac{2x}{3}\frac{dp}{dx} - \frac{9}{4p^2} + \frac{9x}{2p^3}\frac{dp}{dx}$$

**NOTES**

$$\Rightarrow \quad \frac{p}{3}+\frac{9}{4p^2} = \left(\frac{2x}{3}+\frac{9x}{2p^3}\right)\frac{dp}{dx}$$

$$\Rightarrow \quad p\left(\frac{1}{3}+\frac{9}{4p^3}\right) = 2x\left(\frac{1}{3}+\frac{9}{4p^3}\right)\frac{dp}{dx}$$

$$\Rightarrow \quad p = 2x\frac{dp}{dx} \Rightarrow \frac{2dp}{p} = \frac{dx}{x}$$

Integrating, we get

$$2\log p = \log x + \log c \quad \Rightarrow \quad p^2 = cx \quad \Rightarrow \quad p = \sqrt{cx}$$

Putting the value of $p$ in (1), we get

$$(8\,cx\sqrt{cx} - 27)x = 12\,cxy$$

$$\Rightarrow \quad 8\,cx\sqrt{cx} - 27 = 12\,cy \quad \Rightarrow \quad 8\,cx\sqrt{cx} = 12cy + 27$$

$$\Rightarrow \quad 64\,c^3x^3 = (12\,cy + 27)^2 \quad \Rightarrow \quad x^3 = \frac{9}{64c^3}\,(4cy + 9)^2$$

$$\Rightarrow \quad x^3 = \frac{9(16c^2)}{64c^3}\left(y + \frac{9}{4c}\right)^2$$

$$\Rightarrow \quad x^3 = \frac{9}{4c}\left(y + \frac{9}{4c}\right)^2$$

$$\Rightarrow \quad x^3 = c_1(y + c_1)^2, \text{ where } c_1 = \frac{9}{4c}.$$

This represents the general solution of the given equation.

Let $\quad f(x, y, p) = (8p^3 - 27)x - 12p^2y$

$$\therefore \quad \frac{\partial f}{\partial p} = 24p^2x - 24py$$

$$\therefore \quad f(x, y, p) = 0 \quad \Rightarrow \quad (8p^3 - 27)x - 12p^2y = 0 \qquad \ldots(3)$$

and

$$\frac{\partial f}{\partial p} = 0 \quad \Rightarrow \quad 24p^2x - 24\,py = 0 \qquad \ldots(4)$$

$$(4) \Rightarrow p(px - y) = 0 \Rightarrow p = 0, \ \frac{y}{x}$$

$$p = 0 \text{ in } (3) \quad \Rightarrow \quad (0 - 27)x - 0 = 0 \quad \Rightarrow \quad x = 0$$

$$p = \frac{y}{x} \text{ in } (3) \quad \Rightarrow \quad \left(\frac{8y^3}{x^3} - 27\right)x - 12\left(\frac{y^2}{x^2}\right)y = 0$$

$$\Rightarrow \quad 8y^3 - 27x^3 - 12y^3 = 0$$

$$\Rightarrow \quad 4y^3 + 27x^3 = 0$$

$\therefore$ The eliminant of $f(x, y, p) = 0$ and $\dfrac{\partial f}{\partial p} = 0$ is $x(4y^3 + 27x^3) = 0 \qquad \ldots(5)$

Let $\quad \phi(x, y, c_1) = x^3 - c_1(y + c_1)^2$

$$\therefore \quad \frac{\partial \phi}{\partial c_1} = 0 - 1\cdot(y + c_1)^2 - c_1 \cdot 2(y + c_1)\cdot 1$$

$$\therefore \quad \phi(x, y, c_1) = 0 \quad \Rightarrow \quad x^3 - c_1(y + c_1)^2 = 0 \qquad \ldots(6)$$

and

$$\frac{\partial \phi}{\partial c_1} = 0 \quad \Rightarrow \quad -(y + c_1)(y + c_1 + 2c_1) = 0$$

$$\Rightarrow \quad (y + c_1)(y + 3c_1) = 0 \quad ...(7)$$

$$(7) \Rightarrow \quad c_1 = -y, -\frac{y}{3}$$

Putting $c_1 = -y$ in (6) $\Rightarrow$ $x^3 + y(y - y)^2 = 0 \Rightarrow x^3 = 0$

And $c_1 = -\frac{y}{3}$ in (6) $\Rightarrow$ $x^3 + \frac{y}{3}\left(y - \frac{y}{3}\right)^2 = 0 \Rightarrow 4y^3 + 27x^3 = 0$

$\therefore$ The eliminant of $\phi(x, y, c_1) = 0$ and $\frac{\partial\phi}{\partial c} = 0$ is $x^3(4y^3 + 27x^3) = 0$ ...(8)

Using (5), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$$(4y^3 + 27x^3) \cdot 1^2 \cdot x = 0$$

Using (8), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$$(4y^3 + 27x^3) \cdot 1^2 \cdot x^3 = 0$$

$$\therefore \quad E = 0 \quad \Rightarrow \quad 4y^3 + 27x^3 = 0 \quad ...(9)$$

$$(9) \Rightarrow \quad y = -\frac{3}{4^{1/3}}x \quad \text{and} \quad p = -\frac{3}{4^{1/3}}.$$

Putting the values of $y$ and $p$ in (1), we get

$$\left[8\left(-\frac{27}{4}\right) - 27\right]x = 12 \cdot \frac{9}{4^{2/3}}\left(-\frac{3}{4^{1/3}}\right)x$$

$$\Rightarrow \quad -81x = -81x, \text{ which is true.}$$

$\therefore$ $4y^3 + 27x^3 = 0$ satisfies the given equation

$\therefore$ $\mathbf{4y^3 + 27x^3 = 0}$ is the singular solution of the given equation.

**Example 9.** *Find the general and singular solutions of the differential equation $p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$. Also interpret the result geometrically.*

**Sol.** We have $\quad p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0.$ ...(1)

Dividing by $\cos^2 \alpha$, we get $p^2y^2 - 2pxy \tan^2 \alpha + y^2 \sec^2 \alpha - x^2 \tan^2 \alpha = 0$

This is quadratic in $p$.

$$\therefore \quad p = \frac{2xy \tan^2 \alpha \pm \sqrt{4x^2y^2 \tan^4 \alpha - 4y^4 \sec^2 \alpha + 4x^2y^2 \tan^2 \alpha}}{2y^2}$$

$$= \frac{2xy \tan^2 \alpha \pm 2y\sqrt{x^2 \tan^4 \alpha - y^2 \sec^2 \alpha + x^2 \tan^2 \alpha}}{2y^2}$$

$$= \frac{x \tan^2 \alpha \pm \sqrt{x^2 \tan^2 \alpha (1 + \tan^2 \alpha) - y^2 \sec^2 \alpha}}{y}$$

$$= \frac{x \tan^2 \alpha \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}}{y}.$$

$$\Rightarrow \quad yp = x \tan^2 \alpha \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}$$

$$\Rightarrow \quad y\frac{dy}{dx} - x \tan^2 \alpha = \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}$$

$$\Rightarrow \quad \pm \frac{y\,dy - x \tan^2 \alpha\,dx}{\sqrt{x^2 \tan^2 \alpha - y^2}} = \sec \alpha\,dx$$

**NOTES**

or $$\pm \frac{x \tan^2 \alpha\, dx - y\,dy}{\sqrt{x^2 \tan^2 \alpha - y^2}} = -\sec \alpha\, dx$$

or $$\pm \frac{\frac{1}{2} d(x^2 \tan^2 \alpha - y^2)}{\sqrt{x^2 \tan^2 \alpha - y^2}} = -\sec \alpha\, dx$$

Integrating, we get

$$\pm \sqrt{x^2 \tan^2 \alpha - y^2} = -\sec \alpha \cdot x + c$$

Squaring, $$x^2 \tan^2 \alpha - y^2 = \sec^2 \alpha\, x^2 + c^2 - 2cx \sec \alpha$$

or $$\mathbf{x^2 + y^2 - 2cx \sec \alpha + c^2 = 0.}$$

This is the general solution of the given equation.

Let $f(x, y, p) = p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2\sin^2 \alpha$

This is quadratic in $p$.

$\therefore$ Disc. = 0 $\Rightarrow$ $(-2xy \sin^2 \alpha)^2 - 4y^2 \cos^2 \alpha\, (y^2 - x^2 \sin^2 \alpha) = 0$

$\Rightarrow$ $4y^2[x^2 \sin^4 \alpha - y^2 \cos^2 \alpha + x^2 \sin^2 \alpha \cos^2 \alpha] = 0$

$\Rightarrow$ $y^2\, [x^2 \sin^2 \alpha\, (\sin^2 \alpha + \cos^2 \alpha) - y^2 \cos^2 \alpha] = 0$

$\Rightarrow$ $y^2\, (x^2 \sin^2 \alpha - y^2 \cos^2 \alpha) = 0$

$\Rightarrow$ $y^2 \cos^2 \alpha\, (x^2 \tan^2 \alpha - y^2) = 0$

$\Rightarrow$ $y^2(x^2 \tan^2 \alpha - y^2) = 0.$ ...(2)

Let $\phi(x, y, c) = x^2 + y^2 - 2cx \sec \alpha + c^2$

$\Rightarrow$ $\phi(x, y, c) = c^2 - (2x \sec \alpha)\, c + (x^2 + y^2)$

This is quadratic in $c$.

$\therefore$ Disc. = 0 $\Rightarrow$ $4x^2 \sec^2 \alpha - 4 \cdot 1 \cdot (x^2 + y^2) = 0.$

$\Rightarrow$ $4x^2(\sec^2 \alpha - 1) - 4y^2 = 0$ $\Rightarrow$ $x^2 \tan^2 \alpha - y^2 = 0.$ ...(3)

Using (2), the $p$-discriminant relation ($ET^2C = 0$) can be written as

$$(x^2 \tan^2 \alpha - y^2)\, (y)^2 \cdot 1 = 0.$$

Using (3), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$$(x^2 \tan^2 \alpha - y^2) \cdot 1^2 \cdot 1^3 = 0.$$

$\therefore$ E = 0 $\Rightarrow$ $x^2 \tan^2 \alpha - y^2 = 0.$ ...(4)

(4) $\Rightarrow$ $2x \tan^2 \alpha = 2yp$ $\Rightarrow$ $p = \dfrac{x \tan^2 \alpha}{y}$

Putting the value of $p$ in (1), we get

$$\left(\frac{x^2 \tan^4 \alpha}{y^2}\right) y^2 \cos^2 \alpha - 2\left(\frac{x \tan^2 \alpha}{y}\right) xy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$$

$\Rightarrow$ $$\frac{x^2 \sin^4 \alpha}{\cos^2 \alpha} - \frac{2x^2 \sin^4 \alpha}{\cos^2 \alpha} + x^2 \tan^2 \alpha - x^2 \sin^2 \alpha = 0$$

$\Rightarrow$ $x^2 \sin^4 \alpha - 2x^2 \sin^4 \alpha + x^2 \sin^2 \alpha - x^2 \sin^2 \alpha \cos^2 \alpha = 0$

$\Rightarrow$ $x^2 \sin^2 \alpha\, (\sin^2 \alpha - 2 \sin^2 \alpha + 1 - \cos^2 \alpha) = 0$

$\Rightarrow$ $0 = 0$, which is true.

$\therefore$ $x^2 \tan^2 \alpha - y^2 = 0$ satisfies the given equation.

$\therefore$ $\mathbf{x^2 \tan^2 \alpha - y^2 = 0}$ is the singular solution of the given equation. The singular solution represents a pair of straight lines $y = \pm x \tan \alpha$. The general solution represents a family of circles whose envelope is the pair of lines $y = \pm x \tan \alpha$.

NOTES

**Example 10.** *Reduce the equation $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$ to Clairaut's form by substituting $x^2 = u$ and $y^2 = v$. Hence show that the general solution of the equation represents a family of conics touching the four sides of a square.*

**Sol.** Given equation is $\quad xyp^2 - (x^2 + y^2 - 1)\, p + xy = 0.$ ...(1)

We have $\quad u = x^2 \quad \text{and} \quad v = y^2$

$$\therefore \quad \frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = 2y \frac{dy}{dx} = 2yp$$

$$\therefore \quad \frac{dv}{du} = \frac{dv}{dx} \bigg/ \frac{du}{dx} = \frac{2yp}{2x} = \frac{y}{x} p$$

Let $\quad P = \dfrac{dv}{du}. \quad \therefore \quad P = \dfrac{y}{x} p \quad \text{or} \quad p = \dfrac{x}{y} P \quad \text{or} \quad p = \dfrac{\sqrt{u}}{\sqrt{v}} P$

$$\therefore \quad (1) \Rightarrow \sqrt{u}\, \sqrt{v}\, .\, \frac{u}{v}\, P^2 - (u + v - 1) \frac{\sqrt{u}}{\sqrt{v}}\, P + \sqrt{u}\, \sqrt{v} = 0$$

$$\Rightarrow \quad \frac{\sqrt{u}}{\sqrt{v}} (uP^2 - (u + v - 1)\, P + v) = 0$$

$$\Rightarrow \quad uP^2 - (u + v - 1)\, P + v = 0$$

$$\Rightarrow \quad u(P^2 - P) - v(P - 1) + P = 0$$

$$\Rightarrow \quad uP - v + \frac{P}{P - 1} = 0$$

$$\Rightarrow \quad v = Pu + \frac{P}{P-1} \qquad \text{...(2)}$$

This is a Clairaut's equation.

By replacing P by $c$, the general solution of (2) is $\quad v = cu + \dfrac{c}{c-1}.$

$$\Rightarrow \quad y^2 = cx^2 + \frac{c}{c-1} \quad \Rightarrow \quad (y^2 - cx^2)\,(c - 1) = 0$$

$$\Rightarrow \quad (c - c^2)\, x^2 + (c - 1)\, y^2 - c = 0.$$

This is the general solution of (1).

This represents a family of conics.

Let $\quad f(x, y, p) = xyp^2 - (x^2 + y^2 - 1)p + xy.$

This is quadratic in $p$.

$$\therefore \quad \text{Disc.} = 0 \quad \Rightarrow \quad (x^2 + y^2 - 1)^2 - 4\, xy\, .\, xy = 0$$

$$\Rightarrow \quad (x^2 + y^2 - 1 - 2xy)(x^2 + y^2 - 1 + 2xy) = 0$$

$$\Rightarrow \quad ((x - y)^2 - 1)((x + y)^2 - 1) = 0$$

$$\Rightarrow \quad (x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0$$

Let $\quad \phi(x, y, c) = (c - c^2)x^2 + (c - 1)\, y^2 - c$

$$\Rightarrow \quad \phi(x, y, c) = -c^2x^2 + (x^2 + y^2 - 1)\, c - y^2$$

This is quadratic in $c$.

$$\therefore \quad \text{Disc.} = 0 \quad \Rightarrow \quad (x^2 + y^2 - 1)^2 - 4(-x^2)(-y^2) = 0$$

NOTES

$$\Rightarrow \quad (x^2 + y^2 - 1)^2 - 4x^2y^2 = 0$$
$$\Rightarrow \quad (x^2 + y^2 - 1 - 2xy)(x^2 + y^2 - 1 + 2xy) = 0$$
$$\Rightarrow \quad ((x - y)^2 - 1)\,((x + y)^2 - 1) = 0$$
$$\Rightarrow \quad (x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0.$$

Since $p$-discriminant relation and $c$-discriminant relation are same, we have

$$E = 0 \quad \Rightarrow \quad (x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0 \qquad ...(3)$$

$(3) \Rightarrow x - y - 1 = 0$ ...(4) $\qquad x - y + 1 = 0$ ...(5)

$x + y - 1 = 0$ ...(6) and $\qquad x + y + 1 = 0$ ...(7)

$(4) \Rightarrow 1 - \dfrac{dy}{dx} - 0 = 0 \quad \Rightarrow \quad p = 1.$ Also, $(4) \Rightarrow y = x - 1.$

Putting the values of $p$ and $y$ in (1) we get

$$x(x - 1).1^2 - (x^2 + (x - 1)^2 - 1)(1) + x(x - 1) = 0$$

$\Rightarrow \quad x^2 - x - 2x^2 + 2x + x^2 - x = 0,$ which is true.

$\therefore$ (4) satisfies the given equation.

Similarly, (5), (6), (7) satisfy the given equation.

$\therefore$ The singular solution of the given equation is

$$(x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0.$$

This represents four lines $x - y - 1 = 0$, $x - y + 1 = 0$, $x + y - 1 = 0$ and $x + y + 1 = 0$ covering a square.

Since the singular solution is the envelope of the curves given by the general solution, the conics given by the general solution are touched by the lines given by the singular solution.

## EXERCISE 1

*Investigate the following differential equations for singular solutions and extraneous loci (Q. No. 1–12):*

1. $p^3 + px - y = 0$
2. $4xp^2 - (3x - 1)^2 = 0$
3. $y = 2xp - yp^2$
4. $6p^2y^2 + 3px - y = 0$
5. $9yp^2 + 4 = 0$
6. $xp^2 - 2yp + x + 2y = 0$
7. $p^3 - 4xyp + 8y^2 = 0$
8. $x^3p^2 + x^2yp + a^3 = 0$
9. $2p^2 - 2px^2 + 3xy = 0$
10. $(a^2 - x^2)p^2 + 2xyp + b^2 - y^2 = 0$
11. $\sin px \cos y = \cos px \sin y + p$
12. $p = \log (px - y)$.
13. Reduce the equation $y^2(y - xp) = x^4 p^2$ to Clairaut's form by using the substitutions $x = \dfrac{1}{u}$, $y = \dfrac{1}{v}$ and hence find its singular solution and equations of extraneous loci.
14. Transform $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by the transformations $x^2 = u$, $y - x = v$. Hence obtain and interpret the general solution and singular solution of the given equation. Show that the general solution represents a family of parabolas touching a pair of straight lines.
15. Reduce the differential equation $x^2p^2 + yp(2x + y) + y^2 = 0$, where $p = \dfrac{dy}{dx}$, to Clairaut's form by the substitutions $u = y$, $v = xy$. Hence or otherwise solve it. Also prove that $y + 4x = 0$ is singular solution of the given equation.

16. Reduce $(px-y)(x-py)=2p$ to Clairaut's form by using the substitutions $x^2=u$ and $y^2=v$ and find the general solution and its singular solution, if any.

NOTES

## Answers

1. S.S. $27y^2+4x^3=0$
2. S.S. $x=0$, T.L. $3x-1=0$, N.L. $x-1=0$
3. S.S. $x^2-y^2=0$
4. S.S. $3x^2+8y^3=0$
5. C.L. $y=0$
6. S.S. $x^2+2xy-y^2=0$
7. S.S. $y(27y-4x^3)=0$, T.L. $y=0$
8. S.S. $x(xy^2-4a^3)=0$, T.L. $x=0$
9. S.S. $x^3-6y=0$, C.L. $x=0$
10. S.S. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
11. S.S. $y=\sqrt{x^2-1}-\sin^{-1}\dfrac{\sqrt{x^2-1}}{x}$
12. S.S. $x+y-x\log x=0$
13. S.S. $y(y+4x^2)=0$, T.L. $xy=0$
14. G.S. $2c^2x^2-2c(y-x)+1=0$, S.S. $y=(\sqrt{2}+1)x$, $y=(-\sqrt{2}+1)x$
15. G.S. $xy=cy+c^2$
16. G.S. $(1-c)y^2-c(1-c)x^2+2c=0$, S.S. $(x^2+y^2-2)^2-4x^2y^2=0$.

## Hints

6. Try $u=x^2$ and $v=y-x$.

10. Given equation is expressible as $y=px\pm\sqrt{a^2p^2+b^2}$

13. The reduced equation is $v=Pu+P^2$, where $P=\dfrac{dv}{du}$

14. $u=x^2 \Rightarrow \dfrac{du}{dx}=2x$, $v=y-x \Rightarrow \dfrac{dv}{dx}=p-1$

$\therefore \quad P=\dfrac{dv}{du}=\dfrac{p-1}{2x}$ or $p=2xP+1$.

The given equation reduces to $v=Pu+\dfrac{1}{2P}$

15. $u=y \Rightarrow \dfrac{du}{dx}=p$, $v=xy \Rightarrow \dfrac{dv}{dx}=y+xp$

$\therefore \quad P=\dfrac{dv}{du}=\dfrac{y+xp}{p}$ or $p=\dfrac{y}{P-x}$

The given equation reduces to $v=Pu+P^2$.

16. $u=x^2 \Rightarrow \dfrac{du}{dx}=2x$, $v=y^2 \Rightarrow \dfrac{dv}{dx}=2yp$

$\therefore \quad P=\dfrac{dv}{du}=\dfrac{y}{x}p$ or $p=\dfrac{x}{y}P$

The given equation reduces to $v=Pu-\dfrac{2P}{1-P}$

# UNIT 5 ORTHOGONAL TRAJECTORIES

NOTES



## 5.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Define orthogonal trajectories
- Explain working rule to find the equation of orthogonal trajectories

## 5.1. ORTHOGONAL TRAJECTORIES

(*i*) **Trajectory.** A curve which cuts every member of a given family of curves according to some definite law is called a **trajectory** of the family.

(*ii*) **Orthogonal trajectory.** A curve which cuts every member of a given family of curves at right angle is called an **orthogonal trajectory** of the family.

(*iii*) **Self orthogonal family.** A family of curves is called a **self orthogonal family** if the family of orthogonal trajectories of this family coincides with the given family.

## 5.2. WORKING RULES TO FIND THE EQUATION OF ORTHOGONAL TRAJECTORIES

(*a*) **Let the family of cartesian curves be f(x, y, c) = 0.** ...(1)

**Step I.** Differentiate (1) w.r.t. $x$.

**Step II.** Eliminate the arbitrary constant $c$ between (1) and the resulting equation.

This gives the differential equation of the family (1).

Let it be $$F\left(x, y, \frac{dy}{dx}\right) = 0. \qquad ...(2)$$

**Step III.** Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2).

NOTES

The differential equation of the orthogonal trajectories is

$$F\left(x, y, -\frac{dx}{dy}\right) = 0. \qquad ...(3)$$

**Step IV.** Solve (3) to get the equation of the required orthogonal trajectories.

(*b*) **Let the family of polar curves be f (r, θ, c) = 0.** ...(1)

**Step I.** Differentiate (1) w.r.t. θ.

**Step II.** Eliminate the arbitrary constant c between (1) and the resulting equation.

This gives the differential equation of the family (1).

Let it be $$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0. \qquad ...(2)$$

**Step III.** Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (2).

The differential equation of the orthogonal trajectories is

$$F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0. \qquad ...(3)$$

**Step IV.** Solve (3) to get the equation of the required orthogonal trajectories.

**Example 1.** *Find the equation of orthogonal trajectories of the family of parabolas*

$$y^2 = 4ax.$$

**Sol.** The equation of the family of parabolas is

$$y^2 = 4ax. \qquad ...(1)$$

**Step I.** Differentiating (1) with respect to $x$, we get

$$2y\frac{dy}{dx} = 4a \quad i.e. \quad y\frac{dy}{dx} = 2a \qquad ...(2)$$

**Step II.** Eliminating $a$ between (1) and (2), we get

$$y^2 = 2y\frac{dy}{dx}x \quad \text{or} \quad y = 2x\frac{dy}{dx}. \qquad ...(3)$$

**Step III.** Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), the differential equation of orthogonal trajectories is

$$y = -2x\frac{dx}{dy}.$$

$$\Rightarrow \quad y\,dy = -2x\,dx \qquad ...(4)$$

**Step IV.** Integrating (4), we have

$$\int y\,dy = -2\int x\,dx + c_1 \quad \Rightarrow \quad \frac{y^2}{2} = -x^2 + c_1$$

$$\Rightarrow \quad 2x^2 + y^2 = k. \qquad (k = 2c_1)$$

This is the equation of the orthogonal trajectories.

**Example 2.** *Find the orthogonal trajectories of the family of curves* $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$, *where* $\lambda$ *is a parameter.*

**NOTES**

**Sol.** The equation of the given family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1. \qquad \text{...(1)}$$

**Step I.** Differentiating (1) w.r.t. $x$, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

**Step II.** To eliminate the parameter $\lambda$, we equate the values of $b^2 + \lambda$ from (1) and (2).

From (1), $$\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2} \quad \text{or} \quad b^2+\lambda = \frac{a^2y^2}{a^2-x^2}$$

From (2), $$b^2+\lambda = -\frac{a^2y}{x}\frac{dy}{dx}$$

$$\therefore \quad \frac{a^2y^2}{a^2-x^2} = -\frac{a^2y}{x}\frac{dy}{dx} \quad \text{or} \quad \frac{xy}{a^2-x^2} + \frac{dy}{dx} = 0. \qquad \text{...(3)}$$

This is the differential equation of the given family (1).

**Step III.** Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$\frac{xy}{a^2-x^2} - \frac{dx}{dy} = 0 \quad \text{or} \quad y\,dy - \left(\frac{a^2-x^2}{x}\right)dx = 0. \qquad \text{...(4)}$$

This is the differential equation of the orthogonal trajectories.

**Step IV.** Integrating (4), we get

$$\int y\,dy - \int\left(\frac{a^2}{x} - x\right)dx = c \quad \text{or} \quad \frac{y^2}{2} - a^2\log x + \frac{x^2}{2} = c$$

or $$\mathbf{x^2 + y^2 = 2a^2 \log x + C.} \qquad \mathbf{(C = 2c)}$$

This is the equation of the orthogonal trajectories of (1).

**Example 3.** *Prove that the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal.*

**Sol.** The equation of the given family of parabolas is

$$y^2 = 4a(x + a). \qquad \text{...(1)}$$

**Step I.** Differentiating (1) w.r.t. $x$, we get

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad y\frac{dy}{dx} = 2a \qquad \text{...(2)}$$

**Step II.** Eliminating $a$ between (1) and (2), we have

$$y^2 = 4\cdot\frac{y}{2}\frac{dy}{dx}\left[x + \frac{y}{2}\cdot\frac{dy}{dx}\right] = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0. \qquad \text{...(3)}$$

This is the differential equation of the given family (1).

**Step III.** Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get $y\left(-\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0$

or $$y - 2x\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{or} \quad y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0. \qquad ...(4)$$

This is the differential equation of the orthogonal trajectories.

Since (4) is the same as (3), the orthogonal trajectories of the given family (1) are the curves of this family itself.

**NOTES**

$\therefore$ The given family of parabolas is self orthogonal.

**Example 4.** *Prove that the system of confocal conics*

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \text{ is self orthogonal.}$$

**Sol.** The system of confocal conics is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1. \qquad ...(1)$$

**Step I.** Differentiating (1) w.r.t. $x$, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

or $$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\, p = 0, \quad \text{where } p = \frac{dy}{dx} \qquad ...(2)$$

**Step II.** (2) $\Rightarrow$ $(b^2+\lambda)\,x + (a^2+\lambda)\,yp = 0$ or $\lambda(x+yp) = -(a^2yp + b^2x)$

$$\therefore \qquad \lambda = -\frac{a^2yp + b^2x}{x+yp}$$

$$\therefore \qquad a^2+\lambda = \frac{(a^2-b^2)\,x}{x+yp}, \quad b^2+\lambda = -\frac{(a^2-b^2)\,yp}{x+yp} \qquad ...(3)$$

$\therefore$ From (1), eliminating $\lambda$, we get

$$\frac{x^2(x+yp)}{(a^2-b^2)x} - \frac{y^2(x+yp)}{(a^2-b^2)yp} = 1 \quad \text{or} \quad (x+yp)\left(x - \frac{y}{p}\right) = a^2 - b^2. \qquad ...(4)$$

This is the differential equation of the family of curves (1).

**Step III.** Replacing $p$ by $-\frac{1}{p}$, the differential equation of the orthogonal trajectories is

$$\left(x - \frac{y}{p}\right)(x+yp) = a^2 - b^2. \qquad ...(5)$$

This is the same as (4).

Since the differential equation of the family of curves (1) and that of the orthogonal trajectories are same, therefore, the orthogonal trajectories of the given family are the curves of this family itself.

$\therefore$ The family of curves (1) is self orthogonal.

**Example 5.** *Find the equation of orthogonal trajectories of the family of cardioids*

$$r = a(1 - \cos\theta).$$

**Sol.** The equation of the given family of cardioids is

$$r = a(1 - \cos\theta). \qquad ...(1)$$

**NOTES**

**Step I.** Differentiating (1) w.r.t. θ, we get $\frac{dr}{d\theta} = a \sin\theta$ ...(2)

**Step II.** Dividing (2) by (1) [to eliminate $a$], we get

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}. \quad ...(3)$$

**Step III.** Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3) we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta} \Rightarrow \frac{1-\cos\theta}{\sin\theta}\,d\theta = -\frac{dr}{r}. \quad ...(4)$$

This is the differential equation of the orthogonal trajectories.

**Step IV.** Integrating (4), we get

$$\int\frac{1-\cos\theta}{\sin\theta}\,d\theta = -\int\frac{dr}{r} + \log k$$

$$\Rightarrow \quad \int\frac{\sin\theta}{1+\cos\theta}\,d\theta = -\log r + \log k$$

$$\Rightarrow \quad -\log(1+\cos\theta) = -\log r + \log k \Rightarrow \mathbf{r = k(1 + \cos\theta)}.$$

This is the equation of the orthogonal trajectories of (1).

**Example 6.** *Find the equation of orthogonal trajectories of the system of curves $r^n \sin n\theta = a^n$, where $a$ is a parameter.*

**Sol.** The given system of curves is $r^n \sin n\theta = a^n$ ...(1)

**Step I.** Taking 'logs' of both sides, we get

$$n\log r + \log\sin n\theta = n\log a.$$

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{n\cos n\theta}{\sin n\theta} = 0 \quad \text{or} \quad \frac{dr}{d\theta} = -r\cot n\theta. \quad ...(2)$$

This is the differential equation of the family (1) and it does not contain '$a$'.

**Step II.** Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2), we get

$$-r^2\frac{d\theta}{dr} = -r\cot n\theta \text{ i.e., } \quad r\frac{d\theta}{dr} = \cot n\theta$$

$$\therefore \quad \frac{dr}{r} = \tan n\theta\, d\theta \quad ...(3)$$

**Step III.** Integrating (3), we get

$$\int\frac{dr}{r} = \int \tan n\theta\, d\theta + c'$$

$$\therefore \quad \log r = \frac{1}{n}\log\sec n\theta + \log c, \qquad \text{where } c' = \log c$$

or $\quad n\log r = \log\sec n\theta + n\log c \quad$ or $\quad \log r^n = \log\sec n\theta + \log c^n$

i.e., $\quad r^n = c^n\sec n\theta \quad$ or $\quad \mathbf{r^n \cos n\theta = c^n}$.

This is the equation of the orthogonal trajectories of (1).

**Example 7.** *Find the equation of orthogonal trajectories of the family of curves $r = a(\sec\theta + \tan\theta)$ where $a$ is a parameter.*

**Sol.** Given family of curves is $r = a(\sec\theta + \tan\theta)$. ...(1)

**Step I.** Differentiating (1) w.r.t. θ, we get

$$\frac{dr}{d\theta} = a\,(\sec\theta\tan\theta + \sec^2\theta)$$

$$\Rightarrow \qquad \frac{dr}{d\theta} = a\sec\theta\,(\tan\theta + \sec\theta)$$

**Step II.** Putting $a(\tan\theta + \sec\theta)$ equal to $r$, we get

$$\frac{dr}{d\theta} = r\sec\theta \qquad \ldots(2)$$

**Step III.** Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2), we get

$$-r^2\frac{d\theta}{dr} = r\sec\theta \;\Rightarrow\; -\cos\theta\, d\theta = \frac{dr}{r}$$

**Step IV.** Integrating we get

$$-\sin\theta = \log r + \log k$$

$$\Rightarrow \qquad rk = e^{-\sin\theta} \;\Rightarrow\; \mathbf{r} = \frac{\mathbf{1}}{\mathbf{k}}\mathbf{e}^{-\sin\theta}$$

This is the equation of the orthogonal trajectories of (1).

## EXERCISE 1

**1.** Find the orthogonal trajectories of the following families of curves :

(i) $xy = c$ (ii) $y = ax^2$ (iii) $y = ax^3$

(iv) $y = ax^n$ (v) $ay^2 = x^3$ (vi) $x^2 + y^2 = a^2$.

**2.** Find the orthogonal trajectories of the following families of curves :

(i) $x + 2y = c$ (ii) $x^2 + 2y^2 = c$ (iii) $y = ce^{-2x}$

(iv) $y = x - 1 + ce^{-x}$ (v) $y^2 = 2x^2(1 - cx)$ (vi) $y^2 = \frac{x^3}{c - x}$

(vii) $px^2 + qy^2 = a^2$, where $p$ and $q$ are fixed constants.

**3.** Find the orthogonal trajectories of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, where λ is a parameter.

**4.** Show that the family of parabolas $x^2 = 4a(y + a)$ is self-orthogonal.

**5.** Find the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$.

**6.** (i) Find the orthogonal trajectories of the family of circles $x^2 + y^2 + 2gx + c = 0$, $c$ being a parameter.

(ii) Find the orthogonal trajectories of the family of circles $x^2 + y^2 + 2gx + c = 0$, $g$ being a parameter.

**7.** Find the orthogonal trajectories of the following families of curves :

(i) $r = a\cos\theta$ (ii) $r = a(1 + \cos\theta)$ (iii) $r = a + \sin 5\theta$

(iv) $r = a(1 + \sin\theta)$

**8.** Find the orthogonal trajectories of the family of curves $px^2 + qy^2 = a^2$, where $p$ and $q$ are fixed constants.

**NOTES**

**9.** Find the orthogonal trajectories of the following families of curves :

(*i*) $r^2 = a^2 \cos 2\theta$ (*ii*) $r^n = a^n \cos n\theta$ (*iii*) $r = \dfrac{2a}{1 + \cos\theta}$ (*iv*) $r = a^{\theta}$.

**10.** Find the orthogonal trajectories of the family of curves $\left(r + \dfrac{k^2}{r}\right)\cos\theta = a$, where $a$ is a parameter.

## Answers

**1.** (*i*) $x^2 - y^2 = k$ (*ii*) $x^2 + 2y^2 = c$ (*iii*) $x^2 + 3y^2 = c$

(*iv*) $x^2 + ny^2 = c$ (*v*) $2x^2 + 3y^2 = c$ (*vi*) $y = cx$

**2.** (*i*) $y - 2x = k$ (*ii*) $y = kx^2$ (*iii*) $y^2 = x + k$

(*iv*) $x = y - 1 + ke^{-y}$ (*v*) $x^3 + 3y^2 \log (ky) = 0$ (*vi*) $(x^2 + y^2)^2 = k(2x^2 + y^2)$

(*vii*) $y^p = cx^q$

**3.** $x^2 + y^2 = 2a^2 \log x + c$ 5. $x^{4/5} - y^{4/3} = c$ 6. (*i*) $x - ky + g = 0$

(*ii*) $x^2 + y^2 - ky - c = 0$ 7. (*i*) $r = c \sin\theta$ (*ii*) $r = c(1 - \cos\theta)$

(*iii*) $r \log (\sec 5\theta + \tan 5\theta) + cr = 25$ (*iv*) $r = c(1 - \sin\theta)$

**8.** $y^p = cx^q$ 9. (*i*) $r^2 = c^2 \sin 2\theta$ (*ii*) $r^n = c^n \sin n\theta$

(*iii*) $r = \dfrac{c}{1 - \cos\theta}$ (*iv*) $r = e^{\sqrt{k - \theta^2}}$ 10. $r^2 = k^2 + cr \operatorname{cosec}\theta$.

## Hints

**2.** (*vi*) $y^2 = \dfrac{x^3}{c - x} \Rightarrow y^2(c - x) = x^3 \Rightarrow 2yy'(c - x) + y^2(-1) = 3x^2$

$$\Rightarrow \quad 2yy' \cdot \frac{x^3}{y^2} - y^2 = 3x^2 \Rightarrow y' = \frac{3x^2y + y^3}{2x^3}.$$

Replacing $y'$ by $-\dfrac{dx}{dy}$, we get

$$-\frac{dx}{dy} = \frac{3x^2y + y^3}{2x^3} \quad \text{or} \quad \frac{dy}{dx} = -\frac{2x^3}{3x^2y + y^3}.$$

This is a homogeneous differential equation.

**6.** (*i*) $x^2 + y^2 + 2gx + c = 0 \Rightarrow 2x + 2yy' + 2g = 0 \Rightarrow y' = -\dfrac{x + g}{y}$

Replacing $y'$ by $-\dfrac{dx}{dy}$, we get $-\dfrac{dx}{dy} = -\dfrac{x + g}{y}$.

$$\Rightarrow \quad \frac{dx}{x + g} = \frac{dy}{y} \quad \Rightarrow \log (x + g) = \log y + \log k$$

(*ii*) $x^2 + y^2 + 2gx + c = 0 \Rightarrow 2x + 2yy' + 2g = 0$

$\therefore \quad x^2 + y^2 + (-2x - 2yy')\, x + c = 0 \Rightarrow y^2 - x^2 - 2xyy' + c = 0$

Replacing $y'$ by $-\dfrac{dx}{dy}$, we get $y^2 - x^2 + 2xy\dfrac{dx}{dy} + c = 0$ or $2x\dfrac{dx}{dy} - \dfrac{x^2}{y} = -y - \dfrac{c}{y}$

$$z = x^2 \Rightarrow \frac{dz}{dy} = 2x\frac{dx}{dy} \quad \therefore \quad \frac{dz}{dy} + z\left(-\frac{1}{y}\right) = -y - \frac{c}{y}.$$

This is a linear equation.

# UNIT 6 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

**STRUCTURE**



## 6.0. LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Define linear differently equation
- Describe linearly independent functions
- Explain general solution of linear differential equation with constant coefficients
- Describe general method of evaluating particular integral

## 6.1. INTRODUCTION

Till now, we have been discussing ordinary differential equations of order first. These differential equations were of degree one or more than one. Now, we shall consider the solution of differential equations of order more than one. A differential equation of order more than one may or may not be linear. In the present chapter, we shall consider linear differential equations of order more than one and with constant coefficients.

NOTES

## 6.2. LINEAR DIFFERENTIAL EQUATION

A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a **linear differential equation.**

A linear differential equation of order $n$ is of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = Q \qquad \ldots(1)$$

where $P_0 \neq 0, P_1, P_2, \ldots\ldots, P_n, Q$ are functions of $x$.

For example,

$$x^2 \frac{d^3 y}{dx^3} + \sin^2 x \frac{d^2 y}{dx^2} + 6y = 1 + \cos x$$

is a linear differential equation of order 3.

In particular, if $P_0, P_1, P_2, \ldots\ldots, P_n$ are all constants then the equation (1) is called a **linear differential equation with constant coefficients.**

For example,

$$2\frac{d^4 y}{dx^4} + 7\frac{d^3 y}{dx^3} + 9\frac{dy}{dx} + 15y = e^{4x}$$

is a linear differential equation with constant coefficients. The order of this differential equation is 4.

## 6.3. DIFFERENTIAL OPERATOR

In calculus the symbols $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots$ stand respectively for first derivative w.r.t. $x$, second derivative w.r.t. $x$, third derivative w.r.t. $x$, ...... We denote the symbols $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots$ respectively by $D, D^2, D^3, \ldots\ldots$ The symbols $D, D^2, D^3, \ldots\ldots$ are called **operators.** It may be verified that these operators satisfy the following laws :

(*i*) $D^m + D^n = D^n + D^m$ (*ii*) $D^m D^n = D^n D^m = D^{m+n}$

(*iii*) $D(u + v) = Du + Dv$ (*iv*) $(D - \alpha)(D - \beta) = (D - \beta)(D - \alpha)$.

Let us verify law (*iv*)

$$(D - \alpha)(D - \beta)\, y = (D - \alpha)(Dy - \beta y)$$

$$= (D - \alpha)\left(\frac{dy}{dx} - \beta y\right) = D\left(\frac{dy}{dx} - \beta y\right) - \alpha\left(\frac{dy}{dx} - \beta y\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \beta y\right) - \alpha\frac{dy}{dx} + \alpha\beta y = \frac{d^2 y}{dx^2} - \beta\frac{dy}{dx} - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d^2y}{dx^2} - (\beta + \alpha)\frac{dy}{dx} + \alpha\beta y$$

Similarly, we can show that

$$(D - \beta)(D - \alpha)\, y = \frac{d^2y}{dx^2} - (\beta + \alpha)\frac{dy}{dx} + \alpha\beta\,.$$

$\therefore \qquad (D - \alpha)(D - \beta)y = (D - \beta)(D - \alpha)y$

$\Rightarrow \qquad (D - \alpha)(D - \beta) = (D - \beta)(D - \alpha).$

**Remarks** 1. We have shown that $(D - \alpha)(D - \beta)y = (D^2 - (\beta + \alpha)\, D + \alpha\beta)y$.

$\therefore \qquad (D - \alpha)(D - \beta) = D^2 - (\beta + \alpha)D + \alpha\beta$

Thus the factors $D - \alpha$ and $D - \beta$ can be multiplied algebrically.

**2.** In terms of operator D, the differential equation (1) can be written as

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, y = Q.$$

or as $f(D)y = Q$ where $f(D) = P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n$ . $f(D)$ is a polynomial in D.

## 6.4. LINEARLY INDEPENDENT FUNCTIONS

A set of functions, $y_1, y_2, \ldots\ldots, y_n$ is said to be **linearly independent** if the equality $c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n = 0$ holds only when $c_1, c_2, \ldots\ldots, c_n$ are all zero.

Thus, if $y_1, y_2, \ldots\ldots, y_n$ are linearly independent then any one of these functions cannot be expressed as a linear combination of remaining functions.

For example, the functions $e^{3x}, e^{4x}, 5e^{4x}$ are not linearly independent, because we have

$$c_1 e^{3x} + c_2 e^{4x} + c_3 (5e^{4x}) = 0,$$

where $\qquad c_1 = 0,\ c_2 = 10,\ c_3 = -2.$

A necessary and sufficient condition for the functions $y_1, y_2, \ldots\ldots, y_n$ to be linearly independent is

$$\begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ y_1' & y_2' & \ldots\ldots & y_n' \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots\ldots & y_n^{n-1} \end{vmatrix} \neq 0.$$

**Theorem 1.** *If $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are n linearly independent solutions of the linear differential equation*

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = 0$$

*of order n, where $P_0 \neq 0$, $P_1, P_2, \ldots\ldots, P_n$ are functions of x then $y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is the general solution of this differential equation, where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.*

**Proof.** Given equation is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = 0. \qquad \ldots(1)$$

NOTES

$$\Rightarrow \quad \left(P_0 \frac{d^n}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} + P_2 \frac{d^{n-2}}{dx^{n-2}} + \ldots\ldots + P_n\right) y = 0$$

$$\Rightarrow \quad (P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots\ldots + P_n)\, y = 0$$

$$\Rightarrow \quad f(D)\, y = 0 \qquad \ldots(2)$$

where $f(D) = P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots\ldots + P_n$.

$y = y_1, y = y_2, \ldots\ldots, y = y_n$ are solutions of (2).

$\therefore \quad f(D)\, y_1 = 0, \; f(D)\, y_2 = 0, \ldots\ldots, f(D)\, y_n = 0.$

Now $f(D)(c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n)$

$$= f(D)\, c_1y_1 + f(D)c_2y_2 + \ldots\ldots + f(D)c_ny_n$$

$$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + \ldots\ldots + c_n f(D)\, y_n$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots\ldots + c_n \cdot 0 = 0.$$

$$\therefore \quad y = c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n$$

is also a solution of the given differential equation.

Since the functions $y_1, y_2, \ldots\ldots, y_n$ are given to be linearly independent, no function can be expressed as a linear combination of remaining functions.

$\therefore$ No term in $c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n$ can be eliminated.

$\therefore$ The solution $y = c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n$ contains $n$ arbitrary constants $c_1, c_2, \ldots\ldots, c_n$. Also the order of given differential equation is $n$ and the solution $y = c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n$ contains $n$ arbitrary constant.

$\therefore$ The solution $y = c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n$ represents the general solution of the given differential equation.

## 6.5. GENERAL SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS AND SECOND MEMBER ZERO

Let $$P_0 \frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_ny = 0 \qquad \ldots(1)$$

where $P_0 \neq 0, P_1, P_2, \ldots\ldots, P_n$ are constants be a linear differential equation of order $n$ with constant coefficients and second member zero.

(1) can also be written as

$$(P_0\, D^n + P_1\, D^{n-1} + P_2\, D^{n-2} + \ldots\ldots + P_n)\, y = 0 \quad \text{or} \quad f(D)\, y = 0, \qquad \ldots(2)$$

where $$f(D) = P_0\, D^n + P_1\, D^{n-1} + P_2\, D^{n-2} + \ldots\ldots + P_n \qquad \ldots(3)$$

Let $y = e^{mx}$ be a solution of (1).

$$\therefore \quad \frac{d^ky}{dx^k} = m^ke^{mx} \quad \forall \quad k = 1, 2, 3, \ldots\ldots$$

$$\therefore \quad (1) \Rightarrow \quad P_0\, m^ne^{mx} + P_1\, m^{n-1}\, e^{mx} + P_2\, m^{n-2}\, e^{mx} + \ldots\ldots + P_ne^{mx} = 0$$

$$\Rightarrow \quad (P_0\, m^n + P_1\, m^{n-1} + P_2\, m^{n-2} + \ldots\ldots + P_n)\, e^{mx} = 0$$

$$\Rightarrow \quad P_0\, m^n + P_1\, m^{n-1} + P_2\, m^{n-2} + \ldots\ldots + P_n = 0 \qquad \ldots(4)$$

$$(\because \quad e^{mx} \neq 0)$$

This is a polynomial equation of degree $n$ in $m$ and is called the **auxiliary equation** of the differential equation (1).

On comparing (4) and $f(D) = 0$, we see that the roots of the equation $f(D) = 0$ will coincide with the roots of (4). In practice, we call $f(D) = 0$ as the auxiliary equation and solve this equation to find the roots of the equation (4).

Let the roots of the auxiliary equation of (1) be $m_1, m_2, m_3, \ldots\ldots, m_n$.

$\therefore \quad y = e^{m_1 x}, \quad y = e^{m_2 x}, \quad y = e^{m_3 x}, \ldots\ldots \quad y = e^{m_n x}$ are solutions of the given differential equation.

The general solution of the given differential equation will depend upon the nature of the roots $m_1, m_2, m_3, \ldots\ldots m_n$.

**Case I. Roots of real and distinct**

In this case, the solutions

$$y = e^{m_1 x}, y = e^{m_2 x}, y = e^{m_3 x}, \ldots\ldots, y = e^{m_n x} \text{ are linearly independent.}$$

$\therefore$ The general solution of the given equation is

$$\mathbf{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}},$$

where $c_1, c_2, c_3, \ldots\ldots, c_n$ are $n$ arbitrary constants.

**Case II. Roots are distinct and not all real**

The coefficients of the auxiliary equation are real.

$\therefore$ The imaginary roots occurs in conjugate pairs.

Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ and all other roots be real and distinct.

$\therefore$ The solutions $y = e^{m_1 x}, y = e^{m_2 x}, y = e^{m_3 x}, \ldots\ldots, y = e^{m_n x}$ are linearly independent.

$\therefore$ The general solution of the given equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$\Rightarrow \quad y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1 e^{i\beta x} + c_2 e^{-i\beta x}] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$\therefore \quad \mathbf{y = e^{\alpha x}(A \cos \beta x + B \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}},$$

where $A = c_1 + c_2$, $B = i(c_1 - c_2)$.

This is the general solution of the given equation.

**Case III. Roots are real and distinct except two roots**

Let $\quad m_1 = m_2 \neq m_3 \neq \ldots\ldots \neq m_n$

The equation $f(D)\, y = 0$ can be written as

$$P_0 (D - m_1)^2 (D - m_3) \ldots\ (D - m_n)\, y = 0 \qquad (\because \quad m_1 = m_2)$$

The factors on the left side are commutative.

NOTES

$\therefore$ The solution of each of the equations

$$(D - m_1)^2 y = 0, (D - m_2) y = 0 \text{ ......, } (D - m_n) y = 0$$

is a solution of the given equation.

We consider the solution of the equation $(D - m_1)^2 y = 0$. ...(5)

(5) $\Rightarrow$ $(D - m_1)(D - m_1)y = 0$

$\Rightarrow$ $(D - m_1)u = 0$, where $u = (D - m_1)y$

$\Rightarrow$ $\dfrac{du}{dx} - m_1 u = 0 \quad \Rightarrow \quad \dfrac{du}{u} = m_1\, dx$

$\Rightarrow$ $\log u = m_1 x + \log c_1$

$\Rightarrow$ $\dfrac{u}{c_1} = e^{m_1 x} \quad \Rightarrow \quad u = c_1 e^{m_1 x}$

$\therefore$ $(D - m_1)\, y = c_1\, e^{m_1 x} \quad \Rightarrow \quad \dfrac{dy}{dx} - m_1\, y = c_1\, e^{m_1 x}$ ...(6)

This is a linear differential equation

$$\text{I.F.} = e^{\int -m_1\, dx} = e^{-m_1 x}$$

$\therefore$ The solution of (6) is

$$y\,(e^{-m_1 x}) = \int (c_1\, e^{m_1 x})\,(e^{-m_1 x})\, dx + c_2$$

$\Rightarrow$ $y\, e^{-m_1 x} = c_1\, x + c_2$ or $y = (c_1\, x + c_2)\, e^{m_1 x}$

The general solution of the given equation is

$$\mathbf{y = (c_1 x + c_2) e^{m_1 x} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}.}$$

**Remark.** If the root $m_1$ is repeated $r$ times and all other roots are distinct then the general solution takes the form.

$$y = (c_1\, x^{r-1} + c_2\, x^{r-2} + \ldots + c_r)\, e^{m_1 x} + c_{r+1}\, e^{m_{r+1} x} + \ldots\ldots + c_n\, e^{m_n x}.$$

In practical problems, it is convenient to write the general solution in the form

$$y = (c_1 + c_2\, x + \ldots\ldots + c_r\, x^{r-1})\, e^{m_1 x} + c_{r+1}\, e^{m_{r+1} x} + \ldots\ldots + c_n\, e^{m_n x}.$$

**Case IV. Two equal pairs of imaginary roots and all other roots are real and distinct**

Let $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and all other roots are real and distinct.

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)\, e^{(\alpha + i\beta)x} + (c_3 + c_4\, x)\, e^{(\alpha - i\beta)x} + c_5\, e^{m_5 x} + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{\alpha x}[(c_1 + c_2\, x)\, e^{i\beta x} + (c_3 + c_4\, x)\, e^{-i\beta x}] + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{\alpha x}[(c_1 + c_2\, x)\, (\cos \beta x + i \sin \beta x) + (c_3 + c_4 x)(\cos \beta x - i \sin \beta x)]$$
$$+ \ldots\ldots + c_n\, e^{m_n x}$$

$\therefore$ $\mathbf{y = e^{\alpha x}[(A + Bx) \cos \beta x + (C + Dx) \sin \beta x\,] + \ldots\ldots + c_n\, e^{m_n x},}$

where A, B, C, D, ......, $c_n$ are arbitrary constants.

NOTES

## 6.6. WORKING RULES

Let the given differential equation be

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = 0. \quad \ldots(1)$$

where $P_0 \neq 0$, $P_1, P_2, \ldots\ldots, P_n$ are constants.

The auxiliary equation is

$$P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n = 0.$$

Let the roots of this equation be $m_1, m_2, \ldots\ldots, m_n$.

**Case I. Roots are real and distinct**

The general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}.$$

**Case II. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ and all other are real and distinct.**

The general solution is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}.$$

**Case III. Roots are real and $m_1 = m_2 \neq m_3 \neq \ldots\ldots \neq m_n$**

The general solution is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}.$$

**Case IV. $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and all other are real and distinct.**

The general solution is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x]$$
$$+ c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}.$$

**Example 1.** *Solve* $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0.$

**Sol.** We have $\qquad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0 \qquad \ldots(1)$

$\Rightarrow \qquad (D^2 + 4D + 3)\,y = 0.$

$\therefore$ The A.E. is $\qquad D^2 + 4D + 3 = 0 \quad \Rightarrow \quad (D + 1)(D + 3) = 0$

$\Rightarrow \qquad D = -1, -3.$

The roots are real and distinct.

$\therefore$ The general solution of (1) is $y = c_1 e^{-x} + c_2 e^{-3x}$.

**Example 2.** *Solve* $(D^4 + 5D^2 - 36)\,y = 0.$

**Sol.** We have $\qquad (D^4 + 5D^2 - 36)\,y = 0. \qquad \ldots(1)$

$\therefore$ The A.E. is $\qquad D^4 + 5D^2 - 36 = 0$

$\Rightarrow \qquad D^2 = 4, -9. \quad \therefore \quad D = \pm 2, \pm 3i.$

$\therefore$ The general solution of (1) is

$$y = c_1 e^{2x} + c_2 e^{-2x} + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$$

or $\qquad y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 3x + c_4 \sin 3x.$

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**Example 3.** *Solve* $\frac{d^4y}{dx^4}+5\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}-4\frac{dy}{dx}-8y=0.$

**Sol.** The symbolic form of the given equation is

$$(D^4+5D^3+6D^2-4D-8)\,y=0. \quad ...(1)$$

$\therefore$ The A.E. is $D^4+5D^3+6D^2-4D-8=0$ ...(2)

Putting D = 1, we see that (2) is satisfied.

$\therefore$ D = 1 is a solution of (2) by inspection.

$\therefore$ D – 1 is a factor of $D^4+5D^3+6D^2-4D-8$.

The synthetic division by D – 1 gives

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 5 | 6 | – 4 | – 8 |
| | | 1 | 6 | 12 | 8 |
| | 1 | 6 | 12 | 8 | 0 |

$\therefore$ Auxiliary equation (2) can be written as

$$(D-1)(D^3+6D^2+12D+8)=0 \quad \text{or} \quad (D-1)(D+2)^3=0.$$

$\therefore$ $D=1,-2,-2,-2.$

$\therefore$ The general solution of (1) is

$$\mathbf{y=c_1e^x+(c_2+c_3x+c_4x^2)e^{-2x}}.$$

**Example 4.** *Solve* $\frac{d^4y}{dx^4}+m^4y=0.$

**Sol.** The symbolic form of the given equation is

$$(D^4+m^4)y=0. \quad ...(1)$$

$\therefore$ The A.E. is $D^4+m^4=0 \Rightarrow (D^2)^2+(m^2)^2=0$

Completing square by adding and subtracting $2D^2m^2$, we get

$$D^4+2m^2D^2+m^4-2m^2D^2=0$$

$$\Rightarrow \quad (D^2+m^2)^2-2m^2D^2=0$$

$$\Rightarrow \quad [(D^2+m^2)-\sqrt{2}mD]\,[D^2+m^2+\sqrt{2}mD]=0.$$

$$\therefore \quad D^2+m^2-\sqrt{2}mD=0 \quad \text{or} \quad D^2+m^2+\sqrt{2}mD=0$$

$$\Rightarrow \quad D=\frac{m\sqrt{2}\pm\sqrt{2m^2-4m^2}}{2},\frac{-m\sqrt{2}\pm\sqrt{2m^2-4m^2}}{2}$$

$$=\frac{m\sqrt{2}\pm im\sqrt{2}}{2},\frac{-m\sqrt{2}\pm im\sqrt{2}}{2}=\frac{m}{\sqrt{2}}\pm i\frac{m}{\sqrt{2}},\ -\frac{m}{\sqrt{2}}\pm i\frac{m}{\sqrt{2}}.$$

We have two pairs of unequal imaginary roots.

$\therefore$ The general solution of (1) is

$$\mathbf{y=e^{\frac{m}{\sqrt{2}}x}\left[c_1\cos\frac{m}{\sqrt{2}}x+c_2\sin\frac{m}{\sqrt{2}}x\right]+e^{-\frac{m}{\sqrt{2}}x}\left[c_3\cos\frac{m}{\sqrt{2}}x+c_4\sin\frac{m}{\sqrt{2}}x\right]}.$$

**Example 5.** *Solve* $(D^2-8D+65)^2\,y=0.$

**Sol.** We have $(D^2-8D+65)^2\,y=0.$ ...(1)

$\therefore$ The A.E. is $(D^2-8D+65)^2=0$

$$D^2-8D+65=0$$

$$\Rightarrow \quad D=\frac{8\pm\sqrt{64-260}}{2}=4+7i,\ 4-7i$$

∴ The roots of the A.E. are $4 + 7i$, $4 + 7i$, $4 - 7i$, $4 - 7i$.

∴ The general solution of (1) is

$$\mathbf{y = e^{4x} [(c_1 + c_2x) \cos 7x + (c_3 + c_4x) \sin 7x].}$$

**Example 6.** *If $\lambda_1, \lambda_2$ are real and distinct roots of the equation $\lambda^2 + a_1\lambda + a_2 = 0$ and $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$, then prove that $y = c_1y_1 + c_2y_2$ is a solution of the differential equation $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2y = 0$. Is this solution general ?*

**Sol.** $\lambda_1$ and $\lambda_2$ are roots of $\qquad \lambda^2 + a_1\lambda + a_2 = 0. \qquad$ ...(1)

Given equation is $\qquad \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2y = 0.$

$\Rightarrow \qquad (D^2 + a_1D + a_2)\, y = 0 \qquad$ ...(2)

Let $y = e^{mx}$ be a solution of (2).

$\therefore \qquad (D^2 + a_1D + a_2)\, e^{mx} = 0$

$\Rightarrow \qquad (m^2 + a_1m + a_2)\, e^{mx} = 0$

$\Rightarrow \qquad m^2 + a_1m + a_2 = 0. \qquad$ ...(3)

Comparing (1) and (3), we see that $\lambda_1, \lambda_2$ are also roots of (3).

$\therefore \qquad y_1 = e^{\lambda_1 x}, y = e^{\lambda_2 x}$ are solutions of (2).

Let $y = c_1 y_1 + c_2 y_2$, where $c_1$ and $c_2$ are arbitrary constants.

Now $\qquad (D^2 + a_1 D + a_2)(c_1 y_1 + c_2 y_2)$

$$= c_1 (D^2 + a_1D + a_2)\, y_1 + c_2 (D^2 + a_1D + a_2)y_2$$

$$= c_1 (\lambda_1^2 + a_1 \lambda_1 + a_2)\, e^{\lambda_1 x} + c_2 (\lambda_2^2 + a_1\lambda_2 + a_2)\, e^{\lambda_2 x}$$

$$= c_1.0.e^{\lambda_1 x} + c_2.0.e^{\lambda_2 x} = 0$$

$\therefore \qquad y = c_1y_1 + c_2y_2$ is a solution of (2).

Also, $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix}$

$$= e^{\lambda_1 x}\, e^{\lambda_2 x} \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = e^{(\lambda_1 + \lambda_2)x}(\lambda_2 - \lambda_1) \neq 0. \qquad (\because \quad \lambda_1 \neq \lambda_2)$$

$\therefore \quad y_1$ and $y_2$ are linearly independent.

$\therefore \quad y = c_1 y_1 + c_2 y_2$ is the general solution of the given differential equation.

**Example 7.** *If $\lambda_1, \lambda_1$ are real and equal roots of the equation $\lambda^2 + a_1\lambda + a_2 = 0$, then prove that $y = (c_1 + c_2 x)\, e^{\lambda_1 x}$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2\, y = 0.$$

*Is this solution general ?*

**Sol.** $\lambda_1, \lambda_1$ are roots of $\qquad \lambda^2 + a_1\lambda + a_2 = 0. \qquad$ ...(1)

$\therefore \quad$ (1) is same as $\qquad (\lambda - \lambda_1)^2 = 0$

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Given equation is $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$ i.e., $(D^2 + a_1 D + a_2) y = 0$ ...(2)

(1) show that the roots of $D^2 + a_1 D + a_2 = 0$ are also $\lambda_1$, $\lambda_1$ and $D^2 + a_1 D + a_2 = (D - \lambda_1)^2$

$\therefore$ (2) $\Rightarrow (D - \lambda_1)^2 y = 0 \Rightarrow (D - \lambda_1)(D - \lambda_1) y = 0$

$\Rightarrow (D - \lambda_1) u = 0$, where $u = (D - \lambda_1)y$

$$\Rightarrow \quad u = k_1 e^{\lambda_1 x}$$

$$\therefore \quad (D - \lambda_1) y = k_1 e^{\lambda_1 x} \quad \therefore \quad \frac{dy}{dx} - \lambda_1 y = k_1 e^{\lambda_1 x} \quad \text{...(3)}$$

This is a linear differential equation.

$$\text{I.F.} = e^{-\int \lambda_1 dx} = e^{-\lambda_1 x}$$

$\therefore$ The solution of (3) is $y(e^{-\lambda_1 x}) = \int (k_1 e^{\lambda_1 x})(e^{-\lambda_1 x}) dx + k_2$

$$\Rightarrow \quad y e^{-\lambda_1 x} = k_1 x + k_2 \quad \text{or} \quad y = (k_1 x + k_2) e^{\lambda_1 x}$$

Let $k_1 = c_2$, $k_2 = c_1$

$$\therefore \quad y = (c_1 + c_2 x) e^{\lambda_1 x}$$

$\therefore \quad y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$ is a solution of (2).

Let $y_1 = e^{\lambda_1 x}$, $y_2 = x e^{\lambda_1 x}$

$$\therefore \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & x e^{\lambda_1 x} \\ \lambda_1 e^{\lambda_1 x} & (1 + \lambda_1 x) e^{\lambda_1 x} \end{vmatrix}$$

$$= e^{\lambda_1 x} e^{\lambda_1 x} \begin{vmatrix} 1 & x \\ \lambda_1 & 1 + \lambda_1 x \end{vmatrix} = e^{2\lambda_1 x} . 1 = e^{2\lambda_1 x} \neq 0 .$$

$\therefore$ $y_1$ and $y_2$ are linearly independent.

$$\therefore \quad y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} = (c_1 + c_2 x) e^{\lambda_1 x}$$

is the general solution of the given differential equation.

## EXERCISE 1

*Find the general solution of the following differential equations (Q. No. 1–4) :*

1. (*i*) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ (*ii*) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

   (*iii*) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 12\frac{dy}{dx} = 0$ (*iv*) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$ .

2. (*i*) $\frac{d^2y}{dx^2} + b^2 y = 0$ (*ii*) $(D^2 - 2D + 10) y = 0$

   (*iii*) $(D^3 - D^2 + 9D - 9)y = 0$ (*iv*) $\frac{d^4y}{dx^4} - a^4 y = 0$

(v) $\dfrac{d^4y}{dx^4} + 4y = 0$ (vi) $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0$.

**3.** (i) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$ (ii) $(D + 2)(D^2 + 6D + 9)y = 0$

(iii) $(D^3 - 3D^2 + 3D - 1)y = 0$ (iv) $\dfrac{d^4y}{dx^4} + 5\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 8y = 0$

(v) $(D^4 + 6D^3 + 5D^2 - 24D - 36)y = 0$.

**4.** (i) $(D^2 - 2D + 5)^2\, y = 0$ (ii) $\dfrac{d^4y}{dx^4} + 6\dfrac{d^2y}{dx^2} + 9y = 0$

(iii) $(D^2 - 4)(D^2 + 25)^2\, y = 0$ (iv) $(D^6 + 9D^4 + 24D^2 + 16)y = 0$.

**5.** (i) Solve $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = 0$ given that for $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 12$.

(ii) Solve $\dfrac{d^2y}{dx^2} + y = 0$ given that $y = 2$ for $x = 0$ and $y = -2$ for $x = \dfrac{\pi}{2}$.

**6.** Show that the solution of the differential equation $\dfrac{d^2x}{dt^2} + k\dfrac{dx}{dt} + \mu x = 0$ is

$x = e^{(-1/2)kt}(A\cos nt + B\sin nt)$, where $n^2 = \mu - \dfrac{1}{4}k^2$ and $n$ is real.

**7.** Prove that the general solution of $(D - m)^2\, y = 0$ is $y = (c_1 + c_2 x)e^{mx}$, where $D \equiv \dfrac{d}{dx}$ and $c_1, c_2$ are arbitrary constants.

### Answers

**1.** (i) $y = c_1 e^{2x} + c_2 e^{3x}$ (ii) $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

(iii) $y = c_1 + c_2 e^{4x} + c_3 e^{-3x}$ (iv) $y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-3x}$

**2.** (i) $y = c_1 \cos bx + c_2 \sin bx$ (ii) $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$

(iii) $y = c_1 e^x + c_2 \cos 3x + c_3 \sin 3x$ (iv) $y = c_1 \cos ax + c_2 \sin ax + c_3 e^{ax} + c_4 e^{-ax}$

(v) $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$

(vi) $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$

**3.** (i) $y = (c_1 + c_2 x)e^{3x}$ (ii) $y = c_1 e^{-2x} + (c_2 + c_3 x)e^{-3x}$

(iii) $y = (c_1 + c_2 x + c_3 x^2)e^x$ (iv) $y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)e^{-2x}$

(v) $y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 + c_4 x)e^{-3x}$

**4.** (i) $y = e^x[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$

(ii) $y = (c_1 + c_2 x)\cos \sqrt{3}\, x + (c_3 + c_4 x)\sin \sqrt{3}\, x$

(iii) $y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 + c_4 x)\cos 5x + (c_5 + c_6 x)\sin 5x$

(iv) $y = c_1 \cos x + c_2 \sin x + (c_3 + c_4 x)\cos 2x + (c_5 + c_6 x)\sin 2x$

**5.** (i) $x = -6e^{-3t} + 6e^{-t}$ (ii) $y = 2(\cos x - \sin x)$.

## 6.7. GENERAL SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

**Theorem.** If $y = u$ is any particular solution of the linear differential equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_n y = Q$$

NOTES

of order $n$, where $P_0 \neq 0$, $P_1, P_2, \ldots\ldots, P_n$ and Q are functions of the variable $x$ and $y = Y$ is the general solution of the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = 0$$

then prove that $y = Y + u$ is the general solution of the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = Q.$$

**Proof.** $y = u$ is a particular solution of the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = Q$$

or $\quad (P_0 D^n + P_1 D^{n-1} + \ldots\ldots + P_n)y = Q$

or $\quad f(D)y = Q$, say.

$\therefore \quad f(D)u = Q \quad \ldots(1)$

Also, $y = Y$ is a solution of

$$f(D)\, y = 0.$$

$\therefore \quad f(D)Y = 0 \quad \ldots(2)$

Now $\quad f(D)(Y + u) = f(D)Y + f(D)u = 0 + Q \quad [\because$ Using (1) and (2)]

$$= Q$$

$\therefore \quad f(D)(Y + u) = Q.$

$\therefore \quad y = Y + u$ is a solution of the equation $f(D)\, y = Q$.

Since Y is the general solution of the equation $f(D)y = 0$, it must contain $n$ arbitrary constants.

$\therefore$ The solution $y = Y + u$ of the equation $f(D)\, y = Q$ also contain $n$ arbitrary constants.

$\therefore \quad y = Y + u$ is the general solution of the equation $f(D)\, y = Q$.

## 6.8. METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let $\quad f(D)y = Q \quad \ldots(1)$

be a linear differential equation of order $n$ with constant coefficients, where

$$f(D) = P_0 D^n + P_1 D^{n-1} + \ldots + P_n$$

and $\quad P_0 \neq 0, P_1, \ldots\ldots, P_n$ are constants and Q is some function of $x$.

We have proved that if $y = Y$ is the general solution of the equation $f(D)\, y = 0$ and $u$ is any particular solution of the equation $f(D)y = Q$, then $y = Y + u$ is the general solution of the given equation $f(D)y = Q$.

The function Y is called the **complementary function** (C.F.) of the general solution of the equation $f(D)y = Q$. We have already studied the method of finding the complementary function Y for the equation $f(D)y = Q$, which is nothing but the general solution of the equation $f(D)y = 0$.

Now we shall take up the method of finding a particular integral ($u$) of the equation $f(D)y = Q$. We define the operator '$\frac{1}{f(D)}$'.

$\frac{1}{f(D)} Q$ is defined to be that function of $x$ which, when operated upon by $f(D)$, gives Q. The operator $\frac{1}{f(D)}$, according to this definition, is the inverse of the operator $f(D)$. It can be shown that the operator $\frac{1}{f(D)}$ can be broken up into factors which may be taken in any order or into partial fractions.

By the definition of the operator '$\frac{1}{f(D)}$', we have

$$f(D)\left(\frac{1}{f(D)} Q\right) = Q$$

$\therefore \quad y = \frac{1}{f(D)} Q$ is a particular integral of the equation $f(D)y = Q$.

Let the operator $\frac{1}{f(D)}$ may be factored as $\frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdots\cdots \frac{1}{D-m_n}$, then the particular integral may be written as $\frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdots\cdots \frac{1}{D-m_n} Q$.

On operating with the first symbolic factor, beginning at the right, there is obtained $\frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdots\cdots \frac{1}{D-m_{n-1}} e^{m_n x} \int Q e^{-m_n x}\, dx$ then, on operating with the second and remaining factors in succession, taking them from right to left, the value of the particular integral is obtained.

Alternatively, the operator $\frac{1}{f(D)}$ may be decomposed into its partial fractions as $\frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \cdots\cdots + \frac{N_n}{D-m_n}$ and then the particular integral will have the form

$$N_1 e^{m_1 x} \int Q\, e^{-m_1 x}\, dx + N_2\, e^{m_2 x} \int Q\, e^{-m_2 x}\, dx + \cdots\cdots + N_n e^{m_n x} \int Q\, e^{-m_n x}\, dx.$$

Of the above two methods, the latter is generally preferred.

**Remark 1.** In calculating the particular integral of the equation $f(D)y = Q$, no arbitrary constant is introduced.

**Remark 2.** In finding the particular integral, we have used the result $\frac{1}{D-m} Q = e^{mx} \int Q\, e^{-mx}\, dx$. This result is very important. For the sake of completeness, we prove this result

Let $\qquad y = \frac{1}{D-m} Q$

$\therefore$ By definition of the operator $\frac{1}{D-m} Q$, we have $(D-m)\left[\frac{1}{D-m} Q\right] = Q$.

$$\Rightarrow \qquad (D-m)y = Q \quad \Rightarrow \quad \frac{dy}{dx} - my = Q \qquad \ldots(1)$$

NOTES

(1) is a linear differential equation of first order.

$$\text{I.F.} = e^{\int -m\,dx} = e^{-mx}$$

$\therefore$ The solution of (1) is

$$y(e^{-mx}) = \int Q\, e^{-mx}\, dx \quad \text{or} \quad y = e^{mx}\int Q\, e^{-mx}\, dx$$

$$\therefore \qquad \frac{1}{D-m} Q = e^{mx}\int Q\, e^{-mx}\, dx.$$

## 6.9. PARTICULAR INTEGRAL OF f(D)y = Q WHEN Q IS IN SOME STANDARD FORM

When Q is in one of the following standard forms, then the particular integral of the equation $f(D)y = Q$ can be obtained by using shorter methods than by using the general methods given above. We shall discuss the following standard forms of the function Q of the equation $f(D)\, y = Q$

(*i*) $e^{ax}$, where $a$ is any constant

(*ii*) $\sin ax$. $\cos ax$, where $a$ is any constant

(*iii*) $x^m$, where $m$ is any positive integer

(*iv*) $e^{ax}V$, where V is any function of $x$

(*v*) $xV$, where V is any function of $x$.

## 6.10. Q IS OF THE FORM $e^{ax}$, WHERE a IS ANY CONSTANT

Let the linear differential equation be $f(D)y = Q$, where $f(D)$ is a polynomial in D with constant coefficients and $Q = e^{ax}$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)} e^{ax} \qquad \ldots(1)$$

Successive differentiation gives $D^k e^{ax} = a^k e^{ax}$.

Since, $f(D)$ is a polynomial in D with constant coefficients, we have

$$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating on both sides with $\frac{1}{f(D)}$, we get $\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} f(a)\, e^{ax}$

Since $\frac{1}{f(D)}$ and $f(D)$ are inverse operators and $f(a)$ is an algebraic multiplier, we get

$$e^{ax} = f(a)\frac{1}{f(D)} e^{ax} \quad \text{or} \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}.$$

This method fails if $f(a) = 0$.

Let $f(a) = 0$. $\therefore$ $a$ is a root of $f(D) = 0$.

NOTES

**Case I. a is a non-repeated root of f(D) = 0.**

$\therefore$ $D - a$ is a factor of $f(D)$.

Let $$f(D) = (D - a)\,\phi(D) \quad \text{and} \quad \phi(a) \neq 0$$

$$\therefore \quad \frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax}$$

$$= e^{ax} \int \left( \frac{1}{\phi(a)} e^{ax} \right) e^{-ax}\, dx = \frac{xe^{ax}}{\phi(a)}.$$

Also, $$f'(D) = 1 \cdot \phi(D) + (D - a)\,\phi'(D)$$

$$\therefore \quad f'(a) = \phi(a) + (a - a)\,\phi'(a) = \phi(a).$$

$$\therefore \quad \mathbf{\frac{1}{f(D)} e^{ax} = \frac{x\, e^{ax}}{f'(a)}}.$$

**Case II. a is a repeated root of f(D) = 0**

Let $a$ is a twice repeated root of $f(D) = 0$.

$\therefore$ $(D - a)^2$ is a factor of $f(D)$.

Let $$f(D) = (D - a)^2\, \psi(D) \quad \text{and} \quad \psi(a) \neq 0.$$

$$\therefore \quad \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^2} \cdot \frac{1}{\psi(D)} e^{ax} = \frac{1}{(D-a)^2} \cdot \frac{1}{\psi(a)} e^{ax}$$

$$= \frac{1}{D-a} \cdot e^{ax} \int \left( \frac{1}{\psi(a)} e^{ax} \right) e^{-ax}\, dx = \frac{1}{\psi(a)} \cdot \frac{1}{D-a} xe^{ax}$$

$$= \frac{1}{\psi(a)} \cdot e^{ax} \int (x\, e^{ax}) e^{-ax}\, dx = \frac{1}{\psi(a)} \cdot e^{ax} \frac{x^2}{2} = \frac{x^2 e^{ax}}{2\,\psi(a)}.$$

Also $$f'(D) = 2(D - a)\,\psi(D) + (D - a)^2\,\psi'(D)$$

and $$f''(D) = 2\psi(D) + 2(D - a)\,\psi'(D) + 2(D - a)\,\psi'(D) + (D - a)^2\,\psi''(D).$$

$$\therefore \quad f''(a) = 2\psi(a).$$

$$\therefore \quad \mathbf{\frac{1}{f(D)} e^{ax} = \frac{x^2 e^{ax}}{f''(a)}}.$$

Similarly, if $a$ is a root repeated $r$ times, then

$$\mathbf{\frac{1}{f(D)} e^{ax} = \frac{x^r\, e^{ax}}{f^{(r)}(a)}}.$$

**Example 1.** *Solve* $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 3y = e^{2x}$.

**Sol.** We have

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}. \qquad \text{...(1)}$$

$$\Rightarrow \quad (4D^2 + 4D - 3)y = e^{2x}$$

$\therefore$ The A.E. is $$4D^2 + 4D - 3 = 0.$$

$$\therefore \quad D = \frac{-4 \pm \sqrt{16+48}}{8} = \frac{-4 \pm \sqrt{64}}{8} = \frac{-4 \pm 8}{8} = \frac{1}{2}, -\frac{3}{2}$$

$$\therefore \quad \text{C.F.} = c_1\, e^{x/2} + c_2\, e^{-3x/2}.$$

NOTES

$$\text{P.I.} = \frac{1}{4D^2+4D-3}e^{2x} = \frac{1}{4(2)^2+4(2)-3}e^{2x} = \frac{e^{2x}}{21}.$$

$\therefore$ The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \qquad y = c_1 e^{x/2} + c_2 e^{-3x/2} + \frac{e^{2x}}{21}.$$

**Example 2.** *Solve*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{2x} \text{ given that } x = 0, \text{ when } y = 0.$$

**Sol.** The equation in the symbolic form is

$$(D^2 + 2D + 1)y = 2e^{2x} \qquad ...(1)$$

$\therefore$ The A.E. is

$$D^2 + 2D + 1 = 0 \quad i.e., \quad (D+1)^2 = 0$$

$$\therefore \qquad D = -1, -1$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2x)e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}2e^{2x} = 2\frac{1}{(D+1)^2}e^{2x} = 2\frac{1}{(2+1)^2}e^{2x} = \frac{2}{9}e^{2x}$$

$\therefore$ The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \qquad y = (c_1 + c_2x)e^{-x} + \frac{2}{9}e^{2x}$$

It is given that $\qquad y = 0$, when $x = 0$

$$\therefore \qquad 0 = c_1 + \frac{2}{9} \quad \text{or} \quad c_1 = -\frac{2}{9}$$

$$\therefore \qquad y = \left(-\frac{2}{9} + c_2x\right)e^{-x} + \frac{2}{9}e^{2x}$$

or

$$\mathbf{y = \frac{2}{9}(e^{2x} - e^{-x}) + c_2xe^{-x}.}$$

This is the required solution.

**Example 3.** *Solve* $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 104\, e^{3x}$.

**Sol.** We have

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104\,e^{3x} \quad \text{or} \quad (D^2 + 6D + 25)y = 104\,e^{3x}. \qquad ...(1)$$

$\therefore$ The A.E. is $\quad D^2 + 6D + 25 = 0.$

$$\therefore \qquad D = \frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm 4i.$$

$$\therefore \qquad \text{C.F.} = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$\text{P.I.} = \frac{1}{D^2+6D+25}104\,e^{3x} = 104\frac{1}{D^2+6D+25}e^{3x}$$

$$= 104\frac{1}{(3)^2+6(3)+25}e^{3x} = 2e^{3x}.$$

Hence the general solution of (1) is

$$\mathbf{y = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x) + 2e^{3x}.}$$

**Remark.** $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$

**NOTES**

**Example 4.** *Solve* $(D^2 - 3D + 2)\, y = \cosh x$.

**Sol.** We have $(D^2 - 3D + 2)\, y = \cosh x$. ...(1)

$\therefore$ The A.E. is $D^2 - 3D + 2 = 0$. $\therefore$ $D = 1, 2$

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \cosh x = \frac{1}{(D^2 - 3D + 2)}\left(\frac{e^x + e^{-x}}{2}\right).$$

$$= \frac{1}{2}\left(\frac{1}{D^2 - 3D + 2}\, e^x\right) + \frac{1}{2}\left(\frac{1}{D^2 - 3D + 2}\, e^{-x}\right)$$

(case of failure)

$$= \frac{1}{2}\left(\frac{x}{(2D - 3)_{D=1}}\, e^x\right) + \frac{1}{2}\left(\frac{1}{(-1)^2 - 3(-1) + 2}\, e^{-x}\right)$$

(Case of failure)

$$= -\frac{1}{2}\, xe^x + \frac{1}{2}\cdot\frac{1}{6}\, e^{-x} = -\frac{1}{2}\, xe^x + \frac{1}{12} e^{-x}.$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$\therefore$
$$\mathbf{y = c_1 e^x + c_2 e^{2x} - \frac{1}{2} x e^x + \frac{1}{12} e^{-x}}.$$

## EXERCISE 2

*Solve the following differential equations (Q. No. 1–15) :*

1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$
2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$
3. $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = e^x,\ k \neq 1$
4. $(D^3 - 3D^2 + 4)y = e^{3x}$
5. $(D^3 + 2D^2 + D)y = e^{2x}$
6. $\frac{d^3y}{dx^3} + y = 3 + 5e^x$
7. $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2 + q^2)y = e^{ax}$
8. $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$
9. $(D^2 + 4D + 4)y = e^{2x} - e^{-2x}$
10. $(D^2 - a^2)y = e^{ax} + e^{nx}$
11. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = e^x$
12. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2\sinh x$
13. $\frac{d^2y}{dx^2} - y = \cosh x$
14. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2\sinh 2x$
15. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x}\cosh x.$
16. If $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$ ; $a$, $b$ and $g$ being positive numbers and $x = a'$ and $\frac{dx}{dt} = 0$ when $t = 0$ ; show that $x = a + (a' - a)\cos\left(\sqrt{\frac{g}{b}}\, t\right)$.

NOTES

**Answers**

1. $y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2}e^{4x}$

2. $y = c_1e^{x} + c_2e^{2x} + \frac{1}{12}e^{5x}$

3. $y = (c_1 + c_2x)e^{kx} + \frac{1}{(1-k)^2}e^{x}$

4. $y = c_1e^{-x} + (c_2 + c_3x)\,e^{2x} + \frac{1}{4}e^{3x}$

5. $y = c_1 + (c_2 + c_3\,x)e^{-x} + \frac{1}{18}e^{2x}$

6. $y = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^{x}$

7. $y = e^{-px}(c_1\cos qx + c_2\sin qx) + \frac{1}{(a+p)^2 + q^2}e^{ax}$

8. $y = c_1\,e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + 3 + x\frac{e^{-x}}{3} + \frac{5}{9}e^{2x}$

9. $y = (c_1 + c_2x)\,e^{-2x} + \frac{e^{2x}}{16} - \frac{x^2}{2}e^{-2x}$

10. $y = c_1\,e^{ax} + c_2\,e^{-ax} + \frac{xe^{ax}}{2a} + \frac{1}{n^2 - a^2}e^{nx}$

11. $y = (c_1 + c_2x)\,e^{x} + c_3e^{-x} + \frac{1}{4}x^2e^{x}$

12. $y = (c_1 + c_2x)\,e^{-2x} + \frac{1}{9}e^{x} - e^{-x}$

13. $y = c_1\,e^{x} + c_2\,e^{-x} + \frac{x}{2}\sinh x$

14. $y = (c_1 + c_2x)\,e^{-2x} + \frac{1}{16}e^{2x} - \frac{1}{2}x^2\,e^{-2x}$

15. $y = (c_1 + c_2\,x)\,e^{x} + c_3\,e^{3x} + \frac{x}{8}e^{3x} - \frac{1}{8}x^2\,e^{x}$

## 6.11. Q IS OF THE FORM SIN ax OR COS ax

Let the linear differential equation be $f(D)y = Q$, where $f(D)$ is a polynomial in D with constant coefficients and $Q = \sin ax$.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)}\sin ax \qquad \text{...(1)}$$

Successive differentiation of $\sin ax$ gives

$D \sin ax = a \cos ax, \quad D^2 \sin ax = -a^2 \sin ax$

$D^3 \sin ax = -a^3 \cos ax, \quad D^4 \sin ax = a^4 \sin ax = (-a^2)^2 \sin ax$

.............................................................................................

$\therefore$ In general, $(D^2)^k \sin ax = (-a^2)^k \sin ax$

$f(D)$ may or may not be a function of $D^2$.

Let $f(D)$ be a function of $D^2$ and $f(D) = \phi(D^2)$, say.

Since, $\phi(D^2)$ is a polynomial in $D^2$ with constant coefficients, we have

$$\phi(D^2)\sin ax = \phi(-a^2)\sin ax \text{ and thus } \frac{1}{\phi(D^2)}\phi(D^2)\sin ax$$

$$= \frac{1}{\phi(D^2)}\phi(-a^2)\sin ax$$

$$\Rightarrow \qquad \sin ax = \phi(-a^2)\frac{1}{\phi(D^2)}\sin ax$$

or $$\frac{1}{\phi(\mathbf{D}^2)}\ \mathbf{sin\ ax} = \frac{1}{\phi(-\mathbf{a}^2)}\ \mathbf{sin\ ax}.$$

Similarly, if Q is of the form cos $ax$, then

$$\frac{1}{\phi(\mathbf{D}^2)}\ \mathbf{cos\ ax} = \frac{1}{\phi(-\mathbf{a}^2)}\ \mathbf{cos\ ax}.$$

These results fail if $\quad \phi(-a^2) = 0.$

Let $\quad \phi(-a^2) = 0$

We have $\quad e^{iax} = \cos ax + i \sin ax$

$$\therefore \quad \frac{1}{\phi(D^2)}(\cos ax + i \sin ax) = \frac{1}{\phi(D^2)} e^{iax}$$

Now $\quad D = ia \;\Rightarrow\; \phi(D^2) = \phi((ia)^2) = \phi(-a^2) = 0$

$$\therefore \quad \frac{1}{\phi(D^2)} e^{iax} = x \frac{1}{\frac{d}{dD}(\phi(D^2))} e^{iax}$$

$$= x \frac{1}{\frac{d}{dD}(\phi(D^2))}(\cos ax + i \sin ax)$$

Equating real and imaginary parts we get

$$\frac{1}{\phi(\mathbf{D}^2)} \mathbf{cos\,ax} = \mathbf{x} \frac{1}{\frac{\mathbf{d}}{\mathbf{dD}}(\phi(\mathbf{D}^2))} \mathbf{cos\,ax}$$

and $$\frac{1}{\phi(\mathbf{D}^2)} \mathbf{sin\,ax} = \mathbf{x} \frac{1}{\frac{\mathbf{d}}{\mathbf{dD}}(\phi(\mathbf{D}^2))} \mathbf{sin\,ax}.$$

If by using these rules, the denominator again vanishes, we repeat the rule once again.

Now we consider the possibility that $f(D)$ is not a function of $D^2$. In this case, $D^2$ is replaced by $-a^2$ every where in $\frac{1}{f(D)} \sin ax$. In the next step, the operator is rationalised and again $D^2$ is put equal to $-a^2$. This is simplified to get the required particular integral. Same procedure is also followed in case of $\frac{1}{f(D)} \cos ax$.

**Remark.** The following results of Trigonometry are very useful in solving problems :

(*i*) $\sin^2 x = \frac{1 - \cos 2x}{2}$ (*ii*) $\cos^2 x = \frac{1 + \cos 2x}{2}$

(*iii*) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(*iv*) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(*v*) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(*vi*) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

**Example 1.** *Solve* $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x.$

**Sol.** Given equation in the symbolic form is

$$(D^3 + D^2 - D - 1)y = \cos 2x. \qquad \ldots(1)$$

$\therefore$ The A.E. is $\quad D^3 + D^2 - D - 1 = 0.$

NOTES

$$\Rightarrow \quad D^2(D+1)-(D+1)=0 \quad \Rightarrow \quad (D^2-1)(D+1)=0.$$

$$\therefore \quad D = 1, -1, -1$$

$$\therefore \quad \text{C.F.} = c_1 e^x + (c_2 + c_3 x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3+D^2-D-1}\cos 2x = \frac{1}{DD^2+D^2-D-1}\cos 2x$$

$$= \frac{1}{D(-4)+(-4)-D-1}\cos 2x = \frac{1}{-5D-5}\cos 2x$$

$$= -\frac{1}{5(D+1)}\cos 2x = -\frac{1.(D-1)}{5(D+1)(D-1)}\cos 2x$$

$$= -\frac{1}{5(D^2-1)}(D-1)\cos 2x = -\frac{(D-1)}{5}\left(\frac{1}{D^2-1}\cos 2x\right)$$

$$= -\frac{1}{5}(D-1)\left(\frac{1}{-4-1}\cos 2x\right)$$

$$= \frac{1}{25}(D-1)\cos 2x = \frac{1}{25}(D\cos 2x - \cos 2x)$$

$$= \frac{1}{25}(-2\sin 2x - \cos 2x) = -\frac{1}{25}(2\sin 2x + \cos 2x).$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \quad \mathbf{y = c_1e^x + (c_2 + c_3x)e^{-x} - \frac{1}{25}(2\sin 2x + \cos 2x).}$$

**Example 2.** *Solve* $\dfrac{d^2y}{dx^2} - 4y = e^x + \sin 2x.$

**Sol.** Given equation in the symbolic form is

$$(D^2-4)y = e^x + \sin 2x. \qquad \ldots(1)$$

$\therefore$ The A.E. is $D^2 - 4 = 0. \quad \therefore \quad D = \pm 2$

$$\therefore \quad \text{C.F.} = c_1e^{2x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2-4}(e^x + \sin 2x) = \left(\frac{1}{D^2-4}e^x\right) + \left(\frac{1}{D^2-4}\sin 2x\right)$$

$$= \left(\frac{1}{1^2-4}e^x\right) + \left(\frac{1}{-4-4}\sin 2x\right) = -\frac{1}{3}e^x - \frac{1}{8}\sin 2x$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \quad \mathbf{y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x - \frac{1}{8}\sin 2x.}$$

**Example 3.** *Solve* $(D^2 - 4D + 3)y = \sin 2x \cos x.$

**Sol.** Given equation is $(D^2-4D+3)y = 0. \qquad \ldots(1)$

$\therefore$ The A.E. is $D^2 - 4D + 3 = 0. \quad \therefore \quad D = 3, 1$

$$\therefore \quad \text{C.F.} = c_1e^{3x} + c_2e^{x}$$

$$\text{P.I.} = \frac{1}{D^2-4D+3}[\sin 2x\cos x] = \frac{1}{D^2-4D+3}\left[\frac{1}{2}(2\sin 2x\cos x)\right]$$

$$= \frac{1}{D^2-4D+3}\left[\frac{1}{2}(\sin 3x + \sin x)\right]$$

$$= \frac{1}{2}\left(\frac{1}{D^2-4D+3}\sin 3x\right) + \frac{1}{2}\left(\frac{1}{D^2-4D+3}\sin x\right)$$

$$= \frac{1}{2}\left[\frac{1}{-9-4D+3}\sin 3x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-6-4D}\sin 3x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2(3+2D)}\sin 3x + \frac{1}{2(1-2D)}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{3-2D}{(3+2D)(3-2D)}\sin 3x + \frac{1+2D}{(1-2D)(1+2D)}\sin x\right]$$

$$= \frac{1}{4}\left[\frac{-3+2D}{9-4D^2}\sin 3x + \frac{1+2D}{1-4D^2}\sin x\right]$$

$$= \frac{1}{4}\left[\frac{-3+2D}{9-4(-9)}\sin 3x + \frac{(1+2D)\sin x}{1-4(-1)}\right]$$

$$= \frac{1}{4}\left[\frac{(-3+2D)}{45}\sin 3x + \left(\frac{1+2D}{5}\right)\sin x\right]$$

$$= \frac{1}{4}\left[\frac{1}{45}(-3\sin 3x + 2D\sin 3x) + \frac{1}{5}\sin x + \frac{2}{5}D(\sin x)\right]$$

$$= \frac{1}{4}\left[\frac{1}{45}(-3\sin 3x + 6\cos 3x) + \frac{1}{5}\sin x + \frac{2}{5}\cos x\right]$$

$$= \frac{1}{4}\left[-\frac{1}{15}\sin 3x + \frac{2}{15}\cos 3x + \frac{1}{5}\sin x + \frac{2}{5}\cos x\right]$$

$$= \frac{1}{60}[-\sin 3x + 2\cos 3x + 3\sin x + 6\cos x]$$

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$\therefore$ $\quad \mathbf{y = c_1e^{3x} + c_2e^{x} + \frac{1}{60}[-\sin 3x + 2\cos 3x + 3\sin x + 6\cos x].}$

**Example 4.** *Solve* $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x.$

**Sol.** Given equation is

$$\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x \quad \text{or} \quad (D^2+4)\,y = e^x + \sin 2x. \qquad \text{...(1)}$$

$\therefore$ The A.E. is $\quad D^2 + 4 = 0, \quad \therefore \quad D = \pm 2i$

$\therefore$ $\quad$ C.F. $= e^{0x}(c_1\cos 2x + c_2\sin 2x) = c_1\cos 2x + c_2\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}(e^x + \sin 2x) = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{1+4}e^x + x\frac{1}{\frac{d}{dD}(D^2+4)}\sin 2x \qquad \text{[case of failure]}$$

$$= \frac{1}{5}e^x + x\frac{1}{2D}\sin 2x = \frac{1}{5}e^x + x\frac{1}{2D^2}\cdot D\sin 2x$$

$$= \frac{1}{5}e^x + \frac{x}{2}D\left(\frac{1}{D^2}\sin 2x\right) = \frac{1}{5}e^x + \frac{x}{2}D\left(-\frac{1}{4}\sin 2x\right)$$

$$= \frac{e^x}{5} - \frac{x}{8} D(\sin 2x) = \frac{e^x}{5} - \frac{x}{8} 2\cos 2x = \frac{e^x}{5} - \frac{x}{4}\cos 2x.$$

NOTES

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$$\therefore \qquad \mathbf{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x.}$$

**Example 5.** *Solve* $\dfrac{d^4y}{dx^4} + 2n^2 \dfrac{d^2y}{dx^2} + n^4 y = \cos mx$.

**Sol.** The given equation in symbolic form is

$$(D^4 + 2n^2 D^2 + n^4)\, y = \cos mx. \qquad \ldots(1)$$

$\therefore$ The A.E. is $D^4 + 2n^2D^2 + n^4 = 0$.

$$\Rightarrow \qquad (D^2 + n^2)^2 = 0 \quad \Rightarrow \quad D = \pm ni, \pm ni$$

$$\therefore \qquad \text{C.F.} = e^{0x}\,((c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx)$$

$$= (c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx$$

$$\text{P.I.} = \frac{1}{(D^2 + n^2)^2}\cos mx = \frac{1}{((-m^2) + n^2)^2}\cos mx = \frac{\cos mx}{(m^2 - n^2)^2}$$

$\therefore$ The general solution of (1) is $y =$ C.F. + P.I.

$$\therefore \qquad \mathbf{y = (c_1 + c_2x)\cos mx + (c_3 + c_4x)\sin mx + \frac{\cos mx}{(m^2 - n^2)^2}}.$$

## EXERCISE 3

*Solve the following differential equations (Q. No. 1—14) :*

1. $\dfrac{d^3y}{dx^3} + y = \cos 2x$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = \sin 3x$

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = a\cos 2x$

4. $(D^4 - 1)\, y = \sin 2x$

5. $(D^2 - 3D + 2)\, y = 6e^{-3x} + \sin 2x$

6. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = \sin 3x \cos 2x$

7. $(D^2 + 1)\, y = \sin x \sin 2x$

8. $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$

9. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x$

10. $(D^2 - 4D + 4)\, y = e^{-4x} + 5\cos 3x$

11. $\dfrac{d^2y}{dx^2} + 2a\dfrac{dy}{dx} + a^2y = b\cos px$

12. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = \sin^2 x - e^{3x}$

13. $\dfrac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \dfrac{x}{2}$

14. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 2\sin x$

15. Solve $\dfrac{d^2s}{dt^2} + b^2s = k\cos bt$, given that $s = 0, \dfrac{ds}{dt} = 0$ when $t = 0$

16. Solve $(D^4 - m^4)\, y = \sin mx$.

NOTES

### Answers

1. $y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \cos \frac{\sqrt{3}}{2}x\right] + \frac{1}{65}(\cos 2x - 8 \sin 2x)$

2. $y = e^{x}[c_1 \cos 2x + c_2 \sin 2x] + \frac{1}{26}[3 \cos 3x - 2 \sin 3x]$

3. $y = e^{2x}\left[c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}\right] - \frac{a}{73}[8 \sin 2x + 3 \cos 2x]$

4. $y = (c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x) + \frac{1}{15}\sin 2x$

5. $y = c_1 e^{x} + c_2 e^{2x} + \frac{3}{10}e^{-3x} + \frac{1}{20}[3 \cos 2x - \sin 2x]$

6. $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{884}[10 \cos 5x - 11 \sin 5x] + \frac{1}{20}[\sin x + 2 \cos x]$

7. $y = c_1 \cos x + c_2 \sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$

8. $y = c_1 \cos ax + c_2 \sin ax - \frac{x \cos ax}{2a}$

9. $y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2 \cos 2x + 3 \sin 2x)$

10. $y = (c_1 + c_2 x)e^{2x} + \frac{1}{36}e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x)$

11. $y = (c_1 + c_2 x)e^{-ax} + \frac{b}{(a^2 + p^2)^2}[2ap \sin px + (a^2 - p^2)\cos px]$

12. $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{6} + \frac{1}{130}(\cos 2x + 8 \sin 2x) - \frac{1}{2}x e^{3x}$

13. $y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right] + \frac{1}{730}[\sin 3x + 27 \cos 3x] - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$

14. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} - \frac{1}{5}\cos x$

15. $s = \frac{kt}{2b}\sin bt$

16. $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3}\cos mx.$

## 6.12. Q IS OF THE FORM $x^m$, WHERE M IS ANY POSITIVE INTEGER

Let the linear differential equation be $f(D)\,y = Q$, where $f(D)$ is a polynomial in D with constant coefficients and $Q = x^m$.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)}x^m$$

In the first step, take out the lowest degree term from $f(D)$, so that it is reduced to the form $(1 \pm \phi(D))^n$ where $n$ may be 1, 2, ...... . In the second step, we take it to the numerator and get $(1 \pm \phi(D))^{-n}$. Now we expand this by using Binomial theorem, namely

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots$$

**NOTES**

If $Q = x^m$ or a polynomial of degree $m$, then the terms of the expansion beyond $D^m$ need not be written because the result of their operation on $x^m$ would be zero.

**Example 1.** *Solve* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = x.$

**Sol.** The equation in the symbolic form is

$$(D^2 - 3D + 2)y = x. \qquad ...(1)$$

$\therefore$ The A.E. is $D^2 - 3D + 2 = 0$. $\therefore$ $D = 1, 2$.

$\therefore$ $\quad$ C.F. $= c_1e^x + c_2e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}x = \frac{1}{2\left[1 - \frac{3D}{2} + \frac{D^2}{2}\right]}x$$

$$= \frac{1}{2}\left[1 - \frac{3D}{2} + \frac{D^2}{2}\right]^{-1} x = \frac{1}{2}\left[1 - \left(\frac{3D}{2} - \frac{D^2}{2}\right)\right]^{-1} x$$

$$= \frac{1}{2}\left[1 + \left(\frac{3D}{2} - \frac{D^2}{2}\right) + ......\right]x = \frac{1}{2}\left[x + \frac{3}{2}Dx - \frac{1}{2}D^2x + ......\right]$$

$$= \frac{1}{2}\left[x + \frac{3}{2}.1 - 0 + ......\right] = \frac{1}{2}\left[x + \frac{3}{2}\right] = \frac{x}{2} + \frac{3}{4}.$$

The general solution of (1) is $y =$ C.F. + P.I.

$$\therefore \qquad \mathbf{y = c_1e^x + c_2e^{2x} + \frac{x}{2} + \frac{3}{4}.}$$

**Example 2.** *Solve* $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2.$

**Sol.** The symbolic form of the given equation is

$$(D^3 - D^2 - 6D)\, y = 1 + x^2 \qquad ...(1)$$

$\therefore$ The A.E. is $D^3 - D^2 - 6D = 0$.

$\therefore$ $\quad D(D^2 - D - 6) = 0$ or $D(D - 3)(D + 2) = 0$

$\therefore$ $\quad D = 0, 3, -2$

$\therefore$ $\quad$ C.F. $= c_1\, e^{0x} + c_2\, e^{3x} + c_3\, e^{-2x} = c_1 + c_2\, e^{3x} + c_3\, e^{-2x}$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left[1 + \frac{D}{6} - \frac{D^2}{6}\right]}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 + ......\right](1 + x^2)$$

$$[\because \quad (1 + x)^{-1} = 1 - x + x^2 - ......]$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} + \; .....\right](1 + x^2)$$

[Retaining terms only upto $D^2$]

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{7D^2}{36}+\ldots\ldots\right](1+x^2)$$

$$= -\frac{1}{6D}\left[1+x^2-\frac{1}{6}D(1+x^2)+\frac{7}{36}D^2(1+x^2)+\ldots\ldots\right]$$

$$= -\frac{1}{6D}\left[1+x^2-\frac{1}{6}(0+2x)+\frac{7}{36}(0+2)+0+\ldots\ldots\right]$$

$$= -\frac{1}{6D}\left[1+x^2-\frac{x}{3}+\frac{7}{18}\right] = -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\int x^2\,dx-\frac{1}{3}\int x\,dx+\frac{25}{18}\int dx\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right] = -\frac{1}{108}\,[6x^3-3x^2+25x]$$

The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \qquad \mathbf{y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{108}\,[6x^3 - 3x^2 + 25x].}$$

**Example 3.** *Solve* $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x.$

**Sol.** Writing the given equation in the symbolic form, we have

$$(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x \qquad \ldots(1)$$

$\therefore$ The A.E. is $D^2 - 4D + 4 = 0$. $\therefore$ $D = 2, 2$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2x)e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2-4D+4}(x^2+e^x+\cos 2x) = \frac{1}{(D-2)^2}(x^2+e^x+\cos 2x)$$

$$= \frac{1}{(D-2)^2}x^2 + \frac{1}{(D-2)^2}e^x + \frac{1}{(D-2)^2}\cos 2x$$

$$= \frac{1}{4(1-D/2)^2}x^2 + \frac{1}{(D-2)^2}e^x + \frac{1}{D^2-4D+4}\cos 2x$$

$$= \frac{1}{4}\left[\left(1-\frac{D}{2}\right)^{-2}x^2\right] + \frac{e^x}{(1-2)^2} + \frac{1}{-(2)^2-4D+4}\cos 2x$$

$$= \frac{1}{4}\left[1+2\left(\frac{D}{2}\right)+\frac{(-2)(-3)}{2}\left(\frac{-D}{2}\right)^2+\ldots\ldots\right]x^2 + e^x - \frac{1}{4D}\cos 2x$$

$$= \frac{1}{4}\left[1+D+\frac{3}{4}D^2+\ldots\right]x^2 + e^x - \frac{1}{4}\,\frac{\sin 2x}{2}$$

$$= \frac{1}{4}\left[x^2+2x+\frac{3}{4}.2\right]+e^x-\frac{\sin 2x}{8} = \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)+e^x-\frac{1}{8}\sin 2x\,.$$

$\therefore$ The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \qquad \mathbf{y = (c_1 + c_2x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x.}$$

NOTES

## EXERCISE 4

*Solve the following differential equations :*

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x$
2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = x + k$
3. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$
4. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$
5. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$
6. $(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x$
7. $(D^4 - a^4)\, y = x^4 + \sin kx$
8. $\frac{d^2y}{dx^2} + 4y = 3e^x + \sin 3x + x^2.$

### Answers

1. $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{6}\left(x - \frac{1}{6}\right)$
2. $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{6}(x + k + \frac{1}{6})$
3. $y = c_1 + c_2e^{-x} + c_3e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7x}{2}\right]$
4. $y = c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$
5. $y = c_1e^{-2x} + c_2e^{x} - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x)$
6. $y = (c_1 + c_2x)e^{x} + (c_3 + c_4x)e^{-x} + \frac{1}{2}(1 - 2x) - \frac{1}{8}\cos x + \frac{x^2e^x}{8}$
7. $y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax - \frac{1}{a^4}\left(x^4 + \frac{24}{a^4}\right) + \frac{\sin kx}{k^4 - a^4}$
8. $y = c_1\cos 2x + c_2\sin 2x + \frac{3}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}x^2 - \frac{1}{8}$

## 6.13. Q IS OF THE FORM $e^{ax}$ V, WHERE V IS ANY FUNCTION OF $x$

Let the linear differential equation be $f(D)\,y = Q$, where $f(D)$ is a polynomial in D with constant coefficients and $Q = e^{ax}V$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)}e^{ax}V$$

For any function U of $x$, we have

$$De^{ax}U = e^{ax}.\,DU + ae^{ax}.\,U = e^{ax}(D + a)U$$

$$D^2e^{ax}\,U = e^{ax}.\,D(D + a)U + ae^{ax}(D + a)U = e^{ax}(D + a)^2\,U$$

..................................................................................................

$\therefore$ In general $\quad D^k e^{ax}\,U = e^{ax}(D + a)^k\,U$

Since $f(D)$ is a polynomial in D with constant coefficients, we have

$$f(D)e^{ax}\,U = e^{ax}f(D + a)\,U. \qquad ...(1)$$

Let $\quad f(D + a)U = V \quad$ *i.e.,* $\quad U = \frac{1}{f(D + a)}V$

$\therefore \quad (1) \quad \Rightarrow \quad f(D)\, e^{ax} \dfrac{1}{f(D+a)} V = e^{ax} V.$

Operating with $\dfrac{1}{f(D)}$, we get $\quad e^{ax} \dfrac{1}{f(D+a)} V = \dfrac{1}{f(D)} e^{ax} V.$

$$\therefore \qquad \mathbf{\frac{1}{f(D)} e^{ax}\, V = e^{ax} \frac{1}{f(D+a)}\, V.}$$

**Example 1.** *Solve* $(D^2 - 4D + 4)\, y = e^{2x} \cos^2 x.$

**Sol.** We have $(D^2 - 4D + 4)y = e^{2x} \cos^2 x.$ ...(1)

$\therefore$ The A.E. is $D^2 - 4D + 4 = 0. \quad \therefore \quad D = 2, 2.$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} e^{2x} \cos^2 x = e^{2x} \frac{1}{(D+2-2)^2} \cos^2 x$$

$$= e^{2x} \frac{1}{D^2}\left(\frac{1+\cos 2x}{2}\right) = \frac{e^{2x}}{2}\left[\frac{1}{D^2}(1) + \frac{1}{D^2} \cos 2x\right]$$

$$= \frac{e^{2x}}{2}\left[\frac{x^2}{2} + \frac{1}{-(2)^2} \cos 2x\right] = \frac{e^{2x}}{8}[2x^2 - \cos 2x]$$

The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \qquad \mathbf{y = (c_1 + c_2\, x)e^{2x} + \frac{e^{2x}}{8}\, [2x^2 - \cos 2x].}$$

**Example 2.** *Solve* $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = x^2 e^x.$

**Sol.** The given equation in the symbolic form is

$$(D^3 - 3D + 2)\, y = x^2 e^x. \qquad \text{...(1)}$$

$\therefore$ The A.E. is $\quad D^3 - 3D + 2 = 0 \quad$ or $\quad (D-1)(D^2 + D - 2) = 0$

or $\qquad (D-1)(D+2)(D-1) = 0. \quad \therefore \quad D = 1, 1, -2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x + c_3 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^3 - 3D + 2} x^2 e^x = e^x \frac{1}{(D+1)^3 - 3(D+1) + 2} x^2$$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 - 3D - 3 + 2} x^2$$

$$= e^x \frac{1}{D^3 + 3D^2} x^2 = e^x \frac{1}{3D^2\left[1 + \frac{D}{3}\right]} x^2$$

$$= \frac{e^x}{3} \cdot \frac{1}{D^2}\left(1 + \frac{D}{3}\right)^{-1} x^2 = \frac{e^x}{3} \cdot \frac{1}{D^2}\left[1 - \frac{D}{3} + \left(\frac{D}{3}\right)^2 + \ldots\ldots\right] x^2$$

(Retaining terms only upto $D^2$)

$$= \frac{e^x}{3} \cdot \frac{1}{D^2}\left(x^2 - \frac{1}{3} Dx^2 + \frac{1}{9} D^2x^2 + 0 + \ldots\ldots\right)$$

$$= \frac{e^x}{3} \cdot \frac{1}{D^2}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) = \frac{e^x}{3} \frac{1}{D}\left(\int x^2\, dx - \frac{2}{3}\int x dx + \frac{2}{9}\int dx\right)$$

**NOTES**

$$= \frac{e^x}{3} \cdot \frac{1}{D}\left[\frac{x^3}{3} - \frac{x^2}{3} + \frac{2}{9}x\right] = \frac{e^x}{3}\left(\int \frac{x^3}{3}\,dx - \int \frac{x^2}{3}\,dx + \int \frac{2}{9}x\,dx\right)$$

$$= \frac{e^x}{3}\left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9}\right] = \frac{e^x}{108}\,(3x^4 - 4x^3 + 4x^2) = \frac{e^x\,x^2}{108}\,(3x^2 - 4x + 4)$$

The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \quad \mathbf{y = (c_1 + c_2x)\,e^x + c_3e^{-2x} + \frac{1}{108}\,x^2e^x(3x^2 - 4x + 4).}$$

**Example 3.** *Solve* $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x.$

**Sol.** The equation in the symbolic form is

$$(D^2 + 2)\,y = x^2 e^{3x} + e^x \cos 2x. \qquad \ldots(1)$$

$\therefore$ The A.E. is $\quad D^2 + 2 = 0. \quad \therefore \quad D = \pm\sqrt{2}\,i$

$\therefore$ C.F. $= e^{0.x}\,(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$

$$\text{P.I.} = \frac{1}{D^2 + 2}\,(x^2 e^{3x} + e^x \cos 2x)$$

$$= \frac{1}{D^2 + 2}x^2e^{3x} + \frac{1}{D^2 + 2}\,e^x \cos 2x$$

$$= e^{3x}\,\frac{1}{(D + 3)^2 + 2}\,x^2 + e^x\,\frac{1}{(D + 1)^2 + 2}\,\cos 2x$$

$$= e^{3x}\,\frac{1}{D^2 + 6D + 9 + 2}\,x^2 + e^x\,\frac{1}{D^2 + 2D + 3}\,\cos 2x$$

$$= e^{3x}\,\frac{1}{11 + 6D + D^2}\,x^2 + e^x\,\frac{1}{-4 + 2D + 3}\,\cos 2x$$

$$\left[\because \quad \frac{1}{f(D^2)}\,\cos ax = \frac{1}{f(-a^2)}\,\cos ax\right]$$

$$= e^{3x}\,\frac{1}{11\left[1 + \dfrac{6D + D^2}{11}\right]}\,x^2 + e^x\,\frac{1}{-1 + 2D}\,\cos 2x$$

$$= e^{3x}\,\frac{1}{11}\left(1 + \frac{6D + D^2}{11}\right)^{-1} x^2 + \frac{e^x\,(2D + 1)}{(2D - 1)(2D + 1)}\,\cos 2x$$

$$= e^{3x}\,\frac{1}{11}\left[1 - \frac{6D + D^2}{11} + \left(\frac{6D + D^2}{11}\right)^2 - \ldots\ldots\right]x^2 + \frac{e^x\,(2D + 1)}{4D^2 - 1}\,\cos 2x$$

$$= e^{3x}\,\frac{1}{11}\left[1 - \frac{6D + D^2}{11} + \frac{36}{121}\,D^2\right]x^2 + \frac{e^x\,(2D + 1)}{4(-4) - 1}\,\cos 2x$$

(Retaining terms only upto $D^2$)

$$= e^{3x}\,\frac{1}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121}\right]x^2 + \frac{e^x}{-17}\,(2D + 1)\cos 2x$$

$$= e^{3x}\,\frac{1}{11}\left[x^2 - \frac{6}{11}\,Dx^2 + \frac{25}{121}\,D^2x^2\right] + \frac{e^x}{-17}\,(2D \cos 2x + \cos 2x)$$

$$= e^{3x} \cdot \frac{1}{11}\left[x^2 - \frac{12}{11}\,x + \frac{50}{121}\right] - \frac{e^x}{17}\,(-4 \sin 2x + \cos 2x)$$

$$= \frac{1}{11}\,e^{3x}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] - \frac{e^x}{17}\,(-4 \sin 2x + \cos 2x).$$

The general solution of (1) is $y = C.F. + P.I.$

$$\therefore \quad y = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] - \frac{e^x}{17}[-4 \sin 2x + \cos 2x].$$

## EXERCISE 5

*Solve the following differential equations :*

1. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}$
2. $\dfrac{d^2y}{dx^2} + y = x e^{2x}$
3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1 + x)$
4. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$
5. $\dfrac{d^4y}{dx^4} - y = e^x \cos x$
6. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = e^x \cos 2x + \cos 3x$
7. $\dfrac{d^2y}{dx^2} + 2y = e^x x^2 + e^{2x} \sin x$
8. $4\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 9y = 144\, xe^{-3x/2}$
9. $(D^3 + 1)\, y = e^x \cos x$
10. $(D^3 - D^2 - 9D + 9)y = e^x(x^2 + \cos 2x)$
11. $(D^2 - 4D + 4)\, y = e^{2x} \cos^2 x$
12. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin 2x + x^2$
13. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x + x$
14. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = e^{2x} \cos x + k$, where $k$ is a constant.

### Answers

1. $y = (c_1 + c_2 x)\, e^x + \dfrac{1}{4} e^{3x}\left(x^2 - 2x + \dfrac{3}{2}\right)$
2. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{5} e^{2x}\left(x - \dfrac{4}{5}\right)$
3. $y = c_1 e^{3x} + c_2 e^{-x} + c_3 e^{-2x} - \dfrac{1}{12} e^{2x}\left(x + \dfrac{17}{12}\right)$
4. $y = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \dfrac{1}{2} e^x \cos x$
5. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{1}{5}\, e^x \cos x$
6. $y = c_1 e^x + c_2 e^{3x} - \dfrac{e^x}{8}(\cos 2x + \sin 2x) - \dfrac{1}{30}\,(\cos 3x + 2 \sin 3x)$
7. $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + \dfrac{e^x}{3}\left[x^2 - \dfrac{4}{3}x + \dfrac{2}{9}\right] + \dfrac{e^{2x}}{41}\,[5 \sin x - 4 \cos x]$
8. $y = (c_1 + c_2 x)\, e^{-3x/2} + 6x^3 e^{-3x/2}$
9. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \dfrac{\sqrt{3}}{2} x + c_3 \sin \dfrac{\sqrt{3}}{2} x\right) + \dfrac{1}{5} e^x\, (2 \sin x - \cos x)$
10. $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-3x} - \dfrac{e^x}{8}\left(\dfrac{x^3}{3} + \dfrac{x^2}{4} + \dfrac{3x}{8}\right) - \dfrac{e^x}{80}\,(3 \sin 2x + \cos 2x)$
11. $y = (c_1 + c_2 x)\, e^{2x} + \dfrac{1}{8} e^{2x}\, (2x^2 - \cos 2x)$
12. $y = (c_1 + c_2 x)\, e^{-x} - \dfrac{1}{8} e^x\, \cos 2x + x^2 - 4x + 6$

13. $y = e^x (c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x) + \frac{1}{2} e^x \cos x + \frac{x}{4} + \frac{1}{8}$

14. $y = c_1 e^{2x} + c_2 \cos x + c_3 \sin x - \frac{e^{2x}}{8} (\cos x - \sin x) - \frac{x}{2}$

**NOTES**

## 6.14. Q IS OF THE FORM $XV$, WHERE V IS ANY FUNCTION OF X

Let the linear differential equation be $f(D)\, y = Q$, where $f(D)$ is a polynomial in D with constant coefficients and $Q = xV$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)} xV$$

For any function U of $x$, we have

$$D\, xU = x\, DU + U$$

$$D^2\, xU = (x\, D^2U + DU) + DU = x\, D^2U + 2DU$$

..............................................................................

In general, $D^k\, xU = x\, D^k\, U + kD^{k-1}\, U$ or $D^k\, xU = x\, D^k U + \left(\frac{d}{dD} D^k\right) U$

Since $f(D)$ is a polynomial in D with constant coefficients, we have

$$\therefore \qquad f(D)\, xU = x\, f(D)U + f'(D)U. \qquad \ldots(1)$$

Let $\qquad f(D)U = V$ *i.e.*, $U = \frac{1}{f(D)} V$

$$\therefore \quad (1) \Rightarrow f(D)\, x \frac{1}{f(D)} V = xV + f'(D) \frac{1}{f(D)} V$$

Operating with $\frac{1}{f(D)}$, we get

$$x \frac{1}{f(D)} V = \frac{1}{f(D)} xV + \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V$$

$$\Rightarrow \qquad \frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V$$

$$\therefore \qquad \mathbf{\frac{1}{f(D)} xV = \left(x - \frac{1}{f(D)} f'(D)\right) \frac{1}{f(D)} V.}$$

**Corollary.** We have $\frac{1}{f(D)} xV = \left(x - \frac{1}{f(D)} f'(D)\right) \frac{1}{f(D)} V$.

$$\Rightarrow \qquad \frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{(f(D))^2} V$$

$$\Rightarrow \qquad \mathbf{\frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \left[\frac{d}{dD}\left(\frac{1}{f(D)}\right)\right] V.}$$

In practical problems, it is advisable to use this form.

NOTES

**Example 1.** *Solve* $\frac{d^2y}{dx^2} + 4y = x \sin x$.

**Sol.** The equation in the symbolic form is

$$(D^2 + 4)\, y = x \sin x. \qquad ...(1)$$

$\therefore$ The A.E. is $D^2 + 4 = 0$. $\therefore$ $D = \pm 2i$

$\therefore$ $\text{C.F.} = e^{0x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}(x \sin x) = x \frac{1}{D^2+4} \sin x + \left\{\frac{d}{dD}\left(\frac{1}{D^2+4}\right)\right\} \sin x$$

$$= x \frac{1}{-1+4} \sin x + \left(\frac{-1}{(D^2+4)^2} \cdot 2D\right) \sin x$$

$$= \frac{x \sin x}{3} - \frac{2D}{(-1+4)^2} \sin x = \frac{x \sin x}{3} - \frac{2}{9} D \sin x = \frac{x \sin x}{3} - \frac{2}{9} \cos x$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \quad \mathbf{y = c_1 \cos 2x + c_2 \sin 2x + \frac{x \sin x}{3} - \frac{2}{9} \cos x.}$$

**Example 2.** *Solve* $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$.

**Sol.** The equation in the symbolic form is

$$(D^2 + 3D + 2)\, y = x e^x \sin x. \qquad ...(1)$$

$\therefore$ The A.E. is $D^2 + 3D + 2 = 0$. $\therefore$ $D = -1, -2$

$\therefore$ $\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2+3D+2}(xe^x \sin x)$$

$$= e^x \frac{1}{(D+1)^2 + 3(D+1) + 2} x \sin x = e^x \frac{1}{D^2+5D+6}(x \sin x)$$

$$= e^x \left[x \frac{1}{D^2+5D+6} \sin x + \left(\frac{d}{dD} \frac{1}{D^2+5D+6}\right) \sin x\right]$$

$$= e^x \left[x \frac{1}{-1+5D+6} \sin x + \frac{-1(2D+5)}{(D^2+5D+6)^2} \sin x\right]$$

$$= e^x \left[x \frac{1}{5(1+D)} \sin x - \frac{(2D+5)}{(-1+5D+6)^2} \sin x\right]$$

$$= e^x \left[x \frac{(1-D) \sin x}{5(1+D)(1-D)} - \frac{2D+5}{(5+5D)^2} \sin x\right]$$

$$= e^x \left[x \frac{1}{5(1-D^2)}(1-D) \sin x - \frac{2D+5}{25(D+1)^2} \sin x\right]$$

$$= e^x \left[x \frac{1}{5(1+1)}(1-D) \sin x - \frac{2D+5}{25(D^2+2D+1)} \sin x\right]$$

$$= e^x \left[\frac{x}{10}(\sin x - D \sin x) - \frac{1}{25(-1+2D+1)}(2D+5) \sin x\right]$$

$$= e^x \left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{50}\left(2 + \frac{5}{D}\right) \sin x\right]$$

NOTES

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{2}{50}\sin x - \frac{1}{10}\cdot\frac{1}{D}\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$$

$\therefore$ The general solution of (1) is $y$ = C.F. + P.I.

$$\therefore \quad \mathbf{y = c_1e^{-x} + c_2e^{-2x} + e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right].}$$

## EXERCISE 6

*Solve the following differential equations :*

1. $\frac{d^2y}{dx^2} + 4y = x\cos x$
2. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$
3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$
4. $\frac{d^2y}{dx^2} - 9y = x\cos 2x$
5. $\frac{d^2y}{dx^2} - y = x\sin x + e^x(1 + x^2)$
6. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x\sin x$
7. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin 2x$
8. $\frac{d^2y}{dx^2} + y = xe^x\cos 2x.$

### Answers

1. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}x\cos x + \frac{2}{9}\sin x$
2. $y = (c_1 + c_2x)e^x + \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$
3. $y = (c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x)$
4. $y = c_1e^{3x} + c_2e^{-3x} - \frac{x}{13}\cos 2x + \frac{4}{169}\sin 2x$
5. $y = c_1e^x + ce^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)$
6. $y = c_1e^{-x} + c_2e^{-2x} + e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$
7. $y = (c_1 + c_2x)e^x - \frac{e^x}{4}(x\sin 2x + \cos 2x)$
8. $y = c_1\cos x + c_2\sin x + e^x\left[\frac{5x-1}{25}\sin 2x + \frac{11-5x}{50}\cos 2x\right]$

## 6.15. GENERAL METHOD OF EVALUATING PARTICULAR INTEGRAL

In case any of the shorter method is not applicable, then we evaluate the particular integral by using the general method. For this purpose the following formula is used very frequently :

$$\frac{1}{D-\alpha}Q = e^{\alpha x}\int Qe^{-\alpha x}\,dx.$$

**Example 1.** *Solve the differential equation*

$$(D^2 - 3D + 2)\,y = \sin e^{-x}$$

**Sol.** We have $(D^2 - 3D + 2)y = \sin e^{-x}$. ...(1)

$\therefore$ The A.E. is $D^2 - 3D + 2 = 0$. $\therefore$ $D = 1, 2$

$\therefore$ $\quad$ C.F. $= c_1 e^x + c_2 e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \sin e^{-x} = \frac{1}{(D-2)(D-1)} \sin e^{-x} = \frac{1}{D-2}\left(\frac{1}{D-1} \sin e^{-x}\right)$$

$$= \frac{1}{D-2}\left(e^x \int \sin (e^{-x})\, e^{-x}\, dx\right) \qquad \left|\; \because \frac{1}{D-\alpha} Q = e^{\alpha x} \int Q e^{-\alpha x}\, dx \right.$$

Here $\alpha = 1$, $Q = \sin e^{-x}$

Let $e^{-x} = t$. $\therefore$ $-e^{-x}\, dx = dt$ or $e^{-x}\, dx = -dt$

$$\therefore \quad \text{P.I.} = \frac{1}{D-2}\left(e^x \int -\sin t\, dt\right) = \frac{1}{D-2}\, e^x \cos t = \frac{1}{D-2}\, e^x \cos e^{-x}$$

$$= e^{2x} \int e^x \cos e^{-x} \,.\, e^{-2x}\, dx \qquad [\text{Here } \alpha = 2]$$

$$= e^{2x} \int e^{-x} \cos e^{-x} dx = e^{2x} \int -\cos t\, dt, \text{ where } t = e^{-x}$$

$$= -e^{2x} \sin t = -e^{2x} \sin e^{-x}.$$

$\therefore$ The general solution of (1) is $y$ = C.F. + P.I.

$\therefore$ $\quad$ $\mathbf{y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}}$.

**Remark.** We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$

Putting $x = i\theta$, $\quad e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\ldots$

$$\Rightarrow \qquad e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!}$$

$$\Rightarrow \qquad e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots\right)$$

$$\Rightarrow \qquad \mathbf{e^{i\theta} = \cos\theta + i \sin\theta}. \qquad \ldots(1)$$

$\therefore$ $\quad$ $\cos\theta$ is Real part of $e^{i\theta}$ and briefly written as **R.P.** of $e^{i\theta}$

and $\quad$ $\sin\theta$ is Imaginary part of $e^{i\theta}$ and briefly written as **I.P.** of $e^{i\theta}$.

Changing $i$ to $-i$ in equation (1), we get $e^{-i\theta} = \cos\theta - i\sin\theta$.

**Example 2.** *Solve* $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \sin x$.

**Sol.** We have

$$(D^4 + 2D^2 + 1)\, y = x^2 \sin x. \qquad \ldots(1)$$

$\therefore$ The A.E. is $\quad D^4 + 2D^2 + 1 = 0$ or $(D^2 + 1)^2 = 0$.

$\therefore$ $\quad D = \pm i, \; \pm i$

$\therefore$ $\quad$ C.F. $= e^{0.x} ((c_1 + c_2 x) \cos 1.x + (c_3 + c_4 x) \sin 1.x)$

$\therefore$ $\quad$ C.F. $= (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

$$\text{P.I.} = \frac{1}{D^4 + 2D^2 + 1} x^2 \sin x$$

$$= \text{I.P. of } \frac{1}{D^4 + 2D^2 + 1} x^2 e^{ix} \qquad [\because \sin\theta \text{ is I.P. of } e^{i\theta}]$$

**NOTES**

$$= \text{I.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{I.P. of } e^{ix} \frac{1}{((D+i)^2+1)^2} x^2$$

$$= \text{I.P. of } e^{ix} \frac{1}{(D^2+2iD-1+1)^2} x^2 \qquad \left| \because \; i^2 = -1 \right.$$

$$= \text{I.P. of } e^{ix} \frac{1}{(2iD+D^2)^2} x^2 = \text{I.P. of } e^{ix} \frac{1}{4i^2D^2\left(1+\frac{D}{2i}\right)^2} x^2$$

$$= \text{I.P. of } \frac{-1}{4} e^{ix} \frac{1}{D^2}\left(1+\frac{D}{2i}\right)^{-2} x^2$$

$$= \text{I.P of } \frac{-1}{4} e^{ix} \frac{1}{D^2}\left(1-2\left(\frac{D}{2i}\right)+3\left(\frac{D}{2i}\right)^2\right) x^2$$

$$\left| \because \; D^3x^2 = 0, D^4x^2 = 0, \ldots \right|$$

$$= \text{I.P. of } \frac{-1}{4} e^{ix} \left[\frac{1}{D^2} + i\frac{1}{D} - \frac{3}{4}\right] x^2 \qquad \left| \because \; \frac{1}{i} = -i \right.$$

$$= \text{I.P. of } \frac{-1}{4} (\cos x + i \sin x)\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}\left[\frac{x^3}{3}\cos x + \frac{x^4}{12}\sin x - \frac{3}{4}x^2 \sin x\right]$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \quad \mathbf{y = (c_1 + c_2x)\cos x + (c_3 + c_4 x)\sin x}$$

$$\mathbf{-\frac{1}{4}\left[\frac{x^3}{3}\cos x + \frac{x^4}{12}\sin x - \frac{3}{4}x^2 \sin x\right].}$$

**Example 3.** *Solve* $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

**Sol.** We have $(D^2 + a^2)\, y = \sec ax$. ...(1)

$\therefore$ The A.E. is $D^2 + a^2 = 0$. $\therefore$ $D = \pm ai$

$$\therefore \quad \text{C.F.} = e^{0.x}(c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2}\sec ax = \frac{1}{D^2-(-a^2)}\sec ax$$

$$= \frac{1}{D^2-(ia)^2}\sec ax = \frac{1}{(D-ia)(D+ia)}\sec ax$$

$$= \frac{1}{2ia}\frac{(D+ia)-(D-ia)}{(D-ia)(D+ia)}\sec ax$$

or

$$\text{P.I.} = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right] \qquad \ldots(2)$$

Now $\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\, e^{-iax}\, dx$ $\left[\because \; \frac{1}{D-\alpha}Q = e^{\alpha x}\int Q e^{-\alpha x}\, dx \quad \text{Here } \alpha = ia\right]$

$$= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax}\, dx \qquad (\because \quad e^{-i\theta} = \cos\theta - i\sin\theta)$$

$$= e^{iax} \int (1 - i \tan ax)\, dx$$

$$= (\cos ax + i \sin ax)\left(x - \frac{i \log \sec ax}{a}\right)$$

$$\therefore \quad \frac{1}{D - ia} \sec ax = x\cos ax - \frac{i}{a}\cos ax \log \sec ax + ix \sin ax + \frac{1}{a}\sin ax \log \sec ax \qquad ...(3)$$

Changing $i$ to $-i$ in (3), we get

$$\frac{1}{D + ia} \sec ax = x\cos ax + \frac{i}{a}\cos ax \log \sec ax - ix \sin ax + \frac{1}{a}\sin ax \log \sec ax \qquad ...(4)$$

$$(3) - (4) \Rightarrow \frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax = -\frac{2i}{a}\cos ax \log \sec ax + 2ix \sin ax$$

Putting this value in (2), we get

$$\text{P.I.} = \frac{1}{2ia} 2i\left(-\frac{1}{a}\cos ax \log \sec ax + x \sin ax\right)$$

$$= -\frac{1}{a^2}\cos ax \log \sec ax + \frac{x}{a}\sin ax.$$

$\therefore$ The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$$\therefore \quad \mathbf{y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log \sec ax + \frac{x}{a}\sin ax.}$$

## EXERCISE 7

*Solve the following differential equations :*

1. $(D^2 - 3D + 2)\, y = \cos e^{-x}$
2. $\frac{d^2y}{dx^2} - y = x^2 \cos x$
3. $\frac{d^2y}{dx^2} + 16y = \sec 4x$
4. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$
5. $\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$
6. $\frac{d^4y}{dx^4} - y = x \sin x.$

### Answers

1. $y = c_1 e^x + c_2 e^{2x} - e^{2x} \cos e^{-x}$
2. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x^2 - 1)\cos x + x \sin x$
3. $y = c_1 \cos 4x + c_2 \sin 4x - \frac{1}{16}\cos 4x \log \sec 4x + \frac{x}{4}\sin 4x$
4. $y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x$
5. $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}\sin ax \log \sin ax - \frac{x}{a}\cos ax$
6. $y = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} + \frac{1}{8}x^2 \cos x - \frac{3}{8}x \sin x.$

**NOTES**

## MISCELLANEOUS QUESTIONS

*Solve the following differential equations :*

**1.** $\dfrac{d^2y}{dx^2} - y = (1 + x^2)\, e^x$

**2.** $(D^4 + 6D^3 + 11D^2 + 6D)y = 20\, e^{-2x} \sin x$

**3.** $(D^3 + D^2 + D + 1)y = 2e^{-x} + x^2$

**4.** $(D^4 - 2D^2 + 1)y = 40 \cosh x$

**5.** $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x$

**6.** $(D^2 + 1)y = 3 \cos^2 x + 2 \sin^3 x$

**7.** $\dfrac{d^2y}{dx^2} + y = \sec x$

**8.** $(D^2 - 4D + 3)\, y = e^x \cos 2x + \cos 3x.$

### Answers

**1.** $y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{12} xe^x (2x^2 - 3x + 9)$

**2.** $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} + 2e^{-2x} (\sin x - 2 \cos x)$

**3.** $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + xe^{-x} + x^2 - 2x$

**4.** $y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x)\, e^{-x} + 5x^2 \cosh x$

**5.** $y = (c_1 + c_2 x)\, e^{2x} + \dfrac{3}{8} e^{2x} (3 \sin 2x - 4x \cos 2x - 2x^2 \sin 2x)$

**6.** $y = c_1 \cos x + c_2 \sin x + \dfrac{3}{2} - \dfrac{1}{2} \cos 2x - \dfrac{3}{4} x \cos x + \dfrac{1}{16} \sin 3x$

**7.** $y = c_1 \cos x + c_2 \sin x - \cos x \log \sec x + x \sin x$

**8.** $y = c_1 e^x + c_2 e^{3x} - \dfrac{e^x}{8} (\sin 2x + \cos 2x) - \dfrac{1}{30} (2 \sin 3x + \cos 3x).$